Statistical Problems of an Electron Multiplier

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We give a method of determining the distribution functions of impulse amplitudes in an electron multiplier, if we know the probability that n secondary electrons are produced in one elementary act. We answer the question of how to determine the probability of this elementary event, if we know experimentally the distribution curve of the impulse amplitudes in the multiplier.

1. In this article we consider the following two problems:

1) Given a distribution $P(\nu)$, where $P(0)$, $P(1)$, $P(2)$, . . . are the probabilities that the collision of an electron with a multiplier electrode leads to the appearance of 0, 1, 2, . . . secondary electrons. We shall determine the probability $P_N(\nu)$ that ν electrons are ejected from the Nth electrode if a primary electron strikes the first electrode[1]. In addition, we shall consider the quantity $P_N^{(k)}(\nu)$, which denotes the probability of producing ν electrons at the Nth multiplier electrode, if there are k primary electrons.

2) Given a distribution $P_N(\nu)$ of electrons, appearing at the Nth multiplier plate. We shall determine the distribution $P(\nu)$, of the electrons emanating from one electrode, which leads in N steps to the given distribution $P_N(\nu)$.

2. We begin by considering the probability distributions $P_2(\nu)$, $P_3(\nu)$, . . . obtained from the primary distribution $P(\nu)$. Let us extend the definition given above to the cases $N = 0$ and $N = 1$ by writing

$$P_0(\nu) = \begin{cases} 1, & \nu = 1, \\ 0, & \nu \neq 1, \end{cases} \qquad P_1(\nu) = P(\nu). \tag{1}$$

Next we introduce the generating function $G_N(u)$ of the distribution $P_N(\nu)$:

$$G_N(u) = \sum_{\nu=0}^{\infty} u^\nu P_N(\nu). \tag{2}$$

In particular, we have

$$\begin{aligned} G_0(u) &= u, \\ G_1(u) &= \sum u^\nu P(\nu) = G(u), \end{aligned} \tag{3}$$

where $G(u)$ is the generating function of the initial distribution $P(\nu)$.

We construct a recurrence formula for $G_N(u)$ in the following way. Consider the case where not one but $k > 1$ particles impinge on the first electrode. Designate by $P_N^{(k)}(\nu)$ the probability that k primary electrons yield just ν electrons at the Nth electrode. We have

$$P_N^{(k+l)}(\nu) = \sum_{\nu_1+\nu_2=\nu} P_N^{(k)}(\nu_1)\, P_N^{(l)}(\nu_2).$$

Multiplying this equation by u^ν and summing with respect to ν, we obtain in the usual way the corresponding generating function

$$G_N^{(k+l)}(u) = G_N^{(k)}(u)\, G_N^{(l)}(u).$$

Bearing in mind that $G_N^{(1)}(u) = G_N(u)$ by definition, and setting $l = 1$, $k = 1, 2, 3, \ldots$, we obtain, in general,

$$G_N^{(k)}(u) = (G_N(u))^k, \quad k = 0, 1, 2, \ldots, \tag{4}$$

where $G^0(u) = 1$ is the generating function of a distribution such that not a single electron falls on the multiplier, so that $P_N^{(0)} = 0$, $\nu \neq 0$.

To derive a recurrence formula for $G_N(u)$, we observe that ν particles will appear at the Nth stage if at some intermediate stage, say the kth, there are μ particles, and if they give rise to just $\nu - \mu$ particles at the remaining $N - k$ stages. The probability of such an event for a fixed value of μ is

$$P_k(\mu)\, P_{N-k}^{(\mu)}(\nu).$$

Summing over all possible intermediate values of μ, we obtain for the total probability

$$P_N(\nu) = \sum_{\mu=0}^{\infty} P_k(\mu)\, P_{N-k}^{(\mu)}(\nu).$$

[1] L. Janossy, Acta Mathematica **2**, 165 (1951)

Multiplying the last equation by u^{ν} and summing over $\nu = 0, 1, 2, \ldots$, we get, with the aid of Eq. (4),

$$G_N(u) = \sum_{\nu=0}^{\infty} u^{\nu} \sum_{\mu=0}^{\infty} P_k(\mu)\, P_{N-k}^{(\mu)}(\nu)$$

$$= \sum_{\mu=0}^{\infty} P_k(\mu)\, (G_{N-k}(u))^{\mu}.$$

Moreover, using Eq. (2), we have

$$G_N(u) = G_k\, G_{N-k}(u), \tag{5}$$

where we have written for brevity $G_k G_{N-k}(u)$ instead of $G_k[G_{N-k}(u)]$. Setting $k = 1$ in Eq. (5), we obtain in particular

$$G_N(u) = G G_{N-1}(u),$$

whence

$$G_0(u) = u, \quad G_1(u) = G(u),$$

$$G_2(u) = GG(u), \ldots,$$

$$G_N(u) = \underbrace{GG \ldots G}_{N}(u).$$

In this way we have obtained the successive generating functions $G_1(u)$, $G_2(u)$, . . . by iterating the initial function.

3. Thus, we can calculate $G_N(u)$ for a given $G(u)$; the corresponding $P_N(\nu)$ is obtained by a complex integration, namely

$$P_N(\nu) = \frac{1}{2\pi i} \oint_{u=0} u^{-\nu-1} G_N(u)\, du. \tag{6}$$

To simplify the calculations, it is convenient to introduce

$$u = e^{v}, \quad G_N(u) = \exp\{H_N(v)\}. \tag{7}$$

Transforming Eq. (5) with the help of Eq. (7), we obtain

$$H_N(v) = H_k H_{N-k}(v), \quad H_0(v) = v.$$

In this way, the quantities H_N, i.e., the logarithmic generating functions, can be calculated in just the same way as the usual functions G_N.

Moreover, Eq. (6) can be rewritten in the form

$$P_N(\nu) = \frac{1}{2\pi i} \int_{(v_0)} \exp\{-v\nu + H_N(v)\}\, dv, \tag{8}$$

where the integration is to be carried out on a suitable path in the complex plane, a path which intersects the real axis at $v = v_0$. The latter integral can be calculated to a good approximation by the method of steepest descents. Then, designating the first term of the well-known asymptotic expansion of the function $P_N(\nu)$ by $P_N^*(\nu)$, we obtain

$$\nu = H_N'(v),$$

$$P_N(\nu) \sim P_N^*(\nu) \tag{9}$$

$$= \exp\{-v\nu + H_N(v)\} / (2\pi H''(v))^{1/2},$$

where we have written v instead of v_0. The prime designates differentiation with respect to v; the relation (9) furnishes an approximate parametric representation of $P_N(\nu)$, where v plays the role of the independent parameter.

Thus our problem reduces to tabulating the functions $H_N(v)$, $H_N'(v)$, and $H_N''(v)$.

Calculating $H_N(v)$ by iterations, we can obtain $H_N'(v)$ and $H_N''(v)$ by numerical differentiation, and then, with the help of Eq. (9), we can calculate ν and $P_N(\nu)$ as functions of the independent parameter v. The fact that arbitrarily chosen values of v do not generally coincide with integral values of ν does not matter, since, in the cases of practical interest, ν takes on very large values.

4. Before giving a practical method of calculation, we shall introduce simple formulas for some of the moments of the distribution $P_N(\nu)$. The expressions

$$\left(\frac{d^k H_N(v)}{dv^k}\right)_{v=0} = K_k^{(N)} \tag{10}$$

are the so-called semi-invariants of the distribution. In particular

$K_1^{(N)} = \nu_N$, the mean value of ν,

$K_2^{(N)} = \overline{(\nu - \bar{\nu}_N)^2}$, the variance,

$K_3^{(N)} = \overline{(\nu - \bar{\nu}_N)^3}$, the mean cube of the deviation.

Setting $k = N - 1$ in Eq. (5), differentiating with respect to v, and then setting $v = 0$, we obtain (after some simple calculations) recurrence formulas for the moments $K_k^{(N)}$

$$K_1^{(N)} = p^N, \quad K_2^{(N)} = Q\frac{p^{2N} - p^N}{p^2 - p}, \tag{11}$$

$$K_3^{(N)} = \frac{p^N}{p(p^2-1)}\left\{R\left(p^{2N}-1\right)\right.$$

$$\left. + \frac{3Q^2}{p-1}\left(p^{2N-1} - p^N - p^{N-1} + 1\right)\right\},$$

where we have set

$$K_1^{(1)} = p, \quad K_2^{(1)} = Q, \quad K_3^{(1)} = R.$$

The first of the formulas (11) shows that the mean number of electrons at the Nth stage equals the product of the mean multiplication factors p pertinent to the separate stages. The second formula shows that $K_2^{(N)}/(K_1^{(N)})^2 \sim Q/(p^2 - p)$ does not depend on N if $N \gg 1$, $p > 1$. This indicates that the spread of the distribution $P_N(\nu)$ with respect to its mean value does not change greatly as N increases, i.e., if we were to plot $P_N(\nu)$ for $\nu = 1, 2, 3, \ldots$, changing the scale so that the mean values coincide, we would get distributions with the same spread. In other words, the distributions

$$p^N P_N(\nu / p^N) \tag{12}$$

have identical mean values, and moreover, the same spread. This agreement of the distributions does not carry over for the higher moments; the third moments of these distributions will differ considerably.

5. Computing the iterations. The iteration process can be given a simple geometrical interpretation (Fig. 1). On the (u, U) plane we have drawn the curve M, corresponding to the formula $U = G(u)$, and the straight line L corresponding to the formula $U = u$. There are two intersections, one at point A, where $u = U = 1$, and the other at point B, where $u = U = u_1$. There are two and only two such intersections if

1) $G'(1) = p > 1$,

2) $G(0) = P(0) > 0$.

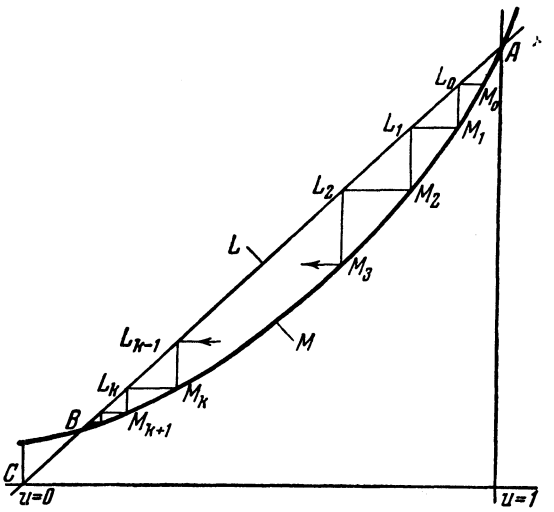

Fig. 1. Geometrical representation of the iteration process.

In fact, at the point A, M has a greater slope than L, and at the point C, where $u = 0$, $U = P(0)$, M lies above L; thus, there must be at least one intersection, say at $u = u_1$, in the interval $0 < u_1 < 1$; moreover, inasmuch as $G(u)$ has only positive derivatives and is concave downward, there can only be one such intersection.

Beginning now with a value of u, for example, $1 - \epsilon$, such that $u_1 < u < 1$, we find the corresponding point M_0 on the curve M. Drawing a horizontal through M_0, we obtain on the left an intersection L_0 with the line L; the coordinates of L_0 will be $u = U = G(1 - \epsilon)$. Now, dropping a vertical through L_0, we intersect the curve M at the point M_1, with coordinates $u = G(1 - \epsilon)$, $U = GG(1 - \epsilon)$. Repeating this procedure, we obtain a broken line. The abscissae of the points $M_0, M_1, M_2, \ldots$ (on the curve M) will be

$$1 - \varepsilon, \quad G(1 - \varepsilon), \quad GG(1 - \varepsilon), \ldots,$$

i.e., the iterations of $1 - \epsilon$.

As the point B is approached, the segments become shorter and shorter, and it can be seen that the sequence $G_n(1 - \epsilon)$, $n = 1, 2, 3, \ldots$, monotonically converges to the value u_1.

In the same way, we see that, starting with a value $0 < 1 - \epsilon < u_1$, we obtain an increasing sequence of points, which also converge to the limit u_1. Starting from the point $u = 1 + \epsilon$, we get a sequence of rapidly increasing values of u.

With a little more argument we can show that

$$\lim_{N=\infty} G_N(u) = G^*(u) = u_1, \quad 0 \leqslant u < 1.$$

Indeed, in the interval $u_1 < u < 1$, we have

$$u_1 < G(u) < u.$$

Iterating this N times, we find [since $G(u)$ is a monotone function] that

$$u_1 < G_{N+1}(u) < G_N(u).$$

Thus we see that the sequence

$$G_1(u), \quad G_2(u), \ldots \tag{13}$$

is monotonically decreasing and bounded, i.e., it must have a limit. The limit can only be u_1.

Thus $G^*(u) = u_1, \quad u_1 \leqslant u < 1.$

Furthermore, starting from

$$u_1 > G(u) > u \text{ for } u_1 > u \geqslant 0,$$

we see that the sequence (13) is monotonically increasing and bounded in the specified interval, and consequently converges to u_1. Thus

$$G^*(u) = u_1 \text{ for } 0 \leqslant u < 1.$$

Let us extend the result obtained above to complex values of u, since complex u are of interest in calculating the integral (9).

We have

$$G_N(u) - G_N(0) = P_N(1)u + P_N(2)u^2 + \ldots;$$

The coefficients are probabilities and, consequently, non-negative. We thus have

$$|G_N(u) - G_N(0)| \leqslant |G_N(|u|) - G_N(0)|.$$

Inasmuch as the right-hand side of this expression has the limit zero for $|u| < 1$, the same must be true for the left-hand side. Thus, we obtain

$$G^*(u) = u \text{ for } |u| < 1. \tag{14}$$

6. It is of some interest to determine the size of the steps of the iterational procedure in a very small neighborhood of the points of intersection A and B. Consider the intersection A; in its neighborhood the curve M can be replaced to a good approximation by a straight line M' with slope p.

Carrying out the iteration between the two straight lines, we see that the abscissae of the steps are, to a good approximation,

$$1 - \varepsilon, \quad 1 - p\varepsilon, \quad 1 - p^2\varepsilon, \ldots$$

Thus, to a first approximation, the distances from $u = 1$ constitute an increasing geometrical progression (Fig. 2). This simple behavior changes for orders of iteration such that there is an appreciable difference between the directions of the curve M and the tangent M'. In most of the interval between A and B, the curve bends away strongly from its tangent, and the steps are given by a complicated function of their index. Near B the steps again approximate a geometrical progression, so that for a large enough number k of steps, the abscissae are given to a good approximation by an expression of the form $u_1 + p_1^k \epsilon$, where $p_1 = G'(u_1)$, so that $0 < p_1 < 1$.

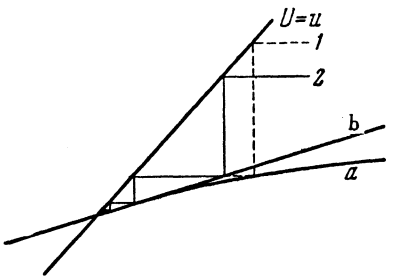

Fig. 2. *a*. curve, *b*. tangent; 1. exact steps, 2. approximate steps.

From the picture introduced above, we can derive a simple practical method of tabulating $G_N(u)$ for a given value of N. Let us draw, as was done above, a broken line $M_1, L_1; M_2, L_2, \ldots$, starting within a small neighborhood of the point A, so that the abscissa of the first point M_1 must be

$$u(0) = 1 - \varepsilon,$$

where ϵ is positive, but very small. Designating the subsequent abscissae by $u(1), u(2), \ldots$, we have

$$u(k) = G_k(1 - \varepsilon).$$

Comparing the kth and $(N + k)$th steps, we have

$$u(N + k) = G_{k+N}(1 - \varepsilon) = G_N(u(k)).$$

Thus, taking k as an independent parameter, we obtain a parametric representation of the function $G_N(u)$; indeed, we have

$$u = u(k), \quad G_N(u) = u(N+k).$$

Consequently, the tabulation of $G_N(u)$ consists in reading the kth abscissa as the value of the variable, and the $(k+N)$th abscissa as the corresponding value of the function, and repeating this operation for $k = 1, 2, \ldots$, etc.

Tabulation by the means just indicated has, nevertheless, the defect that the steps are comparatively large in the intermediate region between A and B. However, to get a more exact tabulation, we can construct several broken lines with mutually related steps. A more staisfactory procedure consists in interpolating between the first and second steps with a geometrical progression; in this way, we can determine a certain value u between $u(0)$ and $u(1)$, by setting

$$u(x) = 1 - p^x\varepsilon.$$

Starting from an iteration of the form

$$u(x+k) = G_k(u(x))$$

for

$$x = 0;\ 0{,}2;\ 0{,}4;\ 0{,}6;\ 0{,}8;\ k = 1, 2, \ldots,$$

we determine the function

$$G^*_{x+k}(1-\varepsilon) = G_k(u(x))$$

for the step 0.2, so that the parametric representation of $G_N(u)$ can be written as

$$u = G^*_{k+x}(1-\varepsilon), \quad G_N(u) = G^*_{k+x+N}(1-\varepsilon). \tag{15}$$

We have tabulated $G_{10}(u)$ by the means indicated, assuming that $P(\nu)$ is a Poisson distribution. Thus, taking

$$P(\nu) = e^{-p}\frac{\nu^p}{\nu!} \qquad \text{and } G(u) = \exp\{p(u-1)\},$$

we found that for the indicated distribtuion, with $p = 5$,

$$u_1 = 0{,}006997.$$

Then, starting form $\epsilon = 2.358 \times 10^{-7}$, and dividing the first interval $u(0)$, $u(1)$ into five parts, we obtained Tables I and II. In Table I we tabulated $G^*_{k+x}(1 \pm \epsilon)$ according to Eq. (15); in Table II we tabulated the function $G_{10}(u)$ using Table I.

7. We must now determine the derivatives of $G_N(u)$. According to the parametric representation (15), differentiation with respect to u can be carried out as follows:

$$\frac{dG_N(u)}{du} = \frac{dG^*_{k+x+N}(1-\varepsilon)}{dx}\,\frac{dx}{du};$$

whence the derivatives of $G_N(u)$ are obtained in the following parametic representation

$$\frac{dG_N(u)}{du} = \frac{dG^*_{k+x+N}(1-\varepsilon)}{dx} \Big/ \frac{dG^*_{k+x}(1-\varepsilon)}{dx}, \tag{16}$$

$$u = G_{k+x}(1-\varepsilon).$$

The derivatives $dG^*_{k+x}(1-\epsilon)/dx$ can be obtained by numerical differentiation of $G^*_{k+x}(1-\epsilon)$.

Differentiating again, we find

$$\frac{d^2G(u)}{du^2} = \frac{d}{dx}\left(\frac{dG_N(u)}{du}\right) \Big/ \frac{dG^*_{k+x}(1-\varepsilon)}{dx}. \tag{17}$$

In differentiating the function $dG_N(u)/du$ with respect to x, it must be understood that u itself is a function of x. Equation (17) is given in the form most useful for computation. We also tabulated these functions in Tables I and II.

The functions $G_N(u)$ for $N = 0, 1, 2, \ldots$, are shown in Fig. 3. It is clear that as N increases, the functions rapidly approach their limiting values.

With the help of the tables given, we calculated $P^*_{10}(\nu)$. In Fig. 4 are shown $P^*_{10}(\nu)$ and $P(\nu)$ in the units given by Eq. (12); it is clear that the two distributions are quite alike.

8. Below we shall investigate more rigorously the properties of the iterations $G_k(u)$, and show that the method of calculation used earlier in this paper leads to a generalization of the iterational process. Using this generalized iterational process, we shall give the solution of the inverse multiplier problem, mentioned at the beginning of the paper.

9. Iterations of negative index. We can introduce $G_{-k}(u)$ as the inverse function of $G_k(u)$. Recalling that the $G_k(u)$ are monotonically increasing functions of the argument u, we see that the inverse function is uniquely defined, at least, in the interval $u_1 \le u \le 1$; the function $G_{-k}(u)$ can

TABLE I

$\varepsilon = 2.35785 \cdot 10^{-7}$

$k+x$	$G^*_{k+x}\ (1-\varepsilon)$	$\frac{\partial G^*_{k+x}\ (1-\varepsilon)}{\partial x}$	$G^*_{k+x}\ (1+\varepsilon)$	$\frac{\partial G^*_{k+x}\ (1-\varepsilon)}{\partial x}$
>5	$\sim 1-\varepsilon p^{k+x}$	$\sim 1-\varepsilon \ln p\ p^{k+x}$	$\sim 1+\varepsilon p^{k+x}$	$\sim 1+\varepsilon \ln p\ p^{k+x}$
5.0	0.9993	0.0012	1.0007	0.0012
5.2	0.9990	0.0016	1.0010	0.0016
5.4	0.9986	0.0023	1.0014	0.0023
5.6	0.9981	0.0031	1.0019	0.0031
5.8	0.9973	0.0043	1.0027	0.0043
6.0	0.9963	0.0059	1.0037	0.0060
6.2	9.9949	0.0081	1.0051	0.0082
6.4	0.9930	0.0112	1.0070	0.0114
6.6	0.9904	0.0154	1.0097	0.0157
6.8	0.9868	0.0211	1.0135	0.0219
7.0	0.9818	0.0290	1.0186	0.0303
7.2	0.9750	0.0396	1.0258	0.0422
7.4	0.9657	0.0540	1.0358	0.0589
7.6	0.9531	0.0733	1.0499	0.0826
7.8	0.9359	0.0989	1.0696	0.1168
8.0	0.9130	0.1323	1.0976	0.1663
8.2	0.8824	0.1749	1.1378	0.2398
8.4	0.8424	0.2276	1.1963	0.3517
8.6	0.7908	0.2900	1.2831	0.5285
8.8	0.7259	0.3591	1.4173	0.8227
9.0	0.6472	0.4281	1.6294	1.3410
9.2	0.5555	0.4856	1.9917	2.3361
9.4	0.4547	0.5170	2.6681	4.4319
9.6	0.3513	0.5086	4.1195	8.9592
9.8	0.2540	0.4555	8.0150	
10.0	0.1713	0.3666	23.2646	
10.2	0.1083	0.2634	142.3725	
10.4	0.0654	0.1697	4190.2500	
10.6	0.0390	0.0996		
10.8	0.0240	0.0547		
11.0	0.0159	0.0290		
11.2	0.0116	0.0152		
11.4	0.0093	0.0079		
11.6	0.0082	0.0040		
11.8	0.0076	0.0021		
12.0	0.0073	0.0011		
12.2	0.0071	0.0006		
12.4	0.0071	0.0003		
12.6	0.0070	0.0001		
12.8	0.0070			
13.0	0.0070			

TABLE II

$$u = G^*_{k+x}(1-\varepsilon),\quad v = G^*_{k+x}(1+\varepsilon)$$

$k+x-10$	$G_{10}(u)$	$\frac{\partial G_{10}(u)}{\partial u}\times 5^{-10}$	$\frac{\partial^2 G_{10}(u)}{\partial u^2}\times 5^{-20}$	$G_{10}(v)$	$\frac{\partial G_{10}(v)}{\partial v}\times 5^{-10}$	$\frac{\partial^2 G_{10}(v)}{\partial v^2}\times 5^{-20}$
5.0	0.9993			1.0007		
5.2	0.9990	0.9992		1.0010		
5.4	0.9986	0.9996		1.0014		
5.6	0.9981	0.9980		1.0019	1.0010	
5.8	0.9973	0.9966		1.0027	1.0030	
6.0	0.9963	0.9971		1.0037	1.0042	
6.2	0.9949	0.9936		1.0051	1.0056	
6.4	0.9930	0.9914		1.0070	1.0079	
6.6	0.9904	0.9876		1.0097	1.0109	
6.8	0.9868	0.9837		1.0135	1.0167	
7.0	0.9818	0.9770		1.0186	1.0229	
7.2	0.9750	0.9688		1.0258	1.0316	
7.4	0.9657	0.9571	0.5491	1.0358	1.0438	
7.6	0.9531	0.9415	0.5340	1.0499	1.0610	
7.8	0.9359	0.9205	0.5195	1.0696	1.0870	
8.0	0.9130	0.8924	0.5031	1.0976	1.1219	0.6455
8.2	0.8824	0.8551	0.4785	1.1378	1.1725	0.6811
8.4	0.8424	0.8067	0.4469	1.1963	1.2465	0.7270
8.6	0.7908	0.7449	0.4062	1.2831	1.3573	0.8101
8.8	0.7259	0.6685	0.3596	1.4173	1.5315	0.9209
9.0	0.6472	0.5776	0.3031	1.6294	1.8093	1.0975
9.2	0.5555	0.4749	0.2406	1.9917	2.2844	1.4230
9.4	0.4547	0.3664	0.1769	2.6681	3.1411	
9.6	0.3513	5.2613	0.1182	4.1195	4.6022	
9.8	0.2540	0.1696	0.0704	8.0150		
10.0	0.1713	0.0989	0.0366	23.2646		
10.2	0.1083	0.0515	0.0165	142.3725		
10.4	0.0654	0.0241	0.0064	4190.2500		
10.6	0.0390	0.0102	0.0022			
10.8	0.0240	0.0041	0.0007			
11.0	0.0159	0.0016				
11.2	0.0116	0.0010				
11.4	0.0093	0.0002				
11.6	0.0082	0.0001				
11.8	0.0076					
12.0	0.0073					
12.2	0.0071					
12.4	0.0071					
12.6	0.0070					
12.8						

sometimes be defined in a larger interval. From the definition introduced above, it follows that

$$G_k G_l(u) = G_{k+l}(u) \tag{18}$$

for arbitrary integers k and l.

Iterations of fractional order. We can generalize the definition of $G_k(u)$ to include fractional orders, while retaining the validity of the relation (18) for interpolating values of k. The interpolation can be done by a method like that already used in this paper. We shall show that this method can be made mathematically rigorous.

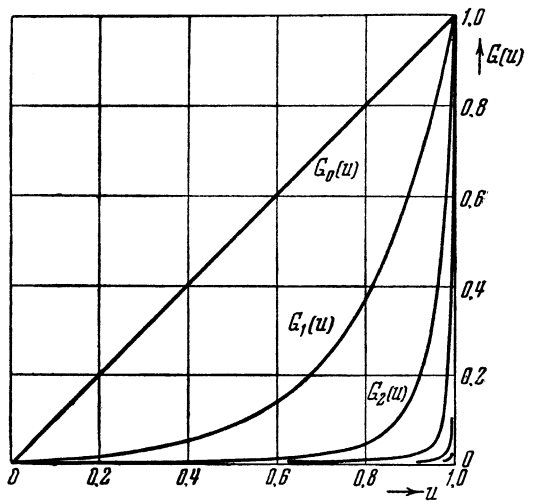

Fig. 3. The function $G_N(u)$ in the case of a Poisson distribution with $p = 5$.

We discovered these iterations of fractional order in considering statistical problems in a multiplier; later our attention was called to the fact that the theory of such iterations had already been given in 1889[2]. In view of the interest of a whole group of questions pertaining to multipliers, we repeat here the theory of such iterations; the present exposition is somewhat more detailed than the original one[2].

In close analogy to Eq. (15), we define the following function

$$\tilde{G}_{x,n}(1-\varepsilon) = G_n(1-\varepsilon p^x). \tag{19}$$

We must perform a suitable passage to the limit $n \to \infty$, $\epsilon \to 0$, and thereby hope to get a generalized iteration. Such a passage to the limit can be achieved by substituting in Eq. (19)

$$\varepsilon = 1 - G_{-n}(u), \quad n \to \infty,$$
where u is fixed

In this way, we define (first for finite values of n)

$$G_{x,n}(u) = G_n[1 - p^x(1 - G_{-n}(u))]. \tag{20}$$

The geometrical meaning of the functions $G_{x,n}(u)$ defined in this way can be shown graphically in Fig. 1 as follows. Beginning with the point $[u, U = G(u)]$ on the curve M, we make a large number n of steps to the right, and finally approach the point with abscissa $G_{-n}(u) \approx 1$ in the neighborhood of A. The abscissa of the latter point differs from unity by the small quantity

$$\varepsilon = 1 - G_{-n}(u).$$

Multiplying this distance by p^x, we arrive approximately at the point x steps back, provided that x is an integer; consequently, for integral x,

$$p^x(1 - G_{-n}(u)) \approx 1 - G_{-x-n}(u).$$

Thus, for integral x we expect that

$$G_{x,n}(u) \approx G_n G_{x-n}(u) = G_x(u).$$

Inasmuch as the approximation should improve with increasing n, we can expect that the limit

$$\lim_{n=\infty} G_{x,n}(u) = G_x^*(u) \tag{21a}$$

exists, and that

$$G_x^*(u) = G_x(u), \tag{21b}$$

if x is integral. Below we shall show that not only Eq. (21a) but also Eq. (21b) actually are valid, and moreover, that for arbitrary real x we have

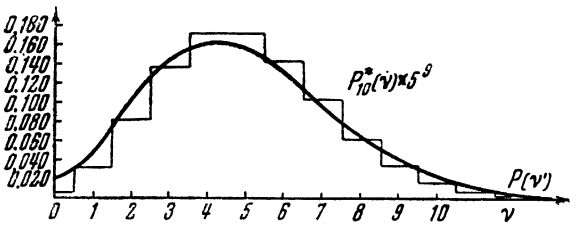

Fig. 4. The distributions $P_{10}^*(\nu)$ and $P(\nu)$ (Poisson distribution), $\nu' = 5^9\nu$.

[2] M. Lémeray, Compt. rend. **128**, 278 (1889)

$$G_x^* G_y^*(u) = G_{x+y}^*(u). \tag{22}$$

10. For the proof, we first investigate $G_{x,n}(u)$, defined by Eq. (20) for finite values of n.

Since $p > 1$, the argument of G_n for $x \leq 0$ in the right-hand side of Eq. (20) lies between zero and one, if $G_{-n}(u)$ is in the same interval. Thus $G_{x,n}(u)$ is completely defined for $u_1 \leq u \leq 1$ and $x < 0$. For $x > 0$, the argument of G_n in the right-hand side of Eq. (20) can be negative, and therefore we must restrict ourselves to a consideration of values of u that are not too small.

Furthermore, from Eq. (20) we have

$$G_{x,n} G_{y,n}(u) = G_{x+y,n}(u), \tag{23}$$

and also

$$G_{0,n}(u) = u. \tag{24}$$

Setting $x = -y$, we obtain

$$G_{x,n} G_{-x,n}(u) = u.$$

Thus $G_{-x,n}$ is the inverse function of $G_{x,n}$. Differentiating Eq. (20) with respect to x, we find

$$\frac{\partial G_{x,n}(u)}{\partial x} = (\ln p)\, p^x (1 - G_{-n}(u))$$

$$G_n' [1 - p^x (1 - G_{-n}(u))] > 0.$$

Consequently, $G_{x,n}(u)$ is a monotone increasing function of x, and therefore has a unique inverse function.

Equations (23) and (24) show that for fixed n and $x = 0, 1, 2, \ldots$, $G_{x,n}(u)$ represents a sequence of iterations, where the iterated function is $G_{1,n}(u)$. It remains to show that the limits exist as $n \to \infty$.

The passage to the limit. To prove the existence of the limits $G_x^*(u)$, it is necessary to assume that the first and second derivatives of $G(u)$ exist at $u = 1$. As before, we write

$$G'(1) = p, \quad G''(1) = Q + p^2 = q.$$

From the fact that $G'(u)$ and $G''(u)$ are nonnegative in the interval $0 \leq u \leq 1$, it is easily found that

$$1 - p(1-u) \leqslant G(u) \tag{25}$$

$$\leqslant 1 - p(1-u) + \tfrac{1}{2} q (1-u)^2.$$

Analogously, for $G_n(u)$ with $0 \leq u \leq 1$, $n = 1, 2, \ldots$, it can be shown that

$$1 - p^n (1-n) \leqslant G_n(u) \tag{26}$$

$$\leqslant 1 - p^n (1-u) + \tfrac{1}{2} \alpha p^{2n} (1-u)^2,$$

where

$$\alpha p^{2n} = G_n''(1).$$

Thus [cf. Eq. (11)]

$$\alpha = \alpha_0 (1 - p^{-n}), \quad \alpha_0 = q / p(p-1).$$

We shall examine the behavior of $1 - G_{-n}(1-\epsilon)$ for small values of ϵ, and large positive values of n. At the beginning of this paper, we arrived qualitatively at the conclusion that $1 - G_{-n}(1-\epsilon)$ approaches its limiting value of zero almost like a geometrical progression. In order to make the qualitative conclusions given above more rigorous, we shall show the existence of a finite limit $A(u)$ such that, for $n > 0$

$$p^n (1 - G_{-n}(u)) < A; \quad u_1 \leqslant u < 1.$$

For the proof, we shall write

$$G_n(1 - \varepsilon) = u \tag{27}$$

or

$$\varepsilon = 1 - G_{-n}(u).$$

Approximating $G_n(1-\epsilon)$ by polynomials in accordance with Eq. (26), we get from Eq. (27)

$$1 - p^n \varepsilon < u \leqslant 1 - p^n \varepsilon + \tfrac{1}{2} \alpha_0 p^{2n} \varepsilon^2.$$

On the left-hand side of the inequality is a monotone decreasing function of ϵ; thus, there exists an $\epsilon' \leq \epsilon$ for which the first equality becomes an equality if ϵ is replaced by ϵ'. Analogously, for not too large ϵ, there exists an $\epsilon'' \geq \epsilon$ for which the second inequality becomes an equality if ϵ is replaced by ϵ''. Thus we can write

$$\varepsilon' \leqslant \varepsilon \leqslant \varepsilon''$$

and

$$1 - p^n \varepsilon' = u = 1 - p^n \varepsilon'' + \tfrac{1}{2} \alpha_0 p^{2n} \varepsilon''^2.$$

Solving these equations with respect to ϵ' and ϵ'', we obtain

$$\varepsilon' = \frac{1-u}{p^n}, \quad \varepsilon'' = \frac{\alpha_0}{p^n}\left(1 - \sqrt{1 - \frac{2(1-u)}{\alpha_0}}\right).$$

Bearing in mind that $1 - \sqrt{1-2h} \leqslant h + 2h^2$ for $h \leqslant 1/2$,

we can write the inequality for ϵ in the form

$$1 - u \leqslant p^n \varepsilon \leqslant 1 - u + \frac{2(1-u)^2}{\alpha_0} = A(u),$$

provided that

$$1 \geqslant u \geqslant 1 - \alpha_0/2, \tag{28}$$

and also

$$1 \geqslant u \geqslant u_1, \tag{29}$$

where we must use the stronger of the stipulated conditions.

Expressing ϵ in the terms of u, we get

$$p^n (1 - G_{-n}(u)) \leqslant A(u), \tag{30}$$

where the boundary value $A(u)$ certainly does not depend on n, if it is assumed that u lies within both intervals (28) and (29). However, it is possible to free ourselves from condition (29). If

$$1 > 1 - \alpha_0/2 > u_1,$$

and u lies in the interval

$$1 - \alpha_0/2 \geqslant u \geqslant u_1,$$

i.e., if u lies outside the interval (29), then there must exist a finite k, dependent on $u > u_1$, such that $1 \geq G_{-k}(u) \geq 1 - \alpha_0/2$; then $v = G_{-k}(u)$ lies inside the interval (29), although u does not lie inside this interval. Substituting $v = G_{-k}(u)$ in the inequality (30) instead of u, and multiplying by p^k, we find that

$$p^{n+k}(1 - G_{-n-k}(u)) \leqslant p^k A(G_{-k}(u));$$

then, setting

$$p^k A(G_{-k}(u)) = B(u),$$

we have

$$p^n (1 - G_{-n}(u)) \leqslant B(u) \tag{31}$$

$$u_1 < u \leqslant 1, \; n > k.$$

Thus we have an upper bound, even if u lies outside the interval (28), but inside the interval (29). For $u = u_1$, the upper bound does not exist, inasmuch as

$$p^n (1 - G_{-n}(u_1)) = p^n (1 - u_1) \to \infty$$

as

$$n \to \infty.$$

With the help of the result just obtained, we can show the existence of the limit of $G_x^*(u)$. To do this, we evaluate the difference

$$h_n = G_{x,n}(u) - G_{x,n+1}(u)$$
$$= G_n[1 - p^x(1 - G_{-n}(u))]$$
$$- G_{n+1}[1 - p^x(1 - G_{-n-1}(u))].$$

Setting

$$1 - G_{-n-1}(u) = \varepsilon$$

or

$$u = G_{n+1}(1 - \varepsilon),$$

we have

$$h_n = G_n(u') - G_n(u''), \tag{32}$$

where

$$u' = 1 - p^x(1 - G(1 - \varepsilon)),$$
$$u'' = G(1 - p^x \varepsilon). \tag{33}$$

From Eq. (32) we have

$$|h_n| \leqslant |u' - u''| p^n. \tag{34}$$

From Eq. (33) we find, with the help of Eq. (25), that

$$1 - p^{x+1}\varepsilon \leqslant u' \leqslant 1 - p^{x+1}\varepsilon + 1/2 q p^x \varepsilon^2,$$
$$1 - p^{x+1}\varepsilon \leqslant u'' \leqslant 1 - p^{x+1}\varepsilon + 1/2 q p^{2x} \varepsilon^2.$$

Thus

$$|u' - u''| \leqslant 1/2 q \varepsilon^2 (p^x + p^{2x}). \tag{35}$$

It was shown above that $\epsilon < B(u)/p^n$; hence, from the inequalities (34) and (35) we have

$$|h_n| \leqslant {}^1/_2 (B(u))^2 q(p^x + p^{2x})/p^n = C(u)/p^n,$$

so that the sequence $G_{x,n}(u)$, $n = 1, 2, \ldots$, converges for sufficiently large values of n, at least as fast as a geometrical series, and hence the limit $G_x^*(u)$ necessarily exists.

For integral x, the value $G_x^*(u)$ can easily be calculated. We shall consider only the case $x = 1$. We introduce two quantities ϵ' and ϵ'' as follows

$$1 - \varepsilon' = G(1 - \varepsilon) \text{ и } 1 - \varepsilon'' = 1 - p\varepsilon. \qquad (36)$$

In analogy with the derivation of the inequality (35), we find that ϵ' and ϵ'' satisfy the following relation

$$|\varepsilon' - \varepsilon''| < q\varepsilon^2.$$

Thus

$$|G_n(1 - \varepsilon') - G_n(1 - \varepsilon'')| \leqslant p^n q \varepsilon^2.$$

Setting

$$\varepsilon = 1 - G_{-n}(u),$$

and taking account of Eq. (36), we can write

$$|G(u) - G_{1,n}(u)| \leqslant p^n q \varepsilon^2.$$

Inasmuch as ϵ^2 is of order p^{-2n}, as shown above [cf. Eq. (31)], the expression on the right-hand side goes to zero, and we have

$$\lim_{n=\infty} G_{1,n}(u) = G(u).$$

11. We note that instead of using expression (20), we could have begun our investigation by introducing the expression

$$\tilde{G}_{x,n}(u) = G_{-n}\{u_1 - p_1^x (u_1 - G_n(u))\}, \qquad (37)$$

where

$$p_1 = G'(u_1).$$

Expressions (20) and (37) are analogous, but nevertheless

$$\lim_{n=\infty} \tilde{G}_{x,n}(u) = \tilde{G}_x^*(u)$$

and

$$\lim_{n=\infty} G_{n,x}(u) = G_x^*(u)$$

are two completely different limiting processes. Therefore, it should by no means be considered trivial that these two processes lead to one and the same limiting function. However, more detailed examination shows that

$$G_x^*(u) = \tilde{G}_x(u).$$

Thus Eq. (37) can be used instead of Eq. (20) for obtaining iterations of fractional order. We note that in just the same way that we proved the relation $G_1^*(u) = G(u)$, we can show that $\tilde{G}_1^*(u) = G(u)$; furthermore

$$\tilde{G}_x^* \tilde{G}_y^*(u) = \tilde{G}_{x+y}^*(u);$$

whence it follows that

$$\tilde{G}_k^*(u) = G_k^*(u) = G_k(u)$$

for an arbitrary integral value of k. The proof can also be extended to non-integral values of the index. Indeed, we shall write

$$g_1(u) = G_x(u),$$

$$g_2(u) = G_{2,x}(u), \ldots, \; g_k(u) = G_{k,x}(u).$$

Here $g_1(u)$, $g_2(u)$, . . . are also a sequence of iterations, which for $x > 1$ can be considered as a basis; we can define

$$g_x^*(u) = \lim_{n=\infty} g_n\{1 - p'^y (1 - g_{-n}(u))\},$$

$$p' = p^x,$$

$$\tilde{g}_y^*(u) = \lim_{n=\infty} g_{-n}\{u_1 - p_1'^y (u_1 - g_n(u))\},$$

$$p_1' = p_1^x.$$

As before, we have

$$g_1^*(u) = \tilde{g}_1^*(u) = g_1(u).$$

Hence

$$\tilde{G}_x^* = G_x^*(u).$$

We see that for $x > 1$, Eqs. (20) and (37) lead to identical results. With the help of Eq. (22), the result obtained above can be extended to arbitrary real values of x.

12. The "inverse" problem of the multiplier, mentioned in Sec. 1, can be formulated as follows.

Given $P_N(\nu) = \pi(\nu)$, the distribution arising at the Nth stage of the multiplier; we want to find $P(\nu) = P_1(\nu)$. With the help of the generating functions, we can write

$$\Gamma(u) = \sum_{\nu=0}^{\infty} u^{\nu}\pi(\nu) = G_N(u).$$

Using the generalized iterational process, we obtain

$$G(u) = \Gamma_{1/N}(u).$$

Indeed

$$G_N(u) = \underbrace{G \ldots G}_{N}(u)$$

$$= \underbrace{\Gamma_{1/N} \ldots \Gamma_{1/N}}_{N}(u) = \Gamma(u).$$

Clearly we have

$$G(u) = \lim_{n=\infty} \Gamma_n \{1 - \pi^{1/N}(1 - \Gamma_{-n}(u))\},$$

where $\pi = \Gamma'(1)$ is the mean number of electrons ejected at the last stage.

Numerical calculations are immediately effective only when the distribution $\pi(\nu)$ is known sufficiently exactly. It goes without saying that we cannot require that $\pi(\nu)$ be known for the separate integral values of ν; the most that can be hoped for is that $\pi(\nu)$ be known over a range of values of ν that is not too broad. If the distribution is given in this way, then $\Gamma(u)$ can be calculated by numerical integration:

$$\Gamma(u) = \int_{-\infty}^{+\infty} u^{\nu}\pi(\nu)\, d\nu.$$

Only those values which are near $u = 1$ are of interest. Indeed, if $|1 - u| \gg 1/p^N$, then the value of the integral is practically equal to either zero or infinity, and the calculation loses its meaning. However, there is a small interval near $u = 1$ where the calculation can be made, which permits the determination of the main part of $\Gamma(u)$, from which we can find the iteration $\Gamma_{1/N}(u)$.

The procedure described above can always be carried out mathematically, and thus the distribution $P_1(\nu)$ can be calculated. However, if we started from an arbitrary distribution $\pi(\nu)$, the calculated result would in general lead to an unacceptable distribution $P_1(\nu)$, containing negative probabilities. Thus if we start with a distribution $\pi(\nu)$ and obtain an acceptable distribution $P(\nu)$, this may be regarded as a check on the correctness of the method used.

Translated by R. Silverman
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