The Coulomb repulsion of the protons, as can be seen from (7), significantly diminishes the spectrum of mesons near the upper limit as compared with the cases considered above. As in the case of two neutrons the spins of the protons are anti-parallel and the cross section is independent of the direction of the vector $\mathbf{P}_n$.

5. The differential cross section σ_1 in the center of the mass system for the absorption of mesons by the deuteron is easily connected with the cross section σ_0, using the principle of detailed balance. A simple calculation gives

$$\sigma_1 = \sigma_0 (E_n^0) \frac{E_n^0}{E_\mu} \frac{[M + (\mu/2)]}{2\mu} \tag{8}$$

$$\times \frac{\sqrt{1 + (E_\mu^0/\mu c^2)}}{\sqrt{1 + (E_\mu/2\mu c^2)}} \frac{[1 + (2E_n^0/\mu c^2)]}{[1 + (E_\mu/\mu c^2)]};$$

$$E_n^0 = \mu c^2 + \frac{P_\mu^2}{4M} + E_\mu' - \varepsilon_0.$$

I express my thanks to N.M. Polievtov-Nikoladze and L.M. Soroke for interesting discussions.

Translated by A. S. Wightman
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SOVIET PHYSICS-JETP VOLUME 1, NUMBER 1 JULY, 1955

Remarks on the Theory of Fusion

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(Submitted to JETP editor February 25, 1953)
J. Exper. Theoret. Phys. USSR 28, 13-16 (January, 1955)

A generalization of De Broglie's theory of fusion is given, which holds for infinite-dimensional as well as finite-dimensional wave equations.

In de Broglie's theory of fusion, which generalizes the idea of the neutrino theory of light [1], a method of constructing particles with higher spin from particles with spin 1/2 was indicated [2].

In the present work, a point of view somewhat more general than the theory of fusion is proposed. The problem is formulated as follows: Consider two relativistically invariant equations, infinite-dimensional in general,

$$\gamma_\mu^{(1)} \frac{\partial\psi^{(1)}}{\partial x_\mu} + \varkappa'\psi^{(1)} = 0, \quad \gamma_\mu^{(2)} \frac{\partial\psi^{(2)}}{\partial x_\mu} + \varkappa''\psi^{(2)} = 0, \tag{1}$$

They will generate two reducible representations of the Lorentz group $\tau^{(1)}$ and $\tau^{(2)}$. It is required to find the invariant equation corresponding to the Kronecker product $(\tau^{(1)} \times \tau^{(2)})$. In our arguments, we will use the results and notation of the work of Gel'fand and Iaglom [3].

[1] A. A. Sokolov, J. Exper. Theoret. Phys. USSR 7, 1055 (1937)

[2] L. De Broglie, Théorie générale des particules a spin (méthode de fusion), Paris, 1943

[3] I. Gel'fand and A. Iaglom, J. Exper. Theoret. Phys. USSR 18, 703 (1948)

1. For the following, we need the reduction formula for the direct product of two irreducible representations of the Lorentz group. The infinitesimal representation of the group is determined by the operators F^+, F^-, F^0, H^+, H^-, H^0. Their form for irreducible representations is given by numbers k_0, k_1 [3]. If k_0, k_1 are simultaneously integral or half integral real numbers, and $k_0 < k_1$, then the representations are finite-dimensional.

The space of the representation is given by basis vectors ξ_p^k, where the total momentum k, appearing as the weight of the sub-group of rotations H^+, H^-, H^0, runs through the series of numbers ($k = k_0$, $k_0 + 1$, $k_0 + 2$, . . .). In the finite-dimensional case, the sequence k breaks off at $k = k_0 - 1$ ($p = k$, $k - 1$, . . . $-k$). It is not difficult to prove that the infinitesimal representations of the group, τ_n, have two scalar operators Δ_1 and Δ_2 which commute with every operator of the representation and have the form

$$\Delta_1 = F^+F^- + F^{02} - H^+H^- - H^{02} - 2iH^0, \qquad \Delta_2 = 2F^0H^0 + F^+H^- + F^-H^+. \tag{2}$$

These operators are given by the formulas

$$\Delta_1 \xi_p^k = (k_0^2 + k_1^2 - 1)\xi_p^k,$$
$$\Delta_2 \xi_p^k = 2i(k_0 k_1)\xi_p^k. \tag{2'}$$

The vectors $\xi^{(1)k_1}_{p_1} \xi^{(2)k_2}_{p_2}$, written in lexicographic order

$$(\xi^{(1)k_1}_{p_1}\xi^{(2)k_2}_{p_2} \prec \xi^{(1)l_1}_{m_1}\xi^{(2)l_2}_{m_2}, \quad \text{if}$$

$$k_1 < l_1 \quad \text{or} \quad k_1 = l_1,\ k_2 < l_2).$$

form a basis for the space of the product representation.

As is easily proved, the commutators among the operators Δ_1, Δ_2, given in (2), and the known scalar operator of the sub-group of rotations $[H^2 + k(k+1)]\,\xi^k_p = 0$, written in the basis $\xi^{(1)k_1}_{p_1} \xi^{(2)k_2}_{p_2}$ are simultaneously reduced to diagonal form by a matrix T. In the general case, T has the form: $T = AC$ where C is a matrix whose elements are finite-dimensional matrices $\| C^{k_1 + k_2 - \lambda}_{p_1 p_2} \|$ indexed by the parameters k_1, k_2. $C^{k_1 + k_2 - \lambda}_{p_1 p_2}$ denotes a Clebsch-Gordon coefficient[4]. A is a matrix commuting with H^2. (The particular form of A has to be determined for each separate case.) Hence, we have $\Delta_1' = T^{-1}\Delta_1 T$ and $\Delta_2' = T^{-1}\Delta_2 T$, where Δ_1' and Δ_2' are diagonal matrices.

From the proper values Δ_1' and Δ_2', we get, using formula (2), the weights $(\pm k_0^m, \pm k_1^m) = \tau_m$ of the representations into which $(\tau_n^{(1)} \times \tau_n^{(2)})$ reduces. The reduction works also for the infinite-dimensional case, but may be written especially simply in the case of finite-dimensional representations:

$$\begin{aligned}[(k_0'k_1') \times (k_0''k_1'')] &= (k_0' + k_0'',\ k_1' + k_1'' - 1) \\ &+ (k_0' + k_0'' + 1,\ k_1' + k_1'' - 2) + (k_0' + k_0'' - 1, \\ &k_1' + k_1'' - 2) + (k_0' + k_0'' + 2,\ k_1' + k_1'' - 3) \\ &+ (k_0' + k_0'',\ k_1' + k_1'' - 3) \\ &+ (k_0' + k_0'' - 2,\ k_1' + k_1'' - 3) + \ldots\end{aligned} \tag{3}$$

In the particular case in which $(k_0'k_1')$ and $(k_0''k_1'')$ are half integers, (3) coincides with a formula of Cartan[5]. Finally, it is impossible, using the spinor formalism, to generalize the development of Cartan to the infinite-dimensional cases.

2. With the use of this formula, we can solve the problem posed above of constructing generalized equations for the product functions $(\psi^{(1)}\psi^{(2)})$. The representations $\tau^{(1)}$ and $\tau^{(2)}$ are reduced into irreducible components $\tau^{(1)} \sim \tau_1^{(1)} + \tau_2^{(1)} + \ldots$; $\tau^{(2)} \sim \tau_1^{(2)} + \tau_2^{(2)} + \ldots$. Taking the product $(\tau^{(1)} \times \tau^{(2)})$, and using the above mentioned reduction into irreducible representations, we get a direct sum of irreducible components τ_i. We use a matrix T, whose elements are the matrices $\| T\,\tau_i\,\tau_j \|$, given in the preceding paragraph, where $(\tau_i\,\tau_j)$ are, as before, in lexicographic order.

As basis for the space of $(\tau_1^{(1)} \times \tau_2^{(2)})$, we take the vectors $g^k_{p\tau}$ connected with the vectors $\xi^{(1)k_1}_{p_1\tau_1}\ \xi^{(2)k_2}_{p_2\tau_2}$ by the formula:

$$g^k_{p\tau} = \sum_{\substack{k_1 k_2 \\ \tau_1\tau_2 \\ p_1 + p_2 = p}} T^{k;\ k_1\ k_2}_{p\tau;\ p_1\tau_1 p_2\tau_2}\, \xi^{(1)k_1}_{p_1\tau_1}\, \xi^{(2)k_2}_{p_2\tau_2}. \tag{4}$$

If, in the expression for the matrix T, we let $A = I$, where I is the unit matrix, we get from (4) the ordinary Clebsch-Gordon reduction. The expansion obtained in (4) determines the fundamental matrix γ_0, giving the generalized equation for the product functions:

$$\gamma^0 = \| c^k_{\tau\tau'}\delta_{pp'}\delta_{kk'} \|; \tag{5}$$

the form of $c^k_{\tau\tau'}$ is found in Gel'fand and Iaglom[3]. The quantity γ^0 can be computed immediately in the basis $\xi^{(1)k_1}_{p_1\tau_1}\ \xi^{(2)k_2}_{p_2\tau_2}$ from the known condition of Lorentz invariance

$$\gamma^0 = [[\gamma^0 F^0]\,F^0]. \tag{5'}$$

To do this, we represent γ^0 in the form

$$\gamma^{0\prime}\xi^{(1)k_1}_{p_1\tau_1}\xi^{(2)k_2}_{p_2\tau_2} = \sum_{\tau_1'\tau_2'} c^{k_1 k_2}_{\tau_1\tau_2\tau_1'\tau_2'}\,\xi^{(1)k_1}_{p_1\tau_1'}\,\xi^{(2)k_2}_{p_2\tau_2'}, \tag{6}$$

and use (5). Then we get 12 homogeneous equations relating the coefficients:

$$c^{k_1 \pm 1 k_2}_{\tau_1\tau_2\tau_1'\tau_2'},\quad c^{k_1\ k_2 \pm 1}_{\tau_1\tau_2\tau_1'\tau_2'},\quad c^{k_1 \pm 1 k_2 \pm 1}_{\tau_1\tau_2\,\tau_1'\tau_2'}, \tag{7}$$
$$c^{k_1 \pm 1 k_2 \mp 1}_{\tau_1\tau_2\,\tau_1'\tau_2'},\quad c^{k_1\ k_2}_{\tau_1\tau_2\tau_1'\tau_2'}.$$

A somewhat tedious calculation shows that this system of equations has non-trivial solutions only when $(\tau_1\ \tau_2)$ and $(\tau_1'\ \tau_2')$ are connected by one of the following possible correspondences:

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[4] van der Waerden, Die Gruppen Theoretische Methode in der Quanten Mechanik, Berlin, 1932

[5] E. Cartan, Leçons sur la Theorie des Spineurs, Paris, 1938

$$k_0'^{(1)} = k_0^{(1)},\ k_0'^{(2)} = k_0^{(2)},\ k_1'^{(1)} = k_1^{(1)} \pm 1,\ k_1'^{(2)} = k_1^{(2)} \pm 1,$$

or

$$k_0'^{(1)} = k_0^{(1)} \pm 1,\ k_0'^{(2)} = k_0^{(2)} \pm 1,\ k_1'^{(1)} = k_1^{(1)},\ k_1'^{(2)} = k_1^{(2)},$$

or

$$k_0'^{(1)} = k_0^{(1)} \pm 1,\ k_0'^{(2)} = k_0^{(2)},\ k_1'^{(1)} = k_1^{(1)},\ k_1'^{(2)} = k_1^{(2)} \pm 1,$$

or

$$k_0'^{(1)} = k_0^{(1)},\ k_0'^{(2)} = k_0^{(2)} \pm 1,\ k_1'^{(1)} = k_1^{(1)} \pm 1,\ k_1'^{(2)} = k_1^{(2)},$$

that is when $(\tau_1\ \tau_2)$ and $(\tau_1'\ \tau_2')$ are "linked" in pairs.

3. We now note the connection between the generalized equations for product functions, given by the formulas (5) and (5′) and the equations of the theory of De Broglie [2]. We consider the particular case of the fusion of two identical finite dimensional equations, whose matrices γ^0 are in diagonal form. Then the matrix β^0 determined by (5) and (5′), can be expressed in the form

$$\beta^0 = {}^1/_2(\gamma^0 I' + \gamma^{0\prime} I); \tag{8}$$

γ^0 and $\gamma^{0\prime}$ are labeled with different sets of indices; I and I' are the corresponding unit matrices. In this case, the proper values μ_i of the matrix β^0 are subject to the condition $\mu_i = 1/2(\lambda_k + \lambda_j)$ ((λ_k, λ_j are the proper values of the matrix γ^0); distinct μ_i correspond to distinct pairs λ_k, λ_j. The condition (8) gives the well known representation of the theory of De Broglie [2].

Thus it is proved that the equations of De Broglie's theory of fusion are obtained from our generalized equations for product functions, while the fusion condition, introduced in (2) is not used in this derivation. As is proved in (2), De Broglie equations consisting of different spins are reducible in the absence of external fields.

As an example, we consider the fusion of two spinor equations, appearing in the basic theory of De Broglie. As is proved in reference 3, the Dirac equation is given by the representation $(-1/2, 3/2) = (1/2, 3/2)$ where γ^0 has the form:

$$\gamma^0 = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

Multiplying the representations and using the reduction formula (3), we get the following irreducible representations (0 1), (0 1), (0 2), (0 2), (−1 2), (1 2). In order that the matrix γ^0 corresponding to this representation be reduced to diagonal form while the energy remains positive in agreement with the theorem of Pauli [6] it is necessary to represent the irreducible components in the form

$$\begin{gathered}(-1\ 2) = (0\ 2) = (1\ 2) \sim 1,\\ (0\ 1) = (0\ 2) \sim 2,\\ (0\ 1) \sim 3.\end{gathered}$$

Thus, the equation for the product functions splits into two equations: a ten dimensional equation giving representations 1, and a five-dimensional giving representations 2. The representation 3 yields the trivial one-dimensional equation.

It is easily shown that the resulting equations coincide with the equations of Duffin [6]. As is well known, those equations describe particles with spin 1 or 0, and their matrices satisfy the relation (8). It is easy to see that the possibility of more general schemes of "coupled" irreducible representations gives irreducible equations. Here the energy will be positive definite only if supplementary conditions are imposed. In that case the equations also describe particles with spin 1 and 0.

We note that in the corresponding generalized equations for product functions consisting of different spins, in general, no reduction takes place, even in the absence of external fields; this follows from (3). This mixing of components can be interpreted as the existence of internal interaction in the "product" particles.

The new type of interaction proposed in the present work differs from ordinary interactions in that internal interaction connects the component irreducible representations of the Lorentz group, but not, as in the usual case, the components of the ψ-functions. Thus, there appears to be a possibility of giving the structure of elementary particles on the basis of representations of the Lorentz group, infinite dimensional ones in general. The connection between internal interaction and positive definiteness conditions has been considered in the same way in the work of Gurevitch [8]. It is necessary to remark, that in the theory of De Broglie, particles of higher spin are described by reducible equations, which are represented some-

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[6] W. Pauli, Revs. Mod. Phys. **13**, 203 (1941)

[8] A. Gurevitch, J. Exper. Theoret. Phys. **24**, 149 (1953)

what artificially from the point of view of Gel'fand and Iaglom's theory of particles of higher spin. It is proposed that the consideration of the problem in the infinite-dimensional case can be connected with the general theory of non-local fields, giving vectors in hilbert space[7].

In conclusion, I want to express my thanks to Professor D. D. Tvanenko for a series of remarks and numerous discussions of the results of this work.

Translated by A. S. Wightman
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[7] D. Ivanenko and A. Sokolov, Klassische Feld. Theorie, Berlin (1953)

SOVIET PHYSICS-JETP VOLUME 1, NUMBER 1 JULY, 1955

On Longitudinal Vibrations of Plasma, 1.

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(Submitted to JETP editor February 17, 1954)
J. Exper. Theoret. Phys. USSR 28, 18-27 (January, 1955)

The problem of propogation of longitudinal waves in a plasma under given boundary conditions is solved. A "dispersion" equation which takes into account motion of the ions is obtained.

1. INTRODUCTION

In studying the processes discovered by Langmuir, of vibrations in a plasma which arise because of the Coulomb interaction of charged particles, one usually considers only the motion of the electrons. The method proposed by Vlasov[1] for investigating specific electrical properties of plasma--the self-consistent field method--basically is applied only to them. Because of their large mass, the ions are considered as a positive background having no effect on the vibration of the electrons.

It is true that recently the necessity for considering ionic vibrations has been pointed out in several papers[2-4] However, no detailed investigations like those for electrons[5,6] exist in the literature*.

Ions can play a significant role in the vibrational properties of a plasma. In fact, since the mean free paths of electrons and ions are comparable, the squares of their transitional velocities are inversely proportional to their masses M/m. Therefore, during the passage of electrons and ions through a region of varying potential (an oscillating electrical double layer, etc.) the amplitude of vibration of the ions may turn out to be comparable to that of the electrons or even to exceed it, if the time for traversal of the region is smaller than the period of the variable potential. This behavior is clear, since the time for traversal of the region by the ions is M/m greater than for the electrons, although the acceleration given to the ion is a factor m/M smaller than that of an electron.

The ions play a special role when we consider the peculiar auto-oscillation process in plasma which leads to the possibility for occurrence of practically undamped waves, despite the occurrence of collisions[3]. The presence of two streams of charged particles (electrons and ions) with different velocities results in the appearance of undamped oscillations if the losses of energy from the wave due to collisions are compensated by a transfer of energy from the directed motion to the wave via the Coulomb interaction of the two particle streams.

A rigorous solution of the problem of longitudinal plasma oscillations (not considering the ions or collisions) was given in a paper of Landau[8] for the case where the initial deviation of the distribu-

[1] A. A. Vlasov, J. Exper. Theoret. Phys. USSR 8, 291 (1938); Scientific Reports, Moscow State University 75, Book II, part I (1945); Theory of Many Particles, State Publishing House, (1950)

[2] G. Ia. Miakishev, Dissertation, Moscow State University, (1952)

[3] M. E. Gertzenshtein, J. Exper. Theoret. Phys. USSR 23, 669 (1952)

[4] V. P. Silin, J. Exper. Theoret. Phys. USSR 23, 649 (1952)

[5] D. Bohm and E. P. Gross, Phys. Rev. 75, 1851, 1864, (1949)

[6] Yu. L. Klimontovich, Dissertation, Moscow State University (1951); J. Exper. Theoret. Phys. USSR 21, 1284, 1292 (1951)

* We consider the paper of Bazarov [e.g., see I. P. Bazarov, J. Exper. Theoret. Phys. USSR 21, 711 (1951)], which investigates the effect of the ions on the propagation of longitudinal waves in a plasma, to be unsatisfactory . In particular, in deriving the initial dispersion equation which is the basis of the investigation, an algebraic error is made as a result of which electronic and ionic terms which are proportional to the square of the charge have different signs in Bazarov's equation.

[8] L. D. Landau, J. Exper. Theoret. Phys. USSR 16, 574 (1946)