$$w_n(E_0, p)\,dp = 2\pi\left(\frac{\pi}{2}\right)^{n-2}[E_0(E_0-2p)^{n-3}p \quad (17)$$

$$\times\sum_{r=0}^{n-1}\frac{C_{n-1}^r E_0^{n-r-1}(E_0-2p)^r}{(n+r-3)!\,(2n-r-4)!}$$

$$\times\left[\frac{E_0}{2n-r-3}-\frac{E_0-2p}{n+r-2}\right]dp$$

(relativistic case)

It should be observed that after the completion of the present work, the paper of Lepore and Stuart[2] appeared in which similar problems were investigated.

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[1] E. Fermi, Prog. Theor. Phys. 5, 570 (1950)

[2] J. V. Lepore and R. N. Stuart, Phys. Rev. 94, 1724 (1954)

* If the particles are identical, then the right side of Eq. (1) reduces to $n!$

The Fermi Distribution at Absolute Zero, Taking into Account the Interaction of Electrons with Zero Point Vibrations of the Lattice

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THE method of Green's function developed in connection with problems of relativistic quantum field theory[1] can also be used in a number of other problems. In particular, the investigation of the distribution function for an electron gas which takes into account the interaction of the electrons with phonons is of considerable interest.
The Green's function is

$$G_{ss'}(x,y) = \frac{i}{\langle S\rangle_0}\langle T\{\psi_s(x)\,\psi^*_{s'}(y)\,S\}\rangle_0. \quad (1)$$

Here S denotes the S matrix, $x=\{\mathbf{x}, x_0\}$, s, s' are spin indices, $\psi_s(x)$ is the wave function of the electronic field in the representation of the interaction. The Green's function (1), being decomposed into an arbitrary complete set of functions $\phi_\lambda(x)$ $\phi^*_{\lambda'}(y)$ characterizes the electron distribution for $x_0 < y_0$, and the hole distribution for $x_0 > y_0$, in the terms of the parameter λ.

The S matrix is given by the usual expression

$$S = T\exp\left\{i\int L(x)\,dx\right\}, \quad (2)$$

where L is the Lagrangian of the interaction (in a system of units in which $\hbar = c = 1$, c = speed of sound). For a system of electrons interacting with acoustic vibrations [2],

$$L = \{g\psi^*_s(x)\,\psi_s(x) + \rho(x)\}\,\varphi(x);\ \varphi = \partial A(x)/\partial x_0, \quad (3)$$

where g is a coupling constant and $\rho(x)$ is the "external charge density",

$$A(x) = \frac{1}{\sqrt{2\pi^3}}\int\frac{d\mathbf{f}}{|\mathbf{f}|}[b_{\mathbf{f}}\exp\{i\mathbf{f}\mathbf{x} - i|\mathbf{f}|x_0\} \quad (4)$$
$$+ b^*_{\mathbf{f}}\exp\{-i\mathbf{f}\mathbf{x} + i|\mathbf{f}|x_0\}],$$

$b^*_{\mathbf{f}}$, $b_{\mathbf{f}}$ are the Bose operators of "creation" and "annihilation" of phonons.

Integration over f is confined to the Debye limiting value of f_0. The equation for the Green's function can easily be found (for example, by the method given by Anderson[3]). It has the form

$$G_{s_1s_2}(x,y) = iK_{s_1s_2}(x,y) \quad (5)$$
$$-ig\int dz\,K_{s_1s'}(x,z)\,G_{s's_2}(z,y)\,a(z)$$
$$-\int dz\,dx'\,K_{s_1s'}(x,z)\,\Delta E_{s's''}(z,x')\,G_{s''s_2}(x',y).$$

Here the summation is carried out over the iterated spin indices;

$$a(z) = i\int F(z,z')\,\rho(z')\,dz'. \quad (6)$$

The functions $K_{ss'}(x,y)$ and $F(x,y)$ are the propagation functions of the "free" electron and phonon fields; E is the analog of the mass operator

$$\Delta E_{s's''}(z,x') \quad (7)$$
$$= -ig\int dz'\,dx''\,\frac{\delta a(z')}{\delta\rho(z)}F(z',z)\,G_{s's''}(z,x'')$$
$$\times[\delta G^{-1}_{s'''s''}(x'',x')/\delta a(z')].$$

We have for the functions $K_{ss'}$ and F (under the condition of complete degeneracy of the electron gas)

$$K_{s_1s_2}(x,y) = \langle T\{\psi_{s_1}(x)\,\psi^*_{s_2}(y)\}\rangle_0 \quad (8)$$
$$= \delta_{s_1s_2}\frac{i}{(2\pi)^4}\int dp\int_L dp_0\,\frac{\exp\{i(\mathbf{p},\mathbf{x}-\mathbf{y}) - ip_0(x_0-y_0)\}}{p_0-(p^2/2m)}$$
$$\equiv \delta_{s_1s_2}K(x-y).$$

$$F(x,y) = \langle T\{\varphi(x)\varphi(y)\}\rangle_0 \tag{9}$$
$$= \frac{1}{2\pi^3}\int d\mathbf{f}\,|\mathbf{f}|\exp\{i(\mathbf{f}, \mathbf{x}-\mathbf{y}) - i|\mathbf{f}|\cdot|x_0 - y_0|\}$$

(The curve L runs from $-\infty$ below the real axis, intersects it at the point $p_0 = E_\Phi$ and goes above the axis to $+\infty$; $E_\Phi = p_\Phi^2/2m$ is the Fermi energy).

In the theory of weak coupling it is customary to carry out an expansion in powers of the constant g. It is essential, however, that one decompose not the Green's function itself (this, generally speaking, is not admissible,) but the equation for it, i.e., the matrix ΔE. Expanding $\Delta E = \Delta E^{(0)} + g\Delta E^{(1)} + g^2\Delta E^{(2)} + \ldots$, we obtain

$$\Delta E^{(0)} = \Delta E^{(1)} = 0;\ \Delta E^{(2)}_{s's''} \tag{10}$$
$$= -iF(x', z)\,K_{s's''}(z, x').$$

Equation (5) can easily be determined from this expression:

$$G_{s's''}(x,y) = \delta_{s's''}\frac{-1}{(2\pi)^4}\int d\mathbf{p} \tag{11}$$
$$\cdot\int_L dp_0 \frac{\exp\{i(\mathbf{p}, \mathbf{x}-\mathbf{y}) - ip_0(x_0 - y_0)\}}{p_0 - (p^2/2m) + ig^2 f(\mathbf{p}, p_0)},$$

where

$$f(\mathbf{p}, p_0) = \int K(x)F(-x)\exp\{i\,\mathbf{px} - ip_0x_0\}\,dx \tag{12}$$

(we note that f is a purely imaginary function).

The energy spectrum of the electron which interacts with the zero point vibrations of the lattice is given by the expression

$$W(\mathbf{p}) = \frac{p^2}{2m} - ig^2 f(\mathbf{p}, p_0^*), \tag{13}$$

where p_0^* is the root of the equation

$$p_0 - \frac{p^2}{2m} + ig^2 f(\mathbf{p}, p_0) = 0.$$

Computing the integral of (12) and integrating over p_0 in (13) (for $x_0 > y_0$) we obtain the energy spectrum and the hole distribution function in terms of the momentum $\Phi(p)$. Denoting by p_Φ the limiting momentum, we get the approximation below, with accuracy to small quantities of order m/p_Φ:

$$\Phi(\mathbf{p}) = 0,\ p < p_c, \tag{14}$$

$$\Phi(\mathbf{p}) = \Bigg[1 + \frac{2g^2m^2}{\pi^2}\Bigg\{\ln\left(1 + \frac{f_0^4}{4p^4}\right)$$
$$+ 2\left[\arctan\left(1 + \frac{f_0}{p}\right) + \arctan\left(1 - \frac{f_0}{p}\right) - \frac{\pi}{2}\right] \tag{15}$$
$$+ \frac{4}{3}\frac{(4p^2 - f_0^2)f_0^2}{4p^4 + f_0^4}\Bigg\}\Bigg]^{-1},$$

where

$$p_c = p_\Phi\Bigg\{1 + \frac{2g^2m^2}{\pi^2}\Bigg[\frac{2}{3}\frac{f_0^2}{p_\Phi^2} \tag{16}$$
$$+ \frac{f_0^3}{3p_\Phi^3}\ln\frac{f_0^2 + 2f_0p_\Phi + 2p_\Phi^2}{f_0^2 - 2f_0p_\Phi + 2p_\Phi^2}$$
$$+ \frac{4}{3}\ln\left(1 + \frac{f_0^4}{4p_\Phi^4}\right) + \frac{8}{3}\left(\mathrm{arctg}\left(1 + \frac{f_0}{p_\Phi}\right)\right.$$
$$\left.+ \mathrm{arctg}\left(1 - \frac{f_0}{p_\Phi}\right) - \frac{\pi}{2}\right)\Bigg]\Bigg\}.$$

$$W(p) = \frac{p^2}{2m} - g^2\frac{m}{\pi^2 p}\Bigg\{\frac{2}{3}pf_0^2 \tag{17}$$
$$+ \frac{f_0^3}{3}\ln\frac{f_0^2 + 2f_0p + 2p^2}{f_0^2 - 2f_0p + 2p^2} + \frac{4}{3}p^3\ln\left(1 + \frac{f_0^4}{4p^4}\right)$$
$$+ \frac{8p^3}{3}\left[\mathrm{arctg}\left(1 + \frac{f_0}{p}\right) + \mathrm{arctg}\left(1 - \frac{f_0}{p}\right) - \frac{\pi}{2}\right]\Bigg\}.$$

These formulas are applicable for $m \ll p < E_\Phi + \frac{1}{2}m$ (i.e., in particular, close to the Fermi surface $p = p_\Phi$). Thus, because of the interaction with phonons, the Fermi distribution at absolute zero is somewhat broken up. Suitable calculations, associated with the application of the Green's function to the determination of distribution functions will be published in a separate paper.

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[1] J. Schwinger, Proc. Nat. Acad. Sci. 37, 452 (1951)
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