$$S = \frac{2\pi^2\varkappa^4}{45h^3v^3}T^3; \quad C = \frac{2\pi^2\varkappa^4}{15h^3v^3}T^3.$$

Here F is the free energy, S the entropy, C the heat capacity excited in the superconducting state. We thus obtain the correct temperature dependence for C. If the value of the velocity of the excitation, $v = H_{km}/\sqrt{4\pi mn_{s0}} \approx 10^5$ cm/sec is substituted in the expression for the heat capacity, then the correct order of magnitude is obtained. The correct value for the electronic heat capacity is obtained without the introduction of the concept of a large dielectric permeability.

If the particles described by the "effective" wave function obeyed Fermi statistics, then the velocity of the phonon type would have been 10^8 cm/sec. Consequently the calculated values of the thermodynamic functions are significantly less than those observed in the superconducting state. Moreover, for very small values of the parameter $\varkappa$ (for low concentration of superconducting electrons), excitations of the Fermi type would arise. Such excitations lead to the appearance of a linear term in T in the expression for the heat capacity.

In the calculation of the interaction, the linear parameter δ enters characteristically. We assume that destruction of the superconducting state by the thermal vibrations of the lattice takes place when $\lambda_{max} \approx \delta$, where λ_{max} is the wavelength which corresponds to the maximum number of phonons for a given temperature. Following the work of De Launay[4], we assume that

$$\lambda_{max} T \approx hs/\kappa,$$

where s is the velocity of sound in the lattice. We then get for the critical temperature the expression

$$\kappa T_k \approx hs/\delta \approx (H_{K,M}/\sqrt{4\pi mn_{s0}})\, s,$$

from which it follows that $T_k \approx 1°$ and $TM^{1/2} = \text{const}$.

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The Asymptote of Green's Function in Quantum Electrodynamics

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1. FROM the system of integral equations for Green's function in the presence of external sources of a photon field, one can show the relation*

$$G^{-1}(p, p'+s) - G^{-1}(p-s, p') = s_\mu \Gamma_\mu(p, p', s). \quad (1)$$

Equation (1) is a generalization of Ward's theorem, and, insofar as it follows from the exact equations, we shall in the future make much use of it. The system of renormalized equations for the Green's function in quantum electrodynamics may be written, to order e^2, as follows (we retain the notation of paper[1]):

$$\Gamma_\sigma(p, p-l, l) = \gamma_\sigma + \frac{e^2}{\pi i} \quad (2)$$

$$\times \int \{\Gamma_\mu(f, p-k, k)\, G(p-k)\, \Gamma_\sigma(p-k,\ p-k-l, l)$$

$$\times G(p-k-l)\, \Gamma_\nu(p-k-l,\ p-l, -k)$$

$$-\Gamma_\mu(p^0, p^0-k, k)\, G(p^0-k)$$

$$\times \Gamma_\sigma(p^0-k,\ p^0-k, 0)\, G(p^0-k)$$

$$\times \Gamma_\nu(p^0-k,\ p^0, -k)\}\, D_{\mu\nu}(k)\, d^4k,$$

$$\frac{\partial G^{-1}(p)}{\partial p_\mu} = \Gamma_\mu(p, p) \text{ or } G^{-1}(p) - G^{-1}(p-k) \quad (3)$$

$$= k_\mu \Gamma_\mu(p, p-k, k),$$

$$\frac{\partial D^{-1}(k^2)}{\partial k^2} = 1 + e^2 P = 1 + \frac{e^2}{6\pi}\,\mathrm{Sp}$$

$$\times \int \Big\{ \frac{\partial}{\partial k_\nu}\Big[\Gamma_\mu(p, p-k, k)\, \frac{\partial G(p-k)}{\partial k_\nu}\, \Gamma_\mu(p-k, p, -k)\Big]$$

$$-\frac{\partial}{\partial k_\nu^0}\Big[\Gamma_\mu(p, p-k^0, k^0)\, \frac{\partial G(p-k^0)}{\partial k_\nu^0} \quad (4)$$

$$\times \Gamma_\mu(p-k^0, p_1-k^0)\Big]\Big\}\, G(p)\, d^4p,$$

where $D_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) D(k^2) + \frac{k_\mu k_\nu}{k^4} d_l(k^2),$

$$k^{0^2} = 0, \; p^{0^2} = m^2.$$

The boundary values for D and G are:

$$(p^0 - m)\, G(p^0) = 1, \quad k^0{}_{\!,}D(k_0^2) = 1. \qquad (5)$$

These equations for D and G differ somewhat from those in reference 1, and are revised so as to satisfy exactly Ward's relation. This allows a significant simplification in further calculations, such that we need not seek the so-called "small" corrections.(See reference 1). Let us look for the asymptote of G and D, at large momenta, in the form:

$$G^{-1}(p) = \alpha(p^2)\, \hat{p}, \quad D^{-1}(k^2) = \frac{k^2}{d(k^2)}, \qquad (6)$$

From Eq. (3) it follows that

$$\Gamma_\mu = \alpha(f)\, \gamma_\mu, \qquad (7)$$

where f is the largest of the momenta.

As a consequence of Eq. (3), Eq. (2) becomes a linear integral equation in α. After a simple calculation, we obtain from Eq. (2)

$$\frac{\partial \alpha(\xi)}{\partial \xi} = \frac{e^2}{4\pi} \alpha(\xi)\, d_l(\xi), \; \xi = \ln\left(-\frac{p^2}{m^2}\right) \qquad (8)$$

with the boundary value

$$\alpha(0) = 1. \qquad (9)$$

From Eqs. (8) and (9) we get

$$\alpha(\xi) = \exp\left\{-\frac{e^2}{4\pi}\int_0^\xi d_l(z)\, dz\right\}. \qquad (10)$$

To this approximation, as a result of Eq. (3), α completely drops out of the expression for the polarization operator P, which goes into the value given it in perturbation theory, i.e., one uses the free particle values of G and Γ. As the result of a simple calculation, we obtain from Eq. (4)

$$\frac{1}{d} = 1 - \frac{e^2}{3\pi}\ln\left(-\frac{p^2}{m^2}\right). \qquad (11)$$

The asymptotes obtained coincide with those previously gotten by Landau, Abrikosov and Khalatnikov [1].

2. We shall show several general properties of renormalized theories.

I. The radiative corrections to the Green's function of Fermi- and Bose- fields, before renormalization, are equal to zero at infinite momenta. Indeed, let us look, as an example, at the Green's function for a photon. According to reference 2,

$$D_F(p) = z_3 D_{FC}(p), \quad G_{FC} = \frac{1}{p^2} + \int_0^\infty \frac{f(\mu^2, e^2)}{p^2 + \mu^2}\frac{d\mu^2}{\mu^2},$$

$$z_3^{-1} = 1 + \int_0^\infty f(\mu^2, e^2)\frac{d\mu^2}{\mu^2}. \qquad (12)$$

Since the theory is normalized, it follows from Eq. (12) that $D_F(p^2) = 1$ ($p^2 \to \infty$), i.e., the radiative corrections actually vanish.

II. If the derivative of the polarization operator with respect to the square of the external momentum is infinite for small momenta (and this occurs in existent calculations), then the renormalized charge is equal to zero. In the renormalized equations, where the self-action is absent and the charge is considered finite, this difficulty is manifested in the change in sign of the Green's function at large momenta, whereupon a fictitious singularity arises. The reason why the difficulty crops up lies in the fact that, since the self-action is infinite, the law of interaction at small distances ($r = 0$) is really not correct; in the renormalized equations the self-action is not used, but is effectively taken into account by the introduction of renormalized masses and charge. Considering the renormalized quantities as finite and equal to the experimental values, we correct the self-action (or the interaction at $r = 0$); however, the interaction at $r \to 0$ remains indefinitely increasing – leading to a contradiction to the finiteness of the self-action. If we use, instead of the renormalized charge, its theoretical value, then the Green's function does not change sign, but the renormalized charge itself proves to be zero. Actually, (see reference 3), D_{FC} is related to the polarization operator in the following way:

$$D_{FC} = \frac{1}{k^2(1 + e^2[\Pi(k^2) + \Pi(0)])}, \qquad (13)$$

$$\Pi(k^2) = \frac{1}{6k^2 z_1}\,\mathrm{Sp} \times \int \gamma_\mu G(p+k)\,\Gamma_\mu(p+k, k)\, G(p)\, d^4p. \qquad (14)$$

If $\Pi(0)$ is infinite ** (and since, according to (1), $\Pi(k^2)$ approaches zero as $k^2 \to \infty$), then D_{FC} changes sign, going through a pole at arbitrary finite e^2. According to theory

$$e^2 = \frac{e_0^2}{1 + e_0^2\,\Pi(0)}. \qquad (15)$$

With this meaning for e^2, D_{FC} nowhere changes sign, but e^2 is equal to zero for any "bare" charge e_0^2.

In the case of small "bare" charge, the asymptote adduced in section I is exact and the renormalized charge is equal to zero. In the case, however, where the "bare" charge is large (which apparently happens in quantum electrodynamics [2]), all the diagrams of higher order than e^2, not considered in section I, become extremely important and may radically alter the asymptote found. Using Eq. (1) and property I, and neglecting terms of the type $(\gamma_\nu \gamma_\mu - \gamma_\mu \gamma_\nu) f(p^2)$ in the expression for $\left.\frac{\delta \Gamma_\mu}{\delta x_\nu}\right|_{k=0}$, one can show that $\Pi(0)$ remains infinite in the general case‡. Therefore, it seems likely that in quantum electrodynamics the renormalized charge is equal to zero. Since the experimental charge in quantum electrodynamics is small, one could eliminate the difficulty of the zero charge by a correction to the interaction at very large momenta, and a calculation of gravitational effects could make it, generally speaking, not inconsistent with the mathematical theory[5,1]

* The proof of this relation will be given separately.

** $\Pi(0)$ is always ≥ 0.

‡ A discussion of the use of the fact that $\Pi(0)$ is infinite is also given in reference 4

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The Optimum Length of an Undulator

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As has been shown by Ginsburg [1], only the utilization of coherent radiation in an undulator [2,3], makes it possible to obtain considerable radiated power. At the same time, the power radiated from an undulator is directly proportional to its length (see, for example, reference 2); therefore it is desirable to increase the length of the undulator. However, these two contributions to the power radiated from an undulator are in contradiction with one another. As the electrons pass through the undulator, the conditions for coherent radiation become worse, in view of the spreading of the bunch. The resolution of this contradiction appears to be the imposition of a restriction on the length of the undulator.

Consider the definiteness, a spherical bunch of N electrons, in a coordinate system moving with the center of the bunch. Making use of the integral of motion $\frac{mv_r^2}{2} + \frac{e^2N}{r} = \frac{e^2N}{r_0}$, we find that the increase of the radius of the bunch by a factor p takes place during proper time $\Delta\tau$, according to the relation

$$\Delta\tau = \frac{1}{e}\sqrt{\frac{m}{2N}}\left[\sqrt{p(p-1)} + \ln\left(\sqrt{p} + \sqrt{p-1}\right)\right] r_0^{3/2}, \tag{1}$$

where r_0 is the initial radius of the bunch, with the initial rate of expansion assumed to be zero, and m and e are the rest mass and charge of the electron, respectively.

The condition for coherent radiation is that the dimensions of the electron bunch be small in comparison with the wavelength of the radiated waves. In order to obtain the wavelength λ in the laboratory system of coordinates, it is necessary to generate the wavelength $\lambda/\sqrt{1-\beta^2}$ in the coordinate system (with velocity $v = c\beta$) moving with the center of the oscillating bunch. In view of this, the condition for coherence is well fulfilled when $r_0 \sim \lambda$, and is completely violated when $p \sim E/mc^2$. When, as was assumed in references 1-3, the fields in the undulator are not too large, the velocities of the electrons are small with respect to a coordinate system moving with the center of the oscillating bunch, and therefore, an interval of proper time of the bunch is approximately equal to an interval of time in the coordinate system moving with the center of the oscillating bunch. Therefore, on the basis of Eq. (1), the time for complete loss of coherence in the laboratory system of coordinates is

$$\Delta t \approx \frac{1}{e}\sqrt{\frac{m}{2N}}\left(\frac{E}{mc^2}\right)^2 \lambda^{3/2}. \tag{2}$$

Denoting the length of the spatial period of the field in the undulator by l_0, and taking into account the equality $\lambda \sim l_0\left(\frac{mc^2}{E}\right)^2$, we find that the number q of spatial periods of the field in the undulator must in any case be no greater than

$$q_{\max} = \frac{c}{e}\sqrt{\frac{m}{2N}}\sqrt{\lambda} \approx \frac{10^6}{\sqrt{N}}\sqrt{\lambda} \quad (\lambda \text{ cm}). \tag{3}$$