Magnetostatics with Ferromagnetics

V. I. SKOBELKIN AND R. N. SOLOMKO
Moscow State University

(Submitted to JETP editor December 8, 1954)
J. Exper. Theoret. Phys. USSR 28, 385-393 (April, 1955)

The formulation and the foundation of a variational principle of magnetostatics with ferromagnetics in a field of currents is given. A direct method of analysis of magnetic fields in a general case of non-linear dependence on the magnetic permeability of a magnetic field is worked out.

THE present article is devoted to the general solution of the problem of magnetostatics with ferromagnetics.

Let us assume a closed magnetostatic system in space, containing n finite ferromagnetic domains $V_1, V_2, \ldots V_n$, limited by the surfaces $S_1, S_2, \ldots, S_n$, and having m finite domains of currents $\omega_1, \omega_2, \ldots \omega_m$, limited by the surfaces $\sigma_1, \sigma_2, \ldots \sigma_m$; in the domains ω_i, the magnetic permeabilities $\mu(H, x, y, z)$ are uniquely determined and the induction field **B** is given by the relation $\mathbf{B} = \mu \mathbf{H}$; also given in the domains ω_i is the current density distribution $\mathbf{j}_i(x, y, z)$ satisfying the condition $\operatorname{div} \mathbf{j}_i = 0$; outside the ferromagnetics $\mu = 1$ (using the absolute system of units).

To a system so defined correspond the following physical conditions: 1) The ferromagnetics are isotropic; 2) μ is determined by the magnetization curves; 3) The ferromagnetics may have inhomogeneously distributed magnetic properties.

The problem is to determine the induction field **B** of the system.

The Maxwell equations satisfying such a magnetostatic system have the form:

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c}\mathbf{j}; \quad \operatorname{div} \mathbf{B} = 0; \quad \mathbf{B} = \mu \mathbf{H},$$

where the parameters μ and j on the surfaces of discontinuity s have to satisfy the boundary conditions*

$$[H_\tau]_s = 0; \quad [B_n]_s = 0,$$

where H_τ is the tangential component of the field **H**, and B_n is the normal component of the field **B**.

The integration of Maxwell's equations when the boundary conditions are given, represent insurmountable difficulties[1,2] because of non-linearity of the relationship between **B** and **H**. Up to now, excluding the case of toroid and ellipsoid forms, the analysis of magnetostatic problems has been confined to idealized ferromagnetics[3], for which $\mu = \text{const}$ or $\mu = \infty$.

1. VARIATIONAL PRINCIPLE**

To solve the given problem it is expedient not to start from Maxwell's equations but to put as a base a variational principle, which can be stated as follows. Among all possible solenoidal induction fields for a real closed magnetostatic system the sum E of its magnetic energy W and the potential function U of the currents has a minimum.

The mathematical expression of this principle leads to the equation

$$\delta E = \delta (W + U) = 0, \tag{1}$$

under the condition

$$\operatorname{div} \mathbf{B} = 0. \tag{2}$$

Here

$$W = \int w \, dV = \int \Big(\frac{1}{4\pi} \int\limits_{\mathbf{H}=B=0}^{B} \mathbf{H} \cdot d\mathbf{B} \Big) dV, \tag{3}$$

where w is the magnetic energy density, dV - the volume element (the volume integration is performed over the whole field of the system);

$$U = -\frac{1}{c}\sum_{i=1}^{m} \int\limits_{0}^{I_i} \Phi \, dI = -\frac{1}{c}\sum_{i=1}^{m} \int\limits_{s_i} \Phi j_i ds, \tag{4}$$

where I_i is the total current ♀ in the domain ω_i, ds- element of the surface normal to j, Φ is the flux of

[1] I. E. Tamm, *Foundations of the Theory of Electricity*, Moscow, 1949; pp. 327, 227

[2] S. V. Vonsovskii and Ia. S. Shur, *Ferromagnetism*, Moscow, 1948; p. 27

* The symbol $[A]_s = 0$ represents the difference of the value of the parameter A on both sides of the discontinuity surfaces.

[3] G. A. Grinberg, *Selected Problems of the Mathematical Theory of Electrical and Magnetic Phenomena*, Akad. Nauk SSSR, 1948

** The variational principle and its formulation giving the possibility of analysis of magnetic fields by the direct methods of variational calculus, were discribed by one of the authors (Skobelkin).

♀ If the current domain ω_i is of the form of a solenoid, then I_i means ampere-turns, and s is the cross-section of the solenoid.

B through the surface, limiting the current density j. The physical meaning of the function U is given[1] by the fact that the work done by the forces due to magnetic field for a virtual change of this field is equal to a decrease of the function U. From Eqs. (1) and (2) in particular, we obtain Maxwell's equation:

$$\text{curl } \mathbf{H} = \frac{4\pi}{c}\mathbf{j}. \tag{5}$$

The application of direct methods to obtain a solution of Eq. (1) under the condition (2), is most effective in conjunction with systems which have a plain-parallel or an axial symmetry. Therefore in the following we are going to investigate only such systems.

1. SYSTEM WITH AN AXIAL SYMMETRY

Let us take the x axis as symmetry axis of a magnetostatic system using cylindrical coordinates r,θ,x.

Let $\Phi(r,x)$ be the flux of B through a circle of radius $r = \sqrt{x^2+y^2}$ in the plane x = const. Let

$$B_x = \frac{1}{2\pi r}\frac{\partial\Phi}{\partial r}; \quad B_r = -\frac{1}{2\pi r}\frac{\partial\Phi}{\partial x}; \tag{6}$$

B, calculated from Eq. (6), satisfies Eq. (2) identically.

For a system with axial symmetry

$$E = \int\left(w - \frac{1}{2\pi c}\frac{j\Phi}{r}\right)2\pi r\, dr\, dx. \tag{7}$$

The integrand in Eq. (7),

$$L = 2\pi r\left(w - \frac{1}{2\pi c}\frac{j\Phi}{r}\right), \tag{8}$$

is the Lagrangian of the magnetostatic system with axial symmetry. Using the identity

$$H dB = dB^2/2\mu \tag{9}$$

and expressing μ through B^2 (e.g., using the magnetization curve), $\mu = B/H = \mu(B^2)$, we can write Eq. (8) as

$$L = 2\pi r\left(\frac{1}{4\pi}\int_0^{B^2}\frac{dB^2}{2\mu(B^2)} - \frac{1}{2\pi c}\frac{j\Phi}{r}\right) \tag{10}$$

$$= L\left(j, r, \Phi, \frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial r}\right),$$

where, as follows from Eq. (6),

$$B^2 = \frac{1}{4\pi^2 r^2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial r}\right)^2\right]. \tag{11}$$

The condition for a minimum gives the Euler's equations for each domain where the flux function Φ is twice differentiable:

$$\frac{\partial L}{\partial\Phi} - \frac{\partial}{\partial x}\left[\frac{\partial L}{\partial(\partial\Phi/\partial x)}\right] - \frac{\partial}{\partial r}\left[\frac{\partial L}{\partial(\partial\Phi/\partial r)}\right] \tag{12}$$

$$= -\frac{1}{c}j + \frac{1}{4\pi}\frac{\partial H_r}{\partial x} - \frac{1}{4\pi}\frac{\partial H_x}{\partial r} = 0.$$

From (12) we then obtain (5).

2. PLANE PARALLEL SYMMETRY

In this case,

$$B_x = \frac{\partial\Phi}{\partial y}; \quad B_y = -\frac{\partial\Phi}{\partial x}; \tag{13}$$

$$B^2 = \left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial r}\right)^2,$$

$$L = \frac{1}{4\pi}\int_0^{B^2}\frac{dB^2}{2\mu(B^2)} - \frac{1}{c}j\Phi = L\left(j, \Phi, \frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}\right).$$

Euler's equation, corresponding to the minimum value of E, then has the form:

$$\frac{\partial L}{\partial\Phi} - \frac{\partial}{\partial x}\left[\frac{\partial L}{\partial(\partial\Phi/\partial x)}\right] - \frac{\partial}{\partial y}\left[\frac{\partial L}{\partial(\partial\Phi/\partial y)}\right]$$

$$= -\frac{1}{c}j + \frac{1}{4\pi}\frac{\partial H_y}{\partial x} - \frac{1}{4\pi}\frac{\partial H_x}{\partial y} = 0,$$

from which Eq. (5) follows.

For the Lagrangeian of type (10) and (13), a strong minimum of corresponding functionals exists and the uniqueness theorem can be applied which makes it possible to postulate a minimizing sequence $\Phi_1, \Phi_2 \ldots \Phi_n \ldots$ which, by Ritz's method, gives as limit Φ, when the complete set of functions of the given magnetostatic system is obtained.

One essential advantage of this direct method of solution of magnetostatic problems on the basis of a variational principle is the non-existence of complicated explicit non-linear boundary conditions:

$$\left[\frac{1}{\mu}B_\tau\right]_{S_i} = 0; \; [B_n]_{S_i} = 0, \tag{14}$$

which have to be taken into account when solving the same problem using Maxwell's equation. The equation div $\mathbf{B} = 0$ leads to the unique linear condition for the function Φ on the discontinuity surfaces S_i:

$$[\Phi]_{S_i} = 0. \tag{15}$$

The conditions $[B_n]_{S_i}$ indeed are consequences of Eq. (15). Considering the problem of determination of **B** from the variational principle (1) under the condition (2) as a discontinuity problem of second order variational calculus[4] and determining

[4] R. Courant and D. Hilbert, *Methods of Mathematical Physics*.

by known methods the conditions on the discontinuity surfaces, we obtain that, for the real field, the conditions $\left[\frac{1}{\mu}B_\tau\right] = 0$ on discontinuity surfaces are identically satisfied.

2. CONSTRUCTION OF THE COMPLETE SET OF FUNCTIONS

We divide the magnetostatic system into such regions δ, where μ and j do not possess discontinuities, in particular, the overlapping of V and ω represent such a region δ. We further assume that on some surface Γ, enclosing the magnetostatic system, the magnetic flux $\Phi = 0$. This boundary condition upon the closed magnetostatic system is always satisfied at infinity. When the systems are completely screened, and if it is possible to neglect the dispersion of flux outside, the condition $\Phi = 0$ is satisfied on the outer surface of the screen.

Let us construct the sequence of functions, satisfying the boundary condition (15), in which the first derivatives may be discontinuous on the limits of the region δ. This set of functions then defines the complete δ set.

Let

$$\varphi_1^{(\delta)}, \varphi_2^{(\delta)}, \ldots, \varphi_n^{(\delta)}, \ldots \tag{16}$$

be a complete set of functions, determined in regions δ, satisfying the conditions $\varphi_k^{(\delta)} = 0 (k = 1,2,\ldots)$ on the surfaces S_δ enclosing regions δ, and let

$$\psi_1, \psi_2, \ldots, \psi_n, \ldots \tag{17}$$

be a complete set of continuosuly differentiable functions, determined in the domain enclosed by surface Γ, on which they vanish.

It then is possible to show that the totality of functions $\{\varphi_k^{(\delta)}, \psi_k\}$ represents the complete δ-set.

For the sake of simplicity of analysis we shall limit our investigation to cases where δ regions are defined by two coordinates e.g., r,x (system with axial symmetry) or x,y (system with plane parallel symmetry). Let Φ be any arbitrary function satisfying following conditions:

1) Φ and its first derivatives are defined and are continuous inside each region δ;

2) On the limits of regions δ condition (15) is satisfied

3) $\Phi = 0$ on the Γ surface.

Let us represent Φ as a sum of two functions Φ_0 and Θ where Φ_0 and its first derivatives are continuous inside each region δ and vanish on the surfaces of δ, and θ and its first derivatives are continuos inside the domain limited by Γ and vanish on that surface. This representation is always possible when the limits of the regions δ are smooth. One of such possible representations leads to the following:

Let us construct a family of straight lines $x =$ const. and let $M_1(x), M_2(x), \ldots, M_k(x)$, where $k = k(x)$, be points of intersection* of the boundaries of surfaces δ with the surface Γ. To each straight line we assign a corresponding polynomial $P(x,y)$ with respect to y, which coincides with the values of Φ at the points $M_1, M_2, \ldots, M_k$. Then $\Theta = P(x,y)$ and $\Phi = \Phi_0 - P(x,y)$, where now $P(x,y)$ and its first derivatives are continuous in the domain limited by Γ, and vanish on that surface.

Since $\{\varphi_k^{(\delta)}\}$ is a complete set of functions in region δ, the function Φ_0 and its first derivatives are uniformly approximated by a linear combination of $\varphi_k^{(\delta)}$. The function Θ and its first derivatives on the other hand are approximated by a linear combination of ψ_k, so that $\Phi = \Phi_0 + \Theta$ and their first derivatives are given by a linear combination of $\varphi_k^{(\delta)}$ and ψ_k; therefore the totality of $\{\varphi_k^{(\delta)}, \psi_k\}$ represents a complete δ-set.

Thus the problem of setting up a complete δ-set is reduced to the writing of a complete set of functions $\varphi_k^{(\delta)}$ and ψ_k. The principle of construction of such complete sets is known[5,6].

Let us now reduce the δ-set to its normal form by introducing a new sequence of functions

$$\Psi_1, \Psi_2, \Psi_3, \ldots,$$

defined in the whole space Ω bounded by the surface Γ. Let us assume
Let us assume

$$\Psi_{(n-1)(p+k)+k} = \begin{cases} \varphi_n^{(\delta_k)} & \text{in region } \delta_k, \\ 0 & \text{outside } \delta_k, \end{cases} \tag{18}$$
$$\Psi_{n(p+1)} = \psi_n,$$

where p gives the number of regions δ of the magnetostatic system. The index $k = 1, 2, \ldots, p$ for each fixed n starting with $n = 1$. Thus the set of functions (18) represents a complete normalized set. An arbitrary function Φ and its first derivatives satisfying conditions 1. to 3. of the second section can be approximated by a linear combination of ψ_k, where derivatives of Φ may have discontinui-

[5] L. V. Kantorovich and V. I. Krylov, *Approximation Methods in Advanced Analysis*, Moscow, 1952

[6] I. Iu. Kharik Doklady, Akad. Nauk SSSR **80**, 25 (1951)

* It is possible that in this case a part of the boundary of δ regions coincides with a straight line, but it is always possible to choose such lines which intersect δ at a finite number of points.

ties on the limits of regions δ.

It is possible to show in particular, using Weierstrass's theorem on uniform approximation of a continuous function by polynomials, that in a general case of magnetostatic problems, the complete system of functions (18) may be determined by following sequences of $\varphi_n^{(\delta_k)}$ and ψ_n:

$$\psi_1 = q,\ \psi_2 = qx,\ \psi_3 = qy,\ \psi_4 = qz,\ \psi_5 = qxy,$$
$$\psi_6 = qxz,\ \psi_7 = qyz,\ \psi_8 = qx^2, \ldots;$$
$$\varphi_1^{(\delta_k)} = q_{\delta_k},\ \varphi_2^{(\delta_k)} = q_{\delta_k}x,\ \varphi_3^{(\delta_k)} = q_{\delta_k}y,\ \varphi_4^{(\delta_k)} = q_{\delta_k}z,$$
$$\varphi_5^{(\delta_k)} = q_{\delta_k}xy, \varphi_6^{(\delta_k)} = q_{\delta_k}xz,\ \varphi_7^{(\delta_k)} = q_{\delta_k}yz, \ldots,$$

where $q_{\delta_k}(x,y,z) = 0$ is the surface equation of δ, and $q(x,y,z) = 0$ is the surface equation of Γ limiting the domain Ω.

Similarly, performing the transformation of variables x,y,z, it is possible to construct other complete sets of functions, e.g., trigonometric polynomials.

3. DETERMINATION OF MAGNETIC INDUCTION FLUX BY DIRECT METHOD

Let us represent the function Φ by the following form:

$$\Phi_n = \sum_{k=1}^{n} \alpha_k \Psi_k, \tag{19}$$

where ψ_k is the complete normalized set of functions and a_k constant coefficients, functions of n.

Φ_n satisfies identically the boundary conditions (15); therefore one can take Φ_n as the nth approximation of the variational problem using Ritz's method[5]. The problem of determination of the nth approximation of Φ reduces to integration of the Lagrangian L over the domain Ω. Substituting Φ_n and its partial derivatives into L and performing the integration, we obtain:

$$E = E\{\alpha_k\}.$$

The condition for a minimum of E leads to transcendental equations for solutions for a_k

$$\partial E / \partial \alpha_k = 0. \tag{20}$$

Let us introduce the components of current potential function U, defined by the following relation:

$$u_k = -\frac{1}{c}\sum_{i=1}^{m} \int_{s_i} \Psi_k(s) j_i ds. \tag{21}$$

Then the nth approximation of the current potential function U is represented as a linear combination of u_k:

$$U_n = \sum_{k=1}^{n} \alpha_k u_k. \tag{22}$$

The system of equations (20) with the condition (22) takes the form:

$$\partial W / \partial \alpha_k = -u_k\ (k = 1, 2, \ldots, n). \tag{23}$$

From Eq. (23) one can see that the unknown coefficients a_k are functions of $u_i (i = 1, 2, \ldots, n)$. This indicates that, for a given geometrical configuration of the magnetostatic system, the flux Φ is determined exclusively by components u_k of the current potential function U. Considering a_k as functions of u_i, and differentiating Eq. (23) with respect to u_k, we obtain

$$\sum_{i=1}^{n} \frac{\partial^2 W}{\partial \alpha_k \partial \alpha_i} \frac{\partial \alpha_i}{\partial u_k} = -1\ (k = 1, 2, \ldots, n) \tag{24}$$

The set of equations (24) is a system of linear equations in $\frac{\delta \alpha_i}{\delta u_k}$ which can be integrated by the method of successive approximations, taking $\alpha_k = 0$ and $u_i = 0$ as initial conditions. The above conditions correspond to vanishing magnetic induction in the absence of the field of currents. In the particular, but practically important case when the current densities are everywhere the same, α_k can be taken as functions of j. Indeed from Eq. (21) it follows:

$$u_k = \Big(-\frac{1}{c}\sum_{i=1}^{m} \int_{s_i} \Psi_k(s) ds\Big) j = \bar{u}_k j,$$

where

$$\bar{u}_k = -\frac{1}{c}\sum_{i=1}^{m} \int_{s_i} \Psi_k(s) ds, \tag{25}$$

$$U_n = j\sum_{k=1}^{n} \alpha_k \bar{u}_k. \tag{26}$$

Let us introduce the specific potential function of currents relative to unit current density

$$\bar{U} = U / j. \tag{27}$$

Then

$$\bar{U}_n = \sum_{k=1}^{n} \alpha_k \bar{u}_k, \tag{28}$$

where u_k represents the components of the specific potential function of currents. In this case the system of equations (24) takes the form:

$$\sum_{i=1}^{n} \frac{\partial^2 W}{\partial \alpha_k \partial \alpha_i} \frac{d\alpha_i}{dj} = -\bar{u}\ \ (k = 1, 2, \ldots, n). \tag{29}$$

We normalize the system (29) by introducing the Jacobian of that system:

$$D = \frac{D\left(\frac{\partial W}{\partial \alpha_1}, \frac{\partial W}{\partial \alpha_2}, \ldots, \frac{\partial W}{\partial \alpha_n}\right)}{D(\alpha_1, \alpha_2, \ldots \alpha_n)}. \quad (30)$$

The Jacobian (30) is not zero identically. Indeed, if the Jacobian (30) were zero, then $a_1, a_2 \ldots a_n$ would be functionally dependent; this contradicts the definitions of $\Psi_1, \Psi_2, \ldots$ as a complete set of functions. Solving Eq. (29) for derivatives we obtain

$$\frac{d\alpha_i}{dj} = -\frac{D\left(\frac{\partial W}{\partial \alpha_1}, \frac{\partial W}{\partial \alpha_2}, \ldots, \frac{\partial W}{\partial \alpha_{i-1}}, U, \frac{\partial W}{\partial \alpha_{i+1}}, \ldots, \frac{\partial W}{\partial \alpha_n}\right) \Big/ D(\alpha_1, \alpha_2, \ldots, \alpha_n)}{D\left(\frac{\partial W}{\partial \alpha_1}, \frac{\partial W}{\partial \alpha_2}, \ldots, \frac{\partial W}{\partial \alpha_n}\right) \Big/ D(\alpha_1, \alpha_2, \ldots, \alpha_n)}. \quad (31)$$

Writing

$$\overline{D}_i = D\left(\frac{\partial W}{\partial \alpha_1}, \frac{\partial W}{\partial \alpha_2}, \ldots, \frac{\partial W}{\partial \alpha_{i-1}}, \overline{U}, \frac{\partial W}{\partial \alpha_{i+1}}, \ldots, \frac{\partial W}{\partial \alpha_n}\right) \Big/ D(\alpha_1, \alpha_2, \ldots, \alpha_n), \quad (32)$$

We get

$$d\alpha_i / dj = -\overline{D}_i / D. \quad (33)$$

Let us now integrate Eqs. (33), taking $a_i = 0$ and $j = 0$ as initial conditions. Using the successive approximation method and taking as zeroth approximation $a_i^0 = 0\,(i = 1,2,\ldots n)$, we get:

$$\alpha_i^{(1)} = -\int_0^j \left(\frac{\overline{D}_i}{D}\right)_0 dj; \qquad \alpha_i^{(2)} = -\int_0^j \left(\frac{\overline{D}_i}{D}\right)_1 dj;$$

$$\ldots;\ \alpha_i^{(m)} = -\int_0^j \left(\frac{\overline{D}_i}{D}\right)_{m-1} dj; \ldots;$$

where $(\overline{D}_i/D)_k = \overline{D}_i/D$ when $a_i = a_i^{(k)}$.

The exact value of a_i is obtained as the limit of sequence $a_i^{(m)}$ when m tends to infinity.

It is possible to apply the same method when $j = j(x,y,z)$. Consider the system of equations:

$$\partial W / \partial \alpha_k = -\lambda u_k \quad (k = 1, 2, \ldots, n), \quad (34)$$

where λ is an arbitrary variable parameter ranging from 0 to 1.

Differentiating Eq. (34) with respect to λ, we get

$$\sum_{i=1}^{n} \frac{\partial^2 W}{\partial \alpha_k \partial \alpha_i} \frac{\partial \alpha_i}{\partial \lambda} = -u_k, \quad (35)$$

from which

$$\frac{\partial \alpha_i}{\partial \lambda} = -\frac{D\left(\frac{\partial W}{\partial a_1}, \frac{\partial W}{\partial a_2}, \ldots, \frac{\partial W}{\partial a_{i-1}}, U, \frac{\partial W}{\partial a_{i+1}}, \ldots, \frac{\partial W}{\partial a_n}\right) \Big/ D(a_1, a_2, \ldots a_n)}{D\left(\frac{\partial W}{\partial a_1}, \frac{\partial W}{\partial a_2}, \ldots, \frac{\partial W}{\partial a_n}\right) \Big/ D(a_1, a_2, \ldots a_n)} \quad (36)$$

Denoting the denominator of Eq. (36) by D_i, we have

$$\alpha_i = -\int_0^\lambda \left(\frac{D_i}{D}\right) d\lambda; \quad (37)$$

when $\lambda = 0$, $a_i = 0$, a_i depend on λ in (37). Again applying the method of successive approximations we obtain:

$$\alpha_i^{(1)}(\lambda) = -\int_0^\lambda \left(\frac{D_i}{D}\right)_0 d\lambda;$$

$$\alpha_i^{(2)}(\lambda) = -\int_0^\lambda \left(\frac{D_i}{D}\right)_1 d\lambda;$$

$$\ldots\ldots\ldots\ldots$$

$$\alpha_i^{(m)}(\lambda) = -\int_0^\lambda \left(\frac{D_i}{D}\right)_{m-1} d\lambda; \ldots$$

The limit of sequence a_i, when $m \to \infty$ gives the exact value of $a_i(\lambda)$, satisfying system (34).

The complete set of a_i, satisfying system of equations (23) is obtained by setting $\lambda = 1$.

The roots a_i of the system of equations (20) then determine the nth approximation of the magnetic induction flux Φ_n, and the exact value of the flux Φ is given by the limit of sequence Φ_n when $n \to \infty$.

Translated by G. Cvijanovich
72