The Quantum Theory of the Radiating Electron, IV

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An expression is derived for the quantum corrections to the trajectory of a relativistic electron moving in an axially symmetric magnetic field.

IN a series of papers[1-3] on the theory of the radiating electron, it was shown that the electromagnetic radiation emitted by an electron moving in a magnetic field becomes extremely intense at high velocities. The radiative energy loss produces a significant contraction of the orbit radius.

A quantum mechanical treatment shows that in addition to the contraction of the orbit the radiation causes radial oscillations of the electron (radiation recoil effect) which leads to a broadening of the trajectory. In our earlier papers[4-6] on the quantum theory of the radiating electron, we investigated these effects for an electron in a constant magnetic field (H = const).

In the present paper we use the methods of our papers[4-6] and generalize the results to the case of of an inhomogeneous magnetic field with axial symmetry, assuming the field in the neighborhood of the stable orbit to be given by

$$H_x = H_y = 0, \quad H_z = H_0 r^{-q}, \tag{1}$$

$$0 < q < 1, \quad z = 0.$$

Our old results will be obtained by putting $q = 0$.

It is convenient to take the components of the vector potential $\mathbf{A}$ in the form

$$A_x = -yH_0/r^q(2-q), \tag{2*}$$

$$A_y = xH_0/r^q(2-q), \quad A_z = 0.$$

Neglecting quantities of order $\hbar^2$, we need not consider spin effects, i.e., we may use the scalar relativistic wave equation

$$\Big\{E^2 + c^2\hbar^2\nabla^2 - e^2A^2 - \frac{2ec\hbar}{i}(\mathbf{A}\overrightarrow{\nabla}) - m^2c^4\Big\}\psi = 0, \tag{3}$$

where E is the energy and m the rest mass of the electron. In what follows we investigate only the motion of the electron in the orbital plane, so that we look for solutions of Eq. (3) of the form

$$\psi = \frac{e^{il\varphi}}{\sqrt{2\pi}}\sqrt{\frac{mc^2}{E}\frac{1}{r}}\,u(r). \tag{4}$$

The radial function u is normalized to unity,

$$\int_0^\infty u^2(r)\,dr = 1. \tag{5}$$

The azimuthal quantum number l takes large integer values ($l \gg 1$) in all cases of interest

The wave equation for the radial function is

$$u'' + f(r)\,u = 0, \tag{6}$$

[1] D. D. Ivanenko and I. Ia. Pomeranchuk, Doklady Akad. Nauk SSSR 44, 343 (1944)

[2] D. D. Ivanenko and A. A. Sokolov, Doklady Akad. Nauk SSSR 59, 1551 (1948)

[3] D. D. Ivanenko and A. A. Sokolov, *Classical Field Theory*, Gostekhisdat, 1951. A detailed bibliography of the classical theory of the radiating electron will be found in this monograph.

[4] A. A. Sokolov, N. P. Klepikov and I. M. Ternov, J. Exper. Theoret. Phys. USSR 23, 632 (1952)

[5] A. A. Sokolov, N. P. Klepikov and I. M. Ternov, J. Exper. Theoret. Phys. USSR 24, 249 (1953); see also Doklady Akad. Nauk SSSR 89, 665 (1953)

[6] A. A. Sokolov and I. M. Ternov, J. Exper. Theoret. Phys. USSR 25, 698 (1953); see also Doklady Akad. Nauk SSSR 92, 537 (1953)

* It is well known that the external currents which maintain the magnetic field must not lie near to the orbital plane $z = 0$, i.e., the condition curl H = 0 must be fulfilled. This is achieved by letting the vector-potential A depend on z:

$$A_x = -\frac{yH_0}{r^q(2-q)}\left(1 + \frac{q(2-q)}{2r^2}z^2\right), \tag{2a}$$

$$A_y = \frac{xH_0}{r^q(2-q)}\left(1 + \frac{q(2-q)}{2r^2}z^2\right), \quad A_z = 0.$$

Then it is easy to see that

$$\operatorname{curl}\mathbf{H}\,|_{z=0} = 0.$$

When $z = 0$, Eq. (2a) reduces to Eq. (2).

with

$$f(r)=\frac{E^2-m^2c^4}{c^2\hbar^2}-\gamma^2 r^{2(1-q)}-2\gamma l r^{-q} \qquad (7)$$
$$-\frac{l^2-{}^1/_4}{r^2},$$

$$\gamma=eH_0/c\hbar(2-q), \quad u'=du/dr.$$

We develop $f(r)$ in a Taylor series in the neighborhood of its maximum given by

$$f'(a)=0. \qquad (8)$$

The last equation fixes the value of the radius of the stable orbit

$$a=\left[\frac{l}{\gamma(1-q)}\right]^{1/(2-q)} \qquad (9)$$

In deriving Eq. (9) we neglected the ¼ in Eq. (7) in comparison with l^2.

Keeping only terms to the order of $x^2=(r-a)^2$ in the Taylor series, the function u satisfies

$$u''+(\alpha-\lambda^2x^2)u=0, \qquad (10)$$

where

$$\alpha=f(a)=\frac{E^2-m^2c^4}{c^2\hbar^2}-\gamma^2(2-q)^2a^{2(1-q)}, \qquad (11)$$
$$\lambda^2=-\frac{1}{2}f''(a)=\gamma^2(2-q)^2(1-q)a^{-2q}.$$

With good accuracy we may assume that x varies between the limits $-\infty$ and $+\infty$, so that Eq. (10) is identical with the equation of motion of a harmonic oscillator. The frequency Ω of radial oscillations of the electron about the stationary orbit is given by

$$\lambda^2=\mu^2\Omega^2/\hbar^2, \qquad (12)$$

where $\mu=E/c^2$ is the relativistic mass of the electron. From Eqs. (11) and (12) we find

$$\Omega=(v/a)\sqrt{1-q} \qquad (13)$$

in agreement with the known result[7] obtained from classical theory.

2. From Eq. (9) it is easy to show that, in the absence of radiation, the radius a will remain constant (independent of the value of the magnetic field) only when Wideroe's condition is satisfied. In fact, by the adiabatic principle of Ehrenfest, the quantum number $(l-l_0)$ must remain constant when the magnetic field increases slowly with time. From Eq. (9) we find

$$=\frac{d(l-l_0)}{dt}=\frac{d}{dt}\frac{ea^2}{c\hbar}\left[H(a)-\frac{1}{2}\overline{H}(a)\right], \qquad (14)$$

where $H(a)$ and $\overline{H}(a)$ are respectively the magnetic field and its mean value over the stable orbit,

$$a^2\overline{H}(a)=2\int_0^a H(r)\,r\,dr+\frac{2c\hbar l_0}{e}, \qquad (15)$$

and the constant term depnds on the law of variation of the magnetic field away from the region of the stationary orbit. From Eq. (14) it follows that the radius a can remain constant while H_0 is varied, only if $H(a)=\frac{1}{2}\overline{H}(a)$.

3. From the wave equation (10) we find the eigenvalues of the energy and the eigenfunctions describing radial oscillations [see also reference 4, Eq. (45)]:

$$\frac{\alpha}{\lambda}=2s+1, \qquad (16)$$

$$E=c\hbar\left((2-q)\frac{\gamma}{a^q}\left[2l+a^{2-q}\,q\gamma\right.\right.$$
$$\left.\left.+(2s+1)\sqrt{1-q}\right]+m^2c^2\hbar^{-2}\right)^{1/2}$$

$$u_s=\sqrt[4]{\frac{\lambda}{\pi}}\frac{1}{\sqrt{2^s s!}}e^{-(\lambda/2)(r-a)^2}H_s(\sqrt{\lambda}(r-a)). \qquad (17)$$

Here H_s is a Chebyshev-Hermite polynomial, and $s=0, 1, 2,\ldots$ is the radial quantum number.

Using the wave-functions (17), it is easy to find the mean-square fluctuation of the radius which defines the breadth of the trajectory:

$$\xi^2=\overline{x^2}=\int_0^\infty x^2u_s^2\,dr\approx\int_{-\infty}^\infty x^2u_s^2\,dx=s/2\gamma', \qquad (18)$$

where

$$\gamma'=\frac{eH(a)}{2c\hbar}\sqrt{1-q}.$$

For a constant magnetic field ($q=0$), the value of ξ^2 reduces to the expression $\xi^2=s/2\gamma$ found by us previously[6].

4. We shall determine the angular frequency $\omega_{\nu\nu'}$, and the probability $w_{\nu\nu'}$ of the radiation associated with a transition of the electron from the state l,s to the state l',s', ($\nu=l-l'$, $\nu'=s-s'$), keeping terms only up to the order ν^2/l. These are

$$\omega_{\nu,\nu'}=\omega_0(\nu+\nu'\sqrt{1-q}), \qquad (19)$$

$$w_{\nu,\nu'}=\frac{W_{\nu,\nu'}}{\hbar\nu\omega_0}=\frac{1}{\hbar\nu\omega_0}I^2_{s,s'}(x)\frac{3\sqrt{3}}{4\pi} \qquad (20)$$
$$\times\frac{e^2c}{a^2}\left(\frac{E}{mc^2}\right)^4 y\,dy\int_y^\infty K_{5/3}(z)\,dz.$$

Here $W_{\nu\nu'}$ is the radiation intensity,

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[7] D. Kerst and R. Serber, Phys. Rev. **60**, 53 (1941); see also the Science-review volume *Betatron*, Moscow, 1948

$$y = \frac{2}{3}\nu\left(\frac{mc^2}{E}\right)^3, \quad (21)$$

$$x = \frac{\nu^2}{2(2-q)\sqrt{1-q}}\,\frac{1}{l}, \qquad \omega_0 = \frac{v}{a},$$

and the function $I_{ss'}(x)$ is connected with the Laguerre polynomial $Q_s^{\nu'}(x)$ by

$$I_{s,s'}(x) = (-1)^{\nu'}\frac{x^{\nu'/2}e^{-x/2}}{\sqrt{s!\,s'!}}\,Q_{s'}^{\nu'}(x). \quad (22)$$

It is easy to see that the intensity (20) differs from the corresponding classical expression (in this approximation) only by the factor $I^2_{ss'}$, which as we repeatedly showed (for example in reference 6) reduces to unity when summed over all final states s'. In this approximation therefore

$$W_\nu = \sum_{\nu'} W_{\nu,\nu'} = W_{\text{class}} \quad (23)$$

5. We shall use the quantum expressions to determine the change in orbit radius produced by the radiation. In a single transition the quantum number l changes by an amount $\Delta l_1 = l' - l = -\nu$. Multiplying this by $w_{\nu\nu'}\,dt$ and summing over all possible transitions, we obtain the change dl occurring in an interval of time dt:

$$dl = -\sum_{\nu,\nu'}\nu w_{\nu,\nu'}dt = -\frac{2}{3}\frac{e^2}{r}\left(\frac{E}{mc^2}\right)^4\frac{1}{\hbar}\,dt. \quad (24)$$

Hence, using the equation

$$e\left\{\frac{\overline{H(r,t)}}{2} - H(r,t)\right\} \quad (25)$$

$$= -\frac{e\Delta r}{a}\,\frac{\partial aH(a,t)}{\partial a} = -\frac{\Delta r}{a}\frac{\partial E}{\partial a}$$

(see reference 3) where $\Delta r = r - a$, and also using Eqs. (14) and (15), we find the well-known classical result for the radiative contraction of the orbit radius (see page 260 of refernce 3),

$$\Delta r = -\frac{2}{3}\left(\frac{e^2}{mc^2}\right)^3\frac{a^2B^3(a)}{mce(1-q)}\,\frac{1}{\omega'}\,F_1(\omega' t), \quad (26)$$

with

$$F_1(y) = \frac{1}{\sin y}\int_0^y \sin^4 z\,dz \quad (27)$$

$$= \frac{3}{8}\left\{\frac{y}{\sin y} - \cos y - \frac{2}{3}\sin^2 y\cos y\right\}.$$

In deriving Eq. (26) we assume that the magnetic field H is increasing slowly with time,

$$H = B\sin\omega' t, \quad E = E_0\sin\omega' t,$$

$$\omega' \ll \omega_0 = v/a,$$

so that the field remains practically constant during the period of one revolution of the electron.

6. We proved[6] in the case of a constant magnetic field ($q = 0$) that the quantum corrections to the intensity of radiation and to the contraction of the orbit are of order $(E/mc^2)^2(\hbar/mca)$ relative to the classical values. Therefore the quantum corrections become an appreciable fraction of the whole effect only at an energy comparable with $E_{½} = mc^2(mca/\hbar)^{½}$. Considering quantum corrections in general, the most interesting effect to investigate is the broadening of the trajectory (fluctuation of the orbit radius) which is characterized by the radial quantum number s.

As we see from Eq. (18), the quantity s remains unchanged as the magnetic field $H(a)$ is increased adiabatically. Therefore the mean-square breadth of the trajectory ξ^2 must decrease[7] inversely to $H(a)$, since, by Eq. (18),

$$\xi^2 H(a) = sc\hbar/e\sqrt{1-q} = \text{const}.$$

However, as we showed in the case of a constant field, at an energy of the order of $E_{1/5} = mc^2(mca/\hbar)^{1/5}$ pure quantum transitions involving a change in the radial quantum number s begin to occur. These transitions, in contrast to the classical theory, must produce an increase in the trajectory breadth $\sqrt{\xi^2}$. The quantum broadening of the trajectory begins to outweigh the classical decrease in breadth at a comparatively low energy $E = aE_{1/5}$ with a about 3. This broadening of the trajectory is an increase in the amplitude of radial oscillations, arising from the recoil when photons are emitted.

The formula for the increase in ξ^2 were derived by us[6] for the case of a constant magnetic field. Here we generalize the results to the case of an axially symmetric field varying adiabatically with time.

By Eqs. (18) and (22) we have

$$ds = -\sum_{\nu,\nu'}\nu' w_{\nu,\nu'}\,dt \quad (28)$$

$$= \frac{55}{48\sqrt{3}}\,\frac{e^2}{mcr^2}\left(\frac{E}{mc^2}\right)^6\frac{dt}{(1-q)^{3/2}},$$

where $\nu' = s - s'$. In addition we had to take into account the expression for the change $\Delta\xi^2$ produced by the emission of a single photon with energy ΔE,

$$\Delta\xi^2 = \frac{1}{2\gamma}\sum_{s'}(s'-s)\,I^2_{s,s'}(x) = \frac{2a^2}{(2-2q)^2}\left(\frac{\Delta E}{E}\right)^2, \quad (29)$$

which will evidently remain valid in the presence of forces acting on the electron parallel to its trajectory, or in the presence of other transient forces. Hence we find the law of variation of the breadth of the trajectory

$$\xi^2 = \xi_0^2 \frac{\beta_0 E_0 r}{E(t)\, a} + \frac{55}{48\sqrt{3}} \frac{e^2 \hbar r}{m(1-q)^2 E(t)} \int_0^t \left(\frac{E(t)}{mc^2}\right)^6 \frac{1}{r^2}\, dt, \quad (30)$$

where $c\beta_0$, E_0 and a represent the initial values of velocity, energy and radius. The last equation gives for $q = 0$ the variation of ξ^2 in a constant magnetic field, agreeing with our earlier result[6].

In particular, if the energy increases like $E = E_0 \sin \omega' t$, the quantum broadening becomes

$$\xi^2_{qv} = \frac{55}{48\sqrt{3}} \frac{1}{(1-q)^2} \frac{e^2}{mc} \frac{\hbar}{mca} \left(\frac{E_0}{mc^2}\right)^5 \frac{1}{\omega'} F_2(\omega' t), \quad (31)$$

with

$$F_2(y) = \frac{1}{\sin y} \int_0^y \sin^6 z\, dz \quad (32)$$

$$= \frac{1}{\sin y} \left\{ \frac{5}{16} y - \frac{15}{64} \sin 2y + \frac{3}{64} \sin 4y - \frac{1}{192} \sin 6y \right\}.$$

The maximum value of F_2 occurs when $\omega' t = \pi/2$ and is equal to $F_2^{MAX} = F_2(\pi/2) = (5/32)\pi$. From Eq. (30) we see that the first term, which comes from the classical theory, gives a mean-square breadth of trajectory decreasing inversely with energy; the second term, which is obtained only from a quantum treatment, gives a mean-square breadth increasing proportional to E^5.

Note added in proof. In a recently published paper [Phys. Rev. **97**, 470 (1955)], Sands investigated the effect of quantum fluctuations on the phase-oscillations of a synchroton. We considered, in our series of papers on the quantum theory of the radiating electron (see for example reference 6), a similar mechanism for the quantum excitation of macroscopic radial oscillations. Sands' final result [Eq. (23)] can be derived from our Eq. (18) [Doklady Akad. Nauk SSSR **97**, 823 (1954)] or from Eq. (30) of this paper, if one restricts the action of the beatron oscillations, which produce the main part of the total effect, and which are correctly described by our theory, to operate only for a time equal to the decay-time of the synchroton oscillations. This decay-time is shorter than the acceleration time of the electrons.

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