SOVIET PHYSICS JETP VOLUME 1, NUMBER 3 NOVEMBER, 1955

Description of the Electromagnetic Field by Means of Matrices

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(Submitted to JETP editor April 3, 1954)
J. Exper. Theoret. Phys. USSR **28**, 524-529 (May, 1955)

The characteristics of matrix-tensors are described. The equations of the electromagnetic field are given in matrix form.

IN three-dimensional space it is convenient to denote vectors by a single letter, e.g., **c**, **d**, etc. On the other hand, in the four-dimensional space of special relativity, tensors usually are given by their components, which depend on the particular coordinate system used. But it is clear that the physical system described by a tensor does not depend on the particular coordinate system used to label the components of the tensor. One should therefore work directly with the tensor as such which then would describe the system in a way valid in any coordinate system.

The tensors of four-dimensional space can be represented by matrices[1]; these are independent of the coordinate system used and therefore are well suited for this purpose. These matrix-tensors are very useful in connection with problems of relativistic invariance. Utilizing them, it is very easy, e.g., to express relations between them, to find their invariants, to determine their character with respect to the transformations of the Lorentz-group, etc.

We shall first investigate the application of matrix-tensors to the description of the classical electromagnetic field. The results thus obtained will be used in following papers[2].

1. For the investigation of four-dimensional matrix-tensors it is useful to start with the properties of matrix-tensors corresponding to vectors in three-dimensional space.

We shall use the following notation: let $\mathbf{d} = \sum_k d^k \mathbf{e}_k$; then the three-dimensional matrix-vector will be denoted by underlining: $\underline{\mathbf{d}} = d_k R^k = d^k R_k$. Furthermore, an underlined letter will denote a matrix (not necessarily a vector) which is a linear combination of R^1, R^2, R^3. So, if $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, respectively, then

$$\underline{\mathbf{E}} = \sum_{k=1}^{3} E_k R^k = E_k R^k, \quad \underline{\mathbf{H}} = H_k R^k = \underline{H} \tag{1}$$

(H_k is a component of a pseudovector), etc.

The following relations hold for three-dimensional matrix-vectors:

$$\underline{\mathbf{c}}\underline{\mathbf{d}} = (1/2)(\underline{\mathbf{c}}\underline{\mathbf{d}} + \underline{\mathbf{d}}\underline{\mathbf{c}}) + (1/2)(\underline{\mathbf{c}}\underline{\mathbf{d}} - \underline{\mathbf{d}}\underline{\mathbf{c}}) \tag{2}$$

$$= (\mathbf{cd}) + R\,\underline{[\mathbf{cd}]}, \quad R = R^1R^2R^3.$$

In particular, taking for $\underline{\mathbf{c}}$ the matrix-operator $\underline{\nabla} = R^k\ \partial/\partial x^k$ which corresponds to the operator ∇, we obtain

$$\underline{\nabla}\underline{\mathbf{d}} = \operatorname{div}\mathbf{d} + R\,\underline{\operatorname{curl}\mathbf{d}} \tag{3}$$

$$= \frac{\partial d^k}{\partial x^k} + RR_k\varepsilon^{kjs}\frac{\partial d_s}{\partial x^j}.$$

Utilizing Eq. (2). it is easy to show that

$$\underline{\mathbf{c}}\underline{\mathbf{b}}\underline{\mathbf{d}} = (\mathbf{cb})\,\underline{\mathbf{d}} + R([\mathbf{cb}]\,\mathbf{d}) + \underline{[\mathbf{d}\,[\mathbf{cb}]]} = \tag{4}$$

$$= (\mathbf{cb})\,\underline{\mathbf{d}} + (\mathbf{bd})\,\underline{\mathbf{c}} - (\mathbf{cd})\,\underline{\mathbf{b}} + R\,(\mathbf{b}\,[\mathbf{dc}])$$

etc.

The components of the electromagnetic field tensor are

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & H_3 & -H_2 & -E_1 \\ -H_3 & 0 & H_1 & -E_2 \\ H_2 & -H_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{pmatrix} \tag{5}$$

$$(\tilde{F}^{\alpha\beta}) = \begin{pmatrix} 0 & E_3 & -E_2 & H_1 \\ -E_3 & 0 & E_1 & H_2 \\ E_2 & -E_1 & 0 & H_3 \\ -H_1 & -H_2 & -H_3 & 0 \end{pmatrix}.$$

[1] see, e.g., G. A. Zaitsev, J. Exper. Theoret. Phys. USSR **25**, 667 (1953). In the following we shall use the notation and the results of the first part of that paper.

[2] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR **28**, 530 (1955); Soviet Phys **1**, 491 (1955)

Here $\tilde{F}^{\alpha\beta}$ are the components of the dual tensor to $F^{\alpha\beta}$ which is defined by $\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$.
The matrix F, representing the field tensor, is given by[1]:

$$F = {}^1/_2 F^{\alpha\beta} R_\alpha R_\beta, \tag{6}$$

or, using Eqs. (1) and (5),

$$F = R\underline{\mathbf{H}} + R_4\underline{\mathbf{E}}. \tag{7}$$

Similarly,

$$\tilde{F} = JF = R\underline{\mathbf{E}} - R_4\underline{\mathbf{H}}. \tag{8}$$

As an example of the application of Eq. (7), we shall obtain from it the law of transformation of $\underline{\mathbf{E}}$ and $\underline{\mathbf{H}}$ under a change of the coordinate system.

If the basis matrix-vectors R^α of the old system of coordinates are connected with the $\tilde{R}^\alpha$ of the new system by

$$\tilde{R}^\alpha = A^{-1}R^\alpha A \quad (A = \ldots A_2A_1, \quad A_i^2 = \pm 1), \tag{9}$$

then the new components $\tilde{H}_k$ and $\tilde{E}_k$ are obtained from

$$F = RR^kH_k + R_4R^kE_k \tag{10}$$

$$= \tilde{R}\tilde{R}^k\tilde{H}_k + \tilde{R}_4\tilde{R}^k\tilde{E}_k \quad (\tilde{R} = \tilde{R}^1\tilde{R}^2\tilde{R}^3)$$

or*

$$AFA^{-1} = RR^k\tilde{H}_k + R_4R^k\tilde{E}_k = R\underline{\tilde{\mathbf{H}}} + R_4\underline{\tilde{\mathbf{E}}}. \tag{11}$$

In particular, for the case of a single symmetry, that is, if

$$A = A_1 = a_\alpha R^\alpha \tag{12}$$

$$= \underline{\mathbf{a}} + a_4R^4 \quad (A_1^2 = \pm 1, \quad A_1^{-1} = \pm A_1),$$

we have

$$A_1FA_1^{-1} = \pm R(\underline{\mathbf{a}} - a_4R^4)\underline{\mathbf{H}}(\underline{\mathbf{a}} + a_4R^4)$$

$$\mp R_4(\underline{\mathbf{a}} - a_4R^4)\underline{\mathbf{E}}(\underline{\mathbf{a}} + a_4R^4),$$

* One can look at the components $\tilde{H}_k$ and $\tilde{E}_k$ either as being the components of the new tensor $\tilde{F} = AFA^{-1}$ obtained from F by rotation or reflection, or as the components of the same tensor F in the new coordinate system. In the latter point of view the basis matrix-vectors of the new system of coordinates are obtained by a four-dimensional rotation or reflection characterized by the matrix A^{-1}.

and, because of

$$(\underline{\mathbf{a}} - a_4R^4)\underline{\mathbf{d}}(\underline{\mathbf{a}} + a_4R^4)$$

$$= \underline{\mathbf{ada}} + a_4R^4(\underline{\mathbf{ad}} - \underline{\mathbf{da}}) - a_4^2\underline{\mathbf{d}},$$

there holds

$$\pm\underline{\mathbf{H}}^+ = \underline{\mathbf{aHa}} - a_4^2\underline{\mathbf{H}} - 2a_4[\underline{\mathbf{aE}}], \tag{13}$$

$$\pm\underline{\mathbf{E}}^+ = -\underline{\mathbf{aEa}} + a_4^2\underline{\mathbf{E}} - 2a_4[\underline{\mathbf{aH}}].$$

By performing two symmetry operations consecutively, namely

$$A_1 = R^1, \quad A_2 = \lambda[R^1\cos(\alpha/2) + R^4\sin(\alpha/2)], \tag{14}$$

$$\cos\alpha = 1/\lambda^2 = +\sqrt{1-(v/c)^2},$$

which corresponds to a Lorentz transformation[1] one obtains

$$\underline{\mathbf{H}}^{++} = [(1+\cos\alpha)/2\cos\alpha]\,\underline{\mathbf{H}} \tag{15}$$

$$-[(1-\cos\alpha)/2\cos\alpha]R^1\underline{\mathbf{H}}R^1 - \mathrm{tg}\,\alpha\,[\underline{\mathbf{e}_1,\ \mathbf{E}}],$$

$$\underline{\mathbf{E}}^{++} = [(1+\cos\alpha)/2\cos\alpha]\,\underline{\mathbf{E}}$$

$$-[(1-\cos\alpha)/2\cos\alpha]R^1\underline{\mathbf{E}}R^1 + \mathrm{tg}\,\alpha\,[\underline{\mathbf{e}_1,\ \mathbf{H}}]$$

Therefore the new components are given by the old components in the following way

$$H_1^{++} = H_1, \tag{16}$$

$$H_2^{++} = [H_2 + (v/c)E_3]/\sqrt{1-(v/c)^2},$$

$$H_3^{++} = [H_3 - (v/c)E_2]/\sqrt{1-(v/c)^2},$$

$$E_1^{++} = E_1,$$

$$E_2^{++} = [E_2 - (v/c)H_3]/\sqrt{1-(v/c)^2},$$

$$E_3^{++} = [E_3 + (v/c)H_2]/\sqrt{1-(v/c)^2}.$$

which are the well known expressions for the components of **E** and **H** in a moving frame of reference (see for example p. 72 of reference 3 or references 4, 5, etc.).

2. We shall now find some relations pertaining to four-dimensional matrix-tensors.

The product of the two matrix-fourvectors $C = c_\beta R^\beta$ and $D = d^\beta R_\beta$ is given by

[3] L. D. Landau, E. M. Lifshitz, *Theory of Fields*, 1948

[4] Ia. I. Frenkel, *Electrodynamics*, 1934, part 1

[5] I. E. Tamm, *Principles of Electricity*, 1949

$$CD = (\underline{\mathbf{c}} + c_4R^4)(\underline{\mathbf{d}} + d^4R_4) = (\mathbf{cd}) + c_4d^4 + R[\underline{\mathbf{cd}}] + R_4(-c_4\underline{\mathbf{d}} - \underline{\mathbf{c}}d^4) \quad (17)$$

or

$$CD = c_\alpha d^\alpha + \frac{1}{2}R^\alpha R^\beta (c_\alpha d_\beta - c_\beta d_\alpha) = c_\alpha d^\alpha - \frac{1}{2}JR_\gamma R_\delta\, \varepsilon^{\gamma\delta\alpha\beta} c_\alpha d_\beta. \quad (18)$$

Equations (17) and (18) hold also for the case that the components of the fourvectors are operators.

A particularly important matrix-operator is

$$\nabla = R^\alpha \frac{\partial}{\partial x^\alpha} = \underline{\nabla} + \frac{1}{c}R^4\frac{\partial}{\partial t}, \quad (19)$$

Application of Eq. (18) yields

$$\nabla^2 = \Box = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad (20)$$

Using Eq. (2) we further obtain

$$DF = -[\underline{\mathbf{dH}}] + d_4\underline{\mathbf{E}} + R^4(\mathbf{dE}) - J([\underline{\mathbf{dE}}] + d_4\underline{\mathbf{H}} + R^4(\mathbf{dH})), \quad (21)$$

or

$$DF = -R_\alpha d_\beta F^{\alpha\beta} - JR_\alpha d_\beta \tilde{F}^{\alpha\beta} \quad (22)$$

Similarly,

$$FD = -[\underline{\mathbf{Hd}}] + \underline{\mathbf{E}}\,d^4 - R^4(\mathbf{Ed}) + J([\underline{\mathbf{Ed}}] + \underline{\mathbf{H}}\,d^4 - R^4(\mathbf{Hd})) \quad (23)$$

or

$$FD = R_\alpha F^{\alpha\beta}d_\beta - JR_\alpha \tilde{F}^{\alpha\beta}d_\beta. \quad (24)$$

Finally, the product of two second rank matrix tensors is given by

$$F_{(1)}F_{(2)} = (R\underline{\mathbf{H}}_{(1)} + R_4\underline{\mathbf{E}}_{(1)})(R\underline{\mathbf{H}}_{(2)} + R_4\underline{\mathbf{E}}_{(2)}) = (\mathbf{E}_{(1)}\mathbf{E}_{(2)}) - (\mathbf{H}_{(1)}\mathbf{H}_{(2)}) + J((\mathbf{H}_{(1)}\mathbf{E}_{(2)}) + (\mathbf{E}_{(1)}\mathbf{H}_{(2)})) + R([\underline{\mathbf{E}_{(1)}\mathbf{E}_{(2)}}] - [\underline{\mathbf{H}_{(1)}\mathbf{H}_{(2)}}]) + R^4([\underline{\mathbf{H}_{(1)}\mathbf{E}_{(2)}}] + [\underline{\mathbf{E}_{(1)}\mathbf{H}_{(2)}}]) \quad (25)$$

or

$$F_{(1)}F_{(2)} = -\frac{1}{2}F_{(1)}^{\alpha\beta}F_{\alpha\beta}^{(2)} + \frac{1}{2}J(F_{(1)}^{\alpha\beta}\tilde{F}_{\alpha\beta}^{(2)}) + \frac{1}{2}R_\beta R_\gamma(F_{(1)}^{\beta\alpha}F_{\alpha(1)}^{\ \ \gamma} - F_{(2)}^{\beta\alpha}F_{\alpha(1)}^{\ \ \gamma}). \quad (26)$$

In particular.

$$F^2 = C_1 + C_2J, \quad (27)$$

where $C_1 = -\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta}$ is an invariant, and $C_2 = \frac{1}{2}F^{\alpha\beta}\tilde{F}_{\alpha\beta}$ is a pseudoinvariant which changes sign by reflection (i.e., after an odd number of inversions); they are invariant under four-dimensional rotations. $F^2 = 0$ means $C_1 = C_2 = 0$.

3. We shall now write down the equations of electrodynamics using the above formulas.

Let $J_{el} = \underline{\mathbf{j}}_{el} + c\rho_{el}R_4$ be the current matrix-fourvector. The equations

$$\operatorname{div}\mathbf{E} = 4\pi\rho_{el}, \quad \operatorname{div}\mathbf{H} = 0, \quad (28)$$

$$\operatorname{curl}\mathbf{H} - \frac{1}{c}\frac{d\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}_{el}, \quad \operatorname{curl}\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} = 0,$$

can be written, putting in Eq. (22) $D = \nabla$,

$$\nabla F = -\frac{4\pi}{c}J_{el}\left(\text{ or },\ \frac{\partial}{\partial x^\alpha}FR^\alpha = \frac{4\pi}{c}J_{el}\right). \quad (29)$$

Maxwell's equations, Eqs. (28), when written in the matrix notation, appear as the single equation (29), where the relativistic invariance is already obvious from the form of the equations.

Maxwell's equations for free space, i.e., if $J_{el} = 0$, are given by $\nabla F = 0$. This is very similar in form to the special case of Maxwell's equations for the electromagnetic field given in reference 6*: $\nabla\psi = 0$. However, this similarity is superficial because here F denotes a matrix-tensor, while ψ is a column composed of the components of spinor .

Let the matrix-fourvector of the potentials be denoted by

$$A = A_\alpha R^\alpha = \underline{\mathbf{A}} + A_4R^4, \quad A^4 = -A_4 = \varphi. \quad (30)$$

[6] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR **25**, 675 (1953)

* We shall use this occasion to remark that in the references given in reference 6, the quoted page numbers should be exchanged: in reference 1, the page number should be 667, in reference 2 it should be 653.

From Eq. (18) we obtain

$$\nabla A\chi = (\partial A^\alpha / \partial x^\alpha)\chi + F\chi + R^\alpha A(\partial / \partial x^\alpha)\chi, \quad (31)$$

where χ is an arbitrary matrix or column vector. Assuming in the usual way $\partial A^\alpha / \partial x^\alpha = 0$, we obtain

$$F = \nabla A. \quad (32)$$

It is easy to obtain in matrix form the forces exerted by the fields on the charges. From Eqs. (21) and (23) we have

$$(1/2)(F\dot{J}_{el} - J_{el}F) = c(\rho_{el}\underline{\mathbf{E}} + (1/c)\underline{[\mathbf{j}_{el}\mathbf{H}]}) + R_4(\mathbf{j}_{el}\mathbf{E}) = cf_{el}, \quad (33)$$

where f_{el} is the matrix-fourvector of the force. The components of the energy-momentum tensor $T^{\alpha\beta}$ must yield $-\partial T^{\alpha\beta}/\partial x^\beta = f^\alpha_{el}$. From Eqs. (29) and (33) we obtain

$$-\frac{\partial}{\partial x^\beta}(R_\alpha T^{\alpha\beta}) = f_{el} = \frac{1}{8\pi}\left(-\frac{\partial(FR^\alpha)}{\partial x^\alpha} - F\nabla F\right) = -\frac{1}{8\pi}\frac{\partial(FR^\alpha F)}{\partial x^\alpha}, \quad (34)$$

and we therefore can put

$$8\pi R_\alpha T^{\alpha\beta} = FR^\beta F. \quad (35)$$

It is immediately possible to show that $T^{\alpha\beta}$ really are the components of the energy-momentum tensor. Indeed, putting $\underline{\mathbf{b}} = R^k$ in Eq. (4), and with Eq. (7), we obtain for Eq. (35) the expressions

$$8\pi T^{ik} = -2H_iH_k - 2E_iE_k + \delta_{ik}(\mathbf{E}^2 + \mathbf{H}^2), \quad 8\pi T^{44} = \mathbf{E}^2 + \mathbf{H}^2, \quad (36)$$

$$8\pi T^{4k} = 8\pi T^{k4} = 2\varepsilon^{kjs}E_jH_s,$$

as it should be.

Up to now, we limited ourselves to the case of electromagnetic fields *in vacuo*. However, the notation of matrix-fourtensors can be used also in the case of the presence of a medium, especially if questions of relativistic invariance are involved. Here one has to introduce to antisymmetric matrix-fourtensors of second rank

$$F = R\underline{\mathbf{B}} + R_4\underline{\mathbf{E}}, \quad \Phi = R\underline{\mathbf{H}} + R_4\underline{\mathbf{D}}, \quad (37)$$

Here $\mathbf{B}$ is the magnetic induction, etc. Instead of Eq. (35) we now have

$$\frac{1}{8\pi}FR^\alpha\Phi = T^{\alpha\beta}_{(c)}R_\beta + J(P^{\alpha\beta}R_\beta),$$

where

$$8\pi T^{ik}_{(c)} = -B_iH_k - H_iB_k - E_iD_k - D_iE_k + \delta_{ik}((\mathbf{BH}) + (\mathbf{ED})),$$

$$8\pi T^{44}_{(c)} = (\mathbf{ED}) + (\mathbf{BH}),$$

$$8\pi T^{4k}_{(c)} = 8\pi T^{k4}_{(c)} = \varepsilon^{kjs}(E_jH_s + D_jB_s).$$

The $T^{\alpha\beta}_{(c)}$ are the components of Minkowski's symmetrized energy-momentum tensor in an arbitrary medium. In order to obtain the components of the general energy-momentum tensor[5] $T^{\alpha\beta}$, i.e., if $T^{\alpha\beta} \neq T^{\beta\alpha}$ one has to use the antisymmetric tensor of second rank $\frac{1}{2}(F\Phi - \Phi F)$.

Finally, we shall give the tensor of the moment of momentum of the field. Putting $X = x^\alpha R_\alpha$ we obtain, according to Eq. (18),

$$\frac{1}{2}(XR_\alpha T^{\alpha\gamma} - R_\alpha T^{\alpha\gamma}X) = \frac{c}{2}R_\alpha R_\beta M^{\alpha\beta\gamma}, \quad (38)$$

where

$$M^{\alpha\beta\gamma} = \frac{1}{c}(x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma}) \quad (39)$$

are the components of the angular momentum density of the field.

Translated by M.Danos
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