Method of Terminated Field Equations and Its Applications to Scattering of Mesons by Nucleons

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Investigations of several general problems involved in the use of a new method of terminated equations are reported. This method is applied to the problem of scattering of mesons by nucleons with a higher degree of approximation than formerly.

1. INTRODUCTION

IN spite of the known shortcomings of modern quantum field theory, all the derivations of quantum electrodynamics that are capable of experimental verification have been fully corroborated experimentally. However, the same cannot be said of meson theory, principally because in meson theory, unlike in electrodynamics, it is impossible to expand the interaction constant in powers. Nevertheless, this does not exclude the possibility that modern meson theory agrees with experience even though in a certain limited energy region only. To resolve this problem it is necessary to solve the field equations by a method other than the conventional perturbation theory. Among the several such methods, the one that is being most intensely developed in recent times is the method of terminating the field equations in accordance with the number of virtual particles (we shall call this for brevity the method of terminated equations).

Its advantages over the so-called four-dimensional methods (for example, the Bethe--Salpeter method[1]) are first that the physical sense of all quantities employed in this method is quite clear, and second, that it permits obtaining directly from the system of four-dimensional covariant equations a relatively easily solvable system of equations in three-dimensional momentum space.

However, the value of this method will, in the final analysis, be determined by whether it is enough, when solving physical problems, even in a limited energy region, to take into account only a relatively small number of virtual particles in the sense that consideration of a larger number of particles will not affect substantially the result of calculations, and that this result will agree with experiment. So far it has not been possible to find a method for the general solution of this problem analytically, so that an answer to this question can apparently be obtained only by direct computation of various effects.

In our opinion, the principal hope for the possible success of this method is based on the fact that its application to the scattering of mesons by nucleons has led to a considerable theoretical success in explaining the experimentally observed resonance of a pi-meson + nucleon system in a $I = J = 3/2$ state at an energy on the order of 180 mev[2].

The method of terminated equations for meson dynamics was independently suggested by Tamm[3] in 1945 and Dancoff[4] in 1950; the latter, however, did not know that an analogous method was employed by Fok[5] in electrodynamics as early as 1934.

Without dwelling on the history of the development of the method, let us note only two of the most important recent investigations that contributed substantially to its perfection. Cini[6] proposed a covariant formulation of the method and first investigated the question of renormalization in this method. Dyson[7] suggested that the actual physical system be described by parameters that characterize the difference between this system and a physical rather than a mathematical vacuum; this enabled him particularly to eliminate from consideration vacuum loops, the treatment of which involves specific difficulties in the old method.

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[2] S. Fubini, Nuovo Cimento **10**, 564 (1953); M. J. Dyson, M. Ross, E. E. Salpeter, S. S. Schweber, M. K. Sundaresan, W. M. Wirschera and H. A. Bethe, Phys. Rev. **95**, 1644 (1954)

[3] I. E. Tamm, J. Phys. USSR **9**, 445 (1945)

[4] S. M. Dancoff, Phys. Rev. **78**, 382 (1950)

[5] V. A. Fok, Phys. Z. Sowjetunion **6**, 425 (1934)

[6] M. Cini, Nuovo Cimento **10**, 526, 624 (1953)

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We shall call the variant of the method as suggested by Dyson the "new method" of terminated equations.

This article first considers several general problems, arising when the "new method" is used, and then applies this method to meson scattering by nucleons, employing a much closer approximation than used in the past. Section 2 contains a formulation of the general rules for obtaining the system of covariant equations by the new method of terminated equations. The essential difference is emphasized between the above formulation of the method and the so called Levy-Klein method[8], which, generally speaking, results in a divergent series in powers of the coupling constant for the unknown amplitude in the kernel. Section 3 discusses the transformation of the covariant equations into the three-dimensional momentum space. Section 4 is devoted principally to the selection of the correct boundary equations when solving various problems by the new method of terminated equations. It is shown that the boundary conditions can be formulated in the new method by using the connection between the amplitudes obtained by the old and new methods.

In Sec. 5, the new method is used to investigate the scattering of pi-mesons by nucleons. An approximation is discussed in which the contribution of the so-called "minus-particles" to the amplitude of the scattering is taken into account. Section 6 contains a report on the basic results of the analysis of the self-energy terms arising in the equations for the meson + nucleon system.

It is quite difficult to eliminate successively the divergent terms in the equation for the meson + nucleon system in the approximation considered in this article. First af all, two renormalized charges[9] result from the renormalization of the self-energy kernel; second, calculation of the finite additions due to these kernels leads to additional poles in these equations (see also reference 10); finally, unlike the solution for the interaction of two nucleons, the solution for the meson + nucleon system with $I = 1/2$ contains singularities, which must be eliminated by additional renormalization[11]. These problems are not considered in this article.

We have carried out numerical calculations for the case $I = 3/2$ without taking into account the self-energy terms, but taking into account the "minus particles",

2. GENERAL FORMULATION OF THE "NEW" METHOD

We shall employ the interaction representation. Let $\Psi(\sigma)$ be the state vector of the system under consideration on a space-like three-dimensional surface σ; let $\Psi_0(\sigma)$ be the corresponding physical-vacuum vector.

Let $H(x)$ be the Hamiltonian of the interaction at the four-dimensional point x lying on the surface σ. The wave functions satisfy the well-known equations:

$$i\frac{\delta\Psi(\sigma)}{\delta\sigma(x)} = H(x)\Psi(\sigma),\qquad -i\frac{\delta\Psi_0^*(\sigma)}{\delta\sigma(x)} = \Psi_0^*(\sigma)H(x). \tag{2.1}$$

Let $\psi_\alpha^\lambda(x)$ and $\overline{\psi}_\alpha^\lambda(x)$ be the nucleon-field operators in the interaction representation and let $\phi^s(x)$ be the meson-field operators. Here α is the spinor index ($\alpha = 1, 2, 3, 4$), λ the isotopic-spin index of the nucleon ($\lambda = 1, 2$), and s the isotopic-spin index of the meson ($s = 1, 2, 3$). We shall frequently drop these indices. Operators ψ, $\overline{\psi}$, and ϕ obey the usual commutation rules:

$$[\psi_\alpha^\lambda(x_1), \overline{\psi}_\beta^\mu(x_2)]_+ = -i\delta_{\lambda\mu}S_{\alpha\beta}(x_1 - x_2), \tag{2.2}$$

$$[\varphi^s(x_1), \varphi^{s'}(x_2)]_- = -i\delta_{ss'}\Delta(x_1 - x_2).$$

We shall term the quantities of the following type:

$$\begin{aligned}
&\langle\psi(x_1)\rangle_\sigma \equiv \Psi_0^*(\sigma)\psi(x_1)\Psi(\sigma),\\
&\langle\psi(x_1)\varphi(x_2)\rangle_\sigma \equiv \Psi_0^*(\sigma)\psi(x_1)\varphi(x_2)\Psi(\sigma),\\
&\dots\dots\dots\dots\dots\dots\\
&\langle\overline{\psi}(x_1)\dots\overline{\psi}(x_n)\psi(y_1)\dots\psi(y_m)\varphi(z_1)\dots\varphi(z_r)\rangle_\sigma \equiv\\
&\equiv \Psi_0^*(\sigma)N\overline{\psi}(x_1)\dots\overline{\psi}(x_n)\psi(y_1)\dots\psi(y_m)\varphi(z_1)\dots\varphi(z_r)\Psi(\sigma),
\end{aligned} \tag{2.3}$$

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the "$n + m + r$ partial amplitudes". The four-dimensional points x_1, x_2, etc. are quite arbitrary and do not have to lie on the surface σ. In Eqs. (2.3), N denotes an ordered operator product, that is, each field operator ψ, $\overline{\psi}$, or ϕ is represented in the form of a sum of the particle absorption and creation operators, while all absorption operators lie to the right of the creation operators in the N product. The following very simple examples show the difference between the N-ordered product and the conventional one:

$$\overline{\psi}^{\lambda}_{\alpha}(x_1)\,\psi^{\mu}_{\beta}(x_2) = N\overline{\psi}^{\lambda}_{\alpha}(x_1)\,\psi^{\mu}_{\beta}(x_2) - i\delta_{\lambda\mu}S^{(-)}_{\beta\alpha}(x_2 - x_1),$$

$$\varphi_s(x_1)\,\varphi_{s'}(x_2) = N\varphi_s(x_1)\,\varphi_{s'}(x_2) - i\delta_{ss'}\Delta^{(+)}(x_1 - x_2),$$

where the $\pm$ indices denote the positive- and negative-frequency portions of the permutation functions S and Δ. To transform analogously more complicated operator products it is necessary to employ the well-known theorem by Wick[12].

It is easy to obtain for Eq. (2.1) an equation for the amplitudes

$$i\frac{\delta}{\delta\sigma(x)}\langle\psi(x_1)\rangle_\sigma = \langle[\psi(x_1),\ H(x)]\rangle_\sigma, \tag{2.4}$$

and analogous equations for the multiple-particle amplit··ᵈes.

If the type of Hamiltonian is made more specific and if we assume

$$H(x) = ig\tau^{s}_{\lambda\mu}\overline{\psi}^{\lambda}(x)\,\gamma^5\,\psi^{\mu}(x)\,\varphi_s(x), \tag{2.5}$$

where $\tau^{s}_{\lambda\mu}$ is the matrix of the isotopic spin of the nucleon, and $\gamma^5 = -i\gamma^1\gamma^2\gamma^3\gamma^4 = \rho_1$ (we use the Feynman form of the Dirac matrix γ^k), we can transform Eq. (2.4) with the aid of Eq. (2.2) into:

$$i\frac{\delta}{\delta\sigma(x)}\langle\psi_\lambda(x_1)\rangle_\sigma = g\tau^{s}_{\lambda\mu}S(x_1 - x)\,\gamma^5\langle\psi^{\mu}(x)\,\varphi_s(x)\rangle_\sigma. \tag{2.6}$$

Analogously, we can also obtain equations for the other amplitudes. In general, several amplitudes appear in the right half. The infinite system of equations that generally results is a rigorous covariant field-equation system. It can be subdivided into independent sub-systems in accordance with the magnitude of the nuclear charge – the number of operators ψ and the number of adjoint operators $\overline{\psi}$ in each of the sub-systems is constant.

To obtain an approximate "terminated" system of equations, it is necessary to discard from the rigorous system of equations all amplitudes for a number of particles exceeding a certain number n_0. As a result we obtain a complete, and as can be shown also a common "terminated" system of equations for determining the ordered amplitudes, with the number of particles not exceeding the number n_0.

When performing the calculations, it is convenient to select the three-dimensional surfaces σ such that t is constant. Integrating the differential equations for the amplitudes of the type (2.4) over the three-dimensional space, we obtain equations of the type.

$$i\frac{\partial}{\partial t}\langle\psi(x_1)\rangle_t = \int d\mathbf{r}\,\langle[\psi(x_1),\ H(x)]\rangle_t, \tag{2.7}$$

where $x = (\mathbf{r}, t)$.

Many investigators[8,13] working with the so-called Tamm-Dancoff equations understand this term to mean one equation for one amplitude, corresponding to the actual particle system specified for the given problem (for example, nucleon + meson in scattering theory or two nucleons in deuteron theory), with this equation being obtained from the above-described system of coupled equations for n various amplitudes by approximate elimination of all amplitudes but one single one. In practice this elimination reduces to the fact that the kernel of the rigorous integral equation for the separated amplitude is expanded in powers of the interaction constant g and is terminated at a certain power of this constant. However, since it is known[8] that the expansion of the kernel in powers of g is generally divergent, such a simplification of the system of terminated equations may lead to incorrect results.

Let us note that approximately half of the

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amplitudes contained in these equations can always be eliminated. In fact, as follows from the above discussion, the derivatives of the amplitudes with odd number of particles are expressible in terms of amplitudes with even number of particles, and vice versa. Therefore, integration of equations of the type (2.7) with respect to time always permits expressing, say, all odd-particle amplitudes in terms of even-particle ones, thus completely eliminating the odd amplitudes from the system of equations.

3. TRANSFORMATION TO MOMENTUM REPRESENTATION

Let us choose the surface σ such that t is constant. Let us restrict our discussion to the stationary states of the physical systems. The state vector of such states has the following form in the interaction representation:

$$\Psi(t) = \exp\{i(H_0 - \mathcal{E})t\}\Psi'. \qquad (3.1)$$

The physical-vacuum vector has an analogous form

$$\Psi_0(t) = \exp\{i(H_0 - \mathcal{E}_0)t\}\Psi_0', \qquad (3.2)$$

where H_0 is the Hamiltonian of the free, non-interacting nucleon and meson fields, $\mathcal{E}$ is the energy of the given system, $\mathcal{E}_0$ is the energy of the physical vacuum, and finally Ψ' and Ψ_0' are constant vectors in the functional space.

Let us expand operators ψ, $\overline{\psi}$, and ϕ in a three-dimensional Fourier series, using the known equations (3.3)

$$\varphi_s(x) = \frac{1}{L^{3/2}}\sum_{\mathbf{k}}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}(Q_s(\mathbf{k})\,e^{-ikx} + Q_s^*(\mathbf{k})\,e^{ikx}),$$

$$\psi_\alpha^\lambda(x) = \frac{1}{L^{3/2}}\sum_{n=1}^{4}\sum_{p} e^{-ipx}B_n^\lambda(\mathbf{p})\;u_\alpha^n(\mathbf{p}),$$

with corresponding expressions for $\overline{\psi} = \psi^*\gamma_4$.

Here

$$kx = \omega_k t - \mathbf{kr}, \quad \omega_k = \sqrt{\mu^2 + \mathbf{k}^2}, \qquad (3.4)$$

Q_s and Q_s^* are respectively the absorption and emission operators for mesons of the s type, whereby

$$[Q_s(\mathbf{k}),\ Q_{s'}^*(\mathbf{k}')]_- = \delta_{ss'}\delta_{\mathbf{kk}'}, \qquad (3.5)$$

$u_\alpha^n(\mathbf{p})$ are four bi-spinors ($n = 1, 2, 3, 4$) satisfying the Dirac equation for the given $\mathbf{p}$ and satisfying the orthogonality conditions

$$\sum_\alpha u_\alpha^{n*}(\mathbf{p})\;u_\alpha^{n'}(\mathbf{p}) = \delta_{nn'}, \qquad (3.6)$$

with the indices $n = 1, 2$ corresponding to solutions with positive energy, and $n = 3, 4$ corresponding to solutions with negative energy. Furthermore,

$$px = p_0 t - \mathbf{pr}, \quad p_0 = \delta_n E_p, \qquad (3.7)$$

$$E_p = \sqrt{M^2 + \mathbf{p}^2},$$

where $\delta_n = +1$ for $n = 1, 2$ and $\delta_n = -1$ for $n = 3, 4$. Finally $B_1^\lambda(\mathbf{p})$ and $B_2^\lambda(\mathbf{p})$ are the absorption operators for nucleons of the λ type ($\lambda = 1$ for a proton and $\lambda = 2$ for a neutron) with momentum $\mathbf{p}$, while $B_3^\lambda(\mathbf{p})$ and $B_4^\lambda(\mathbf{p})$ are the emission operators for anti-nucleons with momentum-$\mathbf{p}$, whereby

$$[B_n^{\lambda*}(\mathbf{p}),\ B_{n'}^{\lambda'}(\mathbf{p}')]_+ = \delta_{\lambda\lambda'}\delta_{nn'}\delta_{\mathbf{pp}'}. \qquad (3.8)$$

With the aid of Eqs. (3.1) and (3.3) it is possible to express the time-dependent amplitudes of the $\langle\psi(x_1)\rangle$ type introduced in Sec. 2 in terms of the time-independent amplitudes of the type

$$\langle B_n^\lambda(\mathbf{p})\rangle \equiv \Psi_0'^*B_n^\lambda(\mathbf{p})\,\Psi'. \qquad (3.9)$$

Thus, for example, taking into account the usual commutation rule of the free-field Hamiltonian H_0 with the operators B_n^λ we obtain from Eqs. (2.7), (3.1) and (3.3)

$$\langle\psi(x_1)\rangle_t = \frac{1}{L^{3/2}}\sum_n\sum_{\mathbf{p}} u_\alpha^n(\mathbf{p}) \qquad (3.10)$$

$$\exp\{-ipx_1 + i(p_0 - W)t\}\,\langle B_n^\lambda(\mathbf{p})\rangle,$$

where $W = \mathcal{E} - \mathcal{E}_0$ is the difference between the energy $\mathcal{E}$ of the system under consideration and the energy $\mathcal{E}_0$ of physical vacuum. Thus W is equal to the observed energy of the system.

To change over to momentum representation it is also necessary to employ the well-known expressions for the permutation functions:

$$\Delta(x) = \frac{i}{(2\pi)^3}\int d^4k\,e^{-ikx}\delta(k^2 - \mu^2)\,\varepsilon(k_0), \qquad (3.11)$$

$$S(x) = \frac{i}{(2\pi)^3}\int d^4p\,e^{-ipx}\delta(p^2 - M^2)\,\varepsilon(p_0)\,(\hat{p} + M),$$

where $\hat{p} = p_\mu\gamma^\mu$ and

$$\varepsilon(k_0) = 1 \quad \text{при } k_0 > 0, \qquad (3.12)$$
$$\varepsilon(k_0) = -1 \quad \text{при } k_0 < 0.$$

With the aid of Eqs. (3.1), (3.3) and (3.11), it is possible to transform the system of covariant terminated equations in four-dimensional coordinate space, described in Sec. 2, into a system of equations for stationary amplitudes of the $\langle B^{\lambda}_{n}(\mathbf{p})\rangle$ type in three-dimensional momentum space. Thus, for example, if this transformation is carried out for Eq. (2.7), we obtain two equations, one for the quantity $\sum_{n=1}^{2} u^{n}_{\alpha}(\mathbf{p})\langle B^{\lambda}_{n}(\mathbf{p})\rangle$ and the other for $\sum_{n=3}^{4} u^{n}_{\alpha}(\mathbf{p})\langle B^{\lambda}_{n}(\mathbf{p})\rangle$. Using the orthogonality of the functions $u^{n}_{\alpha}(\mathbf{p})$ [Eq. (3.6)] we can reduce both equations into the following form

$$(W - \delta_n E_p)\langle B^{\lambda}_{n}(\mathbf{p})\rangle \qquad (3.13)$$
$$= \frac{ig}{2E_p}\tau^{s}_{\lambda\mu}\delta_n u^{n*}(\mathbf{p})(M + \hat{\mathrm{p}})\gamma^5$$
$$\times \int d\mathbf{q} \sum_{n'=1}^{4} u^{n'}(\mathbf{q})\frac{1}{\sqrt{2\omega_{\mathbf{p}-\mathbf{q}}}}$$
$$\times \{\langle B^{\mu}_{n'}(\mathbf{q})Q_s(\mathbf{p}-\mathbf{q})\rangle + \langle B^{\mu}_{n'}(\mathbf{q})Q^{*}_{s}(\mathbf{q}-\mathbf{p})\rangle\},$$

where $\hat{\mathrm{p}} = \delta_n E_n \gamma_4 - \vec{\mathbf{p}}\gamma$.

Let us remark that general transformations of the equations, particularly the elimination of a portion of the amplitudes from the system of equations and their renormalization, is best carried out in the covariant equations written in the coordinate space. On the other hand, elimination of angles and numerical solutions to the equations must be performed in the momentum representation.

4. "MINUS PARTICLES". BOUNDARY CONDITIONS

Let us return to the question of the meaning of the stationary amplitudes of the $\langle B(\mathbf{p})\rangle$ type, which we shall call the Dyson amplitudes.

In the old method of terminated equations, the state of the system Ψ' was characterized by the aggregate of amplitudes of the following type:

$$_0\langle B(\mathbf{p})\ldots Q(\mathbf{k})\ldots\rangle \qquad (4.1)$$
$$\equiv \Phi'^{*}_{0} B(\mathbf{p})\ldots Q(\mathbf{k})\ldots\Psi',$$

where Φ'_0 is the state vector of mathematical vacuum. These amplitudes have the sense of amplitudes of the probability that the state Ψ' has a prescribed number of particles with definite momenta. From the definition of mathematical vacuum it follows that all ordered amplitudes of the type of Eq. (4.1), in which the particle-emission operators are encountered, equal zero.

The Dyson amplitudes (3.9) differ from the old amplitudes (4.1) in that the state vector Φ'_0 of the mathematical vacuum is replaced in them by the state vector Ψ'_0 of the physical vacuum. Therefore, the Dyson amplitudes, generally speaking, also differ from zero in that case when they contain particle-emission operators. Thus, for example, the amplitude $\langle Q^*(\mathbf{k})\rangle = \Psi'^{*}_{0} Q^*(\mathbf{k})\Psi'$ describes the probability that the state Ψ' differs from physical vacuum by the fact that it is short one meson having a momentum $-\mathbf{k}$ contained in the state Ψ'_0, i.e., in vacuum. In this case we shall say that the state Ψ' contains one "minus-meson" (or correspondingly, a "minus nucleon" or a "minus-antinucleon"). Let us note the important fact that if the Dyson amplitudes are found for a given state, it is possible to determine from them both the "old" amplitudes of the (4.1) type, of state Ψ', as well as the vacuum amplitudes *

$$\Phi'^{*}_{0} B(\mathbf{p})\ldots Q(\mathbf{k})\ldots\Psi'_0. \qquad (4.2)$$

Thus the solution of any physical problem obtained by the new method is equivalent to the solution of the same problem by the old method and is simultaneously equivalent to the solution of the problem of determining the amplitudes (4.2) that characterize the state of physical vacuum.

The system of equations, satisfied by the Dyson amplitudes, has the following form:

$$(W - \delta_n E_p - \delta_{n'}E_{p'} - \cdots - \xi_k\omega_k) \qquad (4.3)$$
$$\langle B^{n}(\mathbf{p})B^{n'}(\mathbf{p}')\ldots Q^{\xi_k}(\mathbf{k})\rangle$$
$$= X^{nn'\ldots\xi_k}(\mathbf{p},\mathbf{p}',\ldots,\mathbf{k}),$$

* See reference 7, Eq. (26), from which we can determine the ratio of any two "old" amplitudes for both states Ψ' and Ψ'_0.

in which we introduce the notation $\xi_k = \pm 1$

$$Q^{\xi_k}(\mathbf{k}) = Q(\mathbf{k}), \quad \text{if} \quad \xi_k = +1, \tag{4.4}$$
$$Q^{\xi_k}(\mathbf{k}) = Q^*(\mathbf{k}), \quad \text{if} \quad \xi_k = -1,$$

and where $X(\mathbf{p}, \mathbf{p}', \ldots, \mathbf{k})$ are definite linear functions of the Dyson amplitudes.

If the factor in the left half of Eq. (4.3) does not vanish, this equation has a unique solution

$$\langle B^n(\mathbf{p}) B^{n'}(\mathbf{p}') \ldots Q^{\xi_k}(\mathbf{k})\rangle \tag{4.5}$$
$$= \frac{X^{nn'\ldots\xi_k}(\mathbf{p}, \mathbf{p}', \ldots, \mathbf{k})}{W - \delta_n E_p - \delta_{n'} E_{p'} - \ldots - \xi_k \omega_k}.$$

On the other hand, if the factor in the left half of Eq. (4.3) vanishes, then it is possible to add the quantity $\delta(W - \delta_n E_p - \delta_{n'} E_{p'} - \ldots - \xi_k \omega_k)$, multiplied by an arbitrary factor, to the right half of the solution (4.5), which for definiteness we shall always take in this case to be the principal value. To eliminate such indeterminacy it is necessary, as is customary, to take into account the boundary conditions, without which any physical problem is indeterminate. To formulate correct boundary conditions, we employ the connection between the Dyson amplitudes and the amplitudes of the old method. Namely, we employ Eq. (26) of reference 7:

$$a(N, N') \tag{D}$$
$$= \sum_M \beta^*(N+M)\,\alpha(N'+M)\,C(N, N'; M),$$

where a is the new-method amplitude, α is the old-method amplitude for the same state, β the vacuum-state amplitude, the C's are numerical coefficients, and N and N' denote respectively the number of plus and minus particles. Let us remark first of all that the vacuum amplitudes $\beta(N)$ should not contain delta-functions (if for no other reason than relativistic considerations of the invariance of the energy E_0 of the vacuum state after renormalization, which should be zero). This leads to the fact that in Eq. (D) delta-functions can occur only because of the amplitudes $\delta(N'+M)$ (since $E = W$ when $E_0 = 0$). In fact, in the case of the amplitudes $\alpha(N'+M)$ it is possible to obtain delta-functions in the form $\delta(W - E_{N'} - E_M)$. Consequently, when $N \neq 0$, $a(N, N')$ cannot contain $\delta(W + E_N - E_{N'})$, and when $N = 0$ it can contain $\delta(W - E_{N'})$. However, this appearance of delta-functions for amplitudes of states containing only plus particles is determined by the usual boundary conditions. On the other hand, if singularities of the type $(W + E_N - E_{N'})^{-1}$ occur for the amplitudes of states containing at least one minus particle , they must be considered in the sense of being principle values[7].

Let us make two remarks. First, when dealing with the stationary state, as we are doing at all times, we do not encounter any questions concerning the manner in which the interaction is "turned on". Second, in the method of terminated equations, the equation of a collision between a meson and a nucleon corresponds to the question of collisions in three-dimensional space between real particles (in the given approximation) rather than "bare" ones. To explain the latter, let us consider the amplitudes that are coupled in the system of terminated equations with the zero-approximation amplitudes

$$\langle B^n(\mathbf{p})\,Q(-\mathbf{p})\rangle = \delta(\mathbf{p} - \mathbf{p}_0)\,\chi_n, \tag{4.6}$$

where $\chi_n = 0$ when $n = 3, 4$), describing the motion of the "bare" non-interacting nucleons and meson in terms of their center-of-inertia coordinates. These amplitudes describe not only the collision of the meson with the nucleon, but also the accumulation of a cloud of virtual particles about the "bare" nucleon and "bare" meson. It must be emphasized that such an accumulation takes place also for a meson and nucleon that are infinitely remote from each other (in three-dimensional space, corresponding to momentum space). Thus, for example, substitution of (4.6) into the equation for the amplitude $\langle B(\mathbf{p})\,Q(\mathbf{k})\,Q(\mathbf{l})\rangle$ discloses that the expression for this amplitude contains a term proportional to $\delta(\mathbf{l} + \mathbf{p}_0)\,\delta(\mathbf{p} + \mathbf{k} - \mathbf{p}_0)$ and describing a state in which the primary "bare" electron moves with a momentum $-\mathbf{p}_0$, and the primary nucleon is dissociated into a nucleon $\mathbf{p}$ and a meson $\mathbf{k}$ with a total momentum $\mathbf{p}_0$. What is significant is that here the amplitude $\langle B(\mathbf{p})\,Q(\mathbf{k})\,Q(\mathbf{l})\rangle$ is expressed in terms of the amplitude $\langle B(\mathbf{p}) \times Q(-\mathbf{p})\rangle$ by an equation similar to Eq. (4.5), containing no delta-function whatever in its right half.

Having established the boundary conditions for the equations in momentum space, it is possible to formulate the corresponding conditions for the covariant equation in coordinate space. As we have seen in Sec. 3, equations of the (2.6) type transformed into momentum space have the following form:

$$i\frac{\partial}{\partial t}A(x_1, x_2, \ldots; t) = \int dr\, Y(x_1, x_2, \ldots; \mathbf{r}, t) = \int d\mathbf{r}\int d\nu f(x_1, x_2, \ldots, \mathbf{r}, \nu)e^{-i\nu t}, \tag{4.7}$$

where A denotes any amplitude of the (2.6) type. The solution of this equation is

$$A(x_1, x_2, \ldots, t) = \int d\mathbf{r}\int\frac{d\nu}{\nu} f(x_1, x_2, \ldots \mathbf{r}, \nu)e^{-i\nu t} + a(x_1, x_2, \ldots), \tag{4.8}$$

where $a(x_1, x_2, \ldots)$ is an arbitrary time-independent function, and the integral with respect to ν must be understood in the principal-value sense (ν vanishes within the integration region). It is convenient to rewrite Eq. (4.8) as follows:

$$A(x_1, x_2, \ldots; t) = {}^1\!/_2\int dx'\varepsilon(t-t')\, Y(x_1, x_2, \ldots; x') + a(x_1, x_2, \ldots), \tag{4.9}$$

where the integration is performed over the entire four-dimensional space. However, such a representation must be understood to imply the condition that the value of the integral for $t = \pm\infty$ must be dropped from the resultant expression. The function $a(x_1, x_2, \ldots)$ in momentum space corresponds to a delta-function describing plane and spherical waves with an energy W. From the boundary conditions formulated above for the momentum space, it follows that for all states in which there is at least one minus particle the function $a(x_1, x_2, \ldots)$ must be set equal to zero. For states containing only plus particles, this function must also be set equal to zero if the freqeuncy ν does not vanish within the integration region, otherwise the function a does vanish and should not be so selected that the function A corresponds to plane and outgoing plus-particle waves with a total energy W, corresponding to the conditions of the problem.

In particular, in the scattering of a meson by a nucleon at energies W that are insufficient for the formation of a second free meson, it follows from the above that the function vanishes for all amplitudes except for the state amplitude $< B(\mathbf{p})Q(-\mathbf{p}.)$ (one "bare" meson and one "bare" nucleon).

5. EQUATIONS FOR THE MESON + NUCLEON SYSTEM

Let us restrict ourselves to consideration of the states of a system having not more than three virtual particles. Accordingly, we write down equations similar to Eq. (2.6) for the principal amplitude of the problem $<\psi(x_1)\times\phi(x_2)>_t$ and for the amplitudes that couple with it

$$\langle\psi(x_1)\rangle_t, \quad \langle\psi(x_1)\varphi(x_2)\varphi(x_3)\rangle_t$$

and

$$\langle\overline{\psi}(x_1)\psi(x_2)\psi(x_3)\rangle_t.$$

(nucleon, nucleon + two mesons and nucleon + pair). The right halves of the equations for the latter two amplitudes contain the principal amplitude $<\psi(x_1)\phi(x_2)>_t$ and the four-dimensional amplitudes $<\overline{\psi}(x_1)\psi(x_2)\psi(x_3)\phi(x_4)_t$ and $<\psi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)>_t$; the latter will be omitted in our approximation.

Integrating the differential equations for the single-particle and three-particle amplitudes, we express them in accordance with Eq. (4.9)* in terms of their right halves, i.e., in terms of the amplitude $<\psi(x_1)\phi(x_2)>_t$. The resultant expressions for the single-particle and three-particle amplitudes are inserted into the right half of the equation for the amplitude $<\chi(x_1)\phi(x_2)>_t$ to obtain the final equation for this amplitude:

$$i\frac{\partial}{\partial t}\langle\psi^\lambda(x_1)\varphi_s(x_2)\rangle_t = \frac{1}{2}g^2\int dr dx'\varepsilon(t-t')\{N_1Y_1 + N_2Y_2 + Y_n + Y_m\}, \tag{5.1}$$

where the following notation is used: (5.2)

$$Y_1 = S(x_1 - x)\gamma^5(ZS(x - x')\Delta(x' - x_2))\gamma^5 \times \langle\psi^\mu(x')\varphi_{s'}(x)\rangle_{t'}$$

---

* Following the argument in Sec. 4, it is unnecessary to introduce into this equation additional terms of the type $a(x_1, \ldots)$ if the energy of the system is insufficient for emission of a second free meson.

$$-\Delta(x-x_2)(ZS(x_1-x')\gamma^5 S(x'-x))\gamma^5 \; \langle \psi^{\mu}(x)\varphi_{s'}(x')\rangle_{t'};$$
$$Y_2=(ZS(x_1-x)\Delta(x-x_2))\gamma^5 S(x-x')\gamma^5 \; \langle \psi^{\mu}(x')\varphi_{s'}(x')\rangle_{t'};$$
$$Y_n=3S(x_1-x)\gamma^5(ZS(x-x')\Delta(x'-x))\gamma^5 \; \langle \psi^{\lambda}(x')\varphi_s(x_2)\rangle_{t'};$$
$$Y_m=2\Delta(x-x_2)\,\mathrm{Tr}\,(ZS(x-x')\gamma^5 S(x'-x)\gamma^5) \; \langle \psi^{\lambda}(x_1)\varphi_s(x')\rangle_{t'}.$$

The symbol Z has the following meaning:

$$(ZS(x')\Delta(x''))=S^{(+)}(x')\Delta^{(-)}(x'') - S^{(-)}(x')\Delta^{(+)}(x'') \qquad (5.3)$$

with analogous consideration when $\Delta(x'')$ is replaced by $S(x'')$; the symbol Tr denotes the trace of the matrix relative to the spin indices. Finally, N_1 and N_2 are operators in the isotopic-spin space

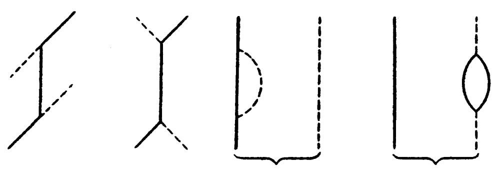
$$N_1=\tau^{s'}_{\lambda\nu}\tau^{s}_{\nu\mu},\quad N_2=\tau^{s}_{\lambda\nu}\tau^{s'}_{\nu\mu}. \qquad (5.4)$$

The eigenvalues of these operators are:

$$\begin{array}{lll} N_1=-1, & N_2=3, & \text{if } I={}^1/_2, \\ N_1=2, & N_2=0, & \text{if } I={}^3/_2, \end{array} \qquad (5.5)$$

where I denotes the total isotopic spin of the meson + nucleon system. From now on we shall no longer write down the isotopic indices λ and s, and will take the operators N_1 and N_2 to mean their eigenvalues (5.5).

Diagrams representing the kernel Y_i of Eq. (5.1) are given in the Figure.



Y_1 $\quad Y_2$ $\quad Y_n$ $\quad Y_m$

The first two kernels correspond to the scattering of a meson by a nucleon, whereby kernel Y_1 corresponds to scattering with the initial emission of a meson ("chain with emission") and Y_2 corresponds to the initial absorption of the meson ("chain with absorption"). These kernels are finite while the kernels Y_n and Y_m corresponding to the self energy of the nucleon and meson, respectively, are infinite and must be renormalized (as incidentally must be the finite kernel Y_2, see below). The renormalization of the self-energy terms will be considered separately; in this section we shall not consider these terms at all and will examine only the finite kernels Y_1 and Y_2. Let us note that Eq. (5.1) takes into account all the processes corresponding to all possible iterations of the diagrams in the Figure.

The transition from Eq. (5.1) to the momentum representation is carried out quite analogously with that performed in Sec. 3. We shall use a system of coordinates with the origin ot the center of inertia of the meson and nucleon. Let $\mathbf{p}_0$ be the momentum of the falling nucleon, and $-\mathbf{p}_0$ that of the falling meson, so that the energy of the system is

$$W=E_0+\omega_0=\sqrt{M^2+\mathbf{p}_0^2}+\sqrt{\mu^2+\mathbf{p}_0^2}. \qquad (5.6)$$

It is convenient to introduce the following designations for the amplitudes

$$b_{\alpha}^{\varepsilon\xi}(\mathbf{p})=\langle B^{n}(\mathbf{p})\,Q^{\xi p}(-\mathbf{p})\rangle, \qquad (5.7)$$

where ϵ, like ξ, assumes the values ± 1 ($\epsilon=+1$ for $n=1$, 2, i.e., for plus nucleons and $\epsilon=-1$ for $n=3$, 4, i.e., for minus antinucleons); the index $\alpha=1$, 2 differentiates among the possible (mechanical) spin directions of the nucleons and antinucleons. Let us also introduce the Pauli matrices σ acting on the spin index α, and finally also the designation

$$a^{\varepsilon\xi}(\mathbf{p})=\left(\frac{\vec{\sigma}\mathbf{p}}{p}\right)^{(1-\varepsilon)/2} b^{\varepsilon\xi}(\mathbf{p}) \qquad (5.8)$$

(we omit the spin indices). Using these designations, Eq. (5.1) (without the Y_n and Y_m terms) assumes the following form in the momentum representation:

$$(W-\varepsilon E-\xi\omega)\,a^{\varepsilon\xi}(\mathbf{p}) = \frac{\lambda}{4\pi}\sum_{\varepsilon'\xi'}\int d\mathbf{p}' R^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\mathbf{p}')\,a^{\varepsilon'\xi'}(\mathbf{p}'), \qquad (5.9)$$

$$R^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\mathbf{p}')=\varphi(p,p')\{N_1 S^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\mathbf{p}') + N_2 T^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\mathbf{p}')\}, \qquad (5.10)$$

where

$$\lambda = \frac{g^2}{8\pi^2}, \quad \varphi(p,\ p') = \frac{1}{2}\sqrt{\frac{(E+M)(E'+M)}{EE'\omega\omega'}}, \tag{5.11}$$

$$S^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\ \mathbf{p}') = \varepsilon\varepsilon'\xi\left(\frac{p}{E+M}\right)^{(1-\varepsilon)/2}\left(\frac{p'}{E'+M}\right)^{(1-\varepsilon')/2}[(W-M)\,m_{\varepsilon\varepsilon'}$$
$$+ n_{\xi\xi'}(\varepsilon E + \varepsilon' E' + \xi\omega + \xi'\omega' - M - W)] + \xi\left(\frac{E+M}{p}\right)^{(1-\varepsilon)/2}\left(\frac{E'+M}{p'}\right)^{(1-\varepsilon')/2}$$
$$\times \frac{\vec{\sigma}\mathbf{p}}{E+M}\,\frac{\vec{\sigma}\mathbf{p}'}{E'+M}[(W+M)\,m_{\varepsilon\varepsilon'} + (\varepsilon E + \varepsilon' E' + \xi\omega + \xi'\omega' + M - W)\,n_{\xi\xi'}],$$
$$T^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\ \mathbf{p}') = (\varepsilon + \xi)\left\{\frac{1}{W+M}\left(\frac{M-E}{p}\right)^{(1-\varepsilon)/2}\left(\frac{M-E'}{p'}\right)^{(1-\varepsilon')/2} + \right.$$
$$\left. + \frac{1}{W-M}\left(\frac{E+M}{p}\right)^{(1-\varepsilon)/2}\left(\frac{E'+M}{p'}\right)^{(1-\varepsilon')/2}\frac{\vec{\sigma}\mathbf{p}}{E+M}\,\frac{\vec{\sigma}\mathbf{p}'}{E'+M}\right\},$$
$$m_{\varepsilon\varepsilon'} = \{E_q(\varepsilon E_q + \varepsilon E + \varepsilon' E' - W)\}^{-1},$$
$$n_{\xi\xi'} = \{E_q(\xi E_q + \xi\omega + \xi'\omega' - W)\}^{-1}, \tag{5.12}$$

and E, E', ω, and ω' denote the energies of the nucleon and the meson with momentum $\mathbf{p}$ and momentum $\mathbf{p}'$, and $E_q = \sqrt{(\mathbf{p}+\mathbf{p}')^2 + M^2}$.

In accordance with what was said in Sec. 4, the asymptotic behavior of the function $a^{+,+}(\mathbf{p})$ in the scattering problems under consideration should correspond to the incident and outgoing waves. We accordingly put

$$a^{+,+}(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}_0) + f^{+,+}(\mathbf{p})\,\delta_+(E + \omega - W), \tag{5.13}$$

where, as usual, $\delta_+(x) = i\pi\delta(x) - (1/x)$ and for the remaining functions

$$a^{\varepsilon\xi}(\mathbf{p}) = \frac{1}{W - \varepsilon E - \xi\omega} f^{\varepsilon\xi}(\mathbf{p}). \tag{5.14}$$

Inserting Eqs. (5.13) and (5.14) into (5.9), we obtain

$$f^{\varepsilon\xi}(\mathbf{p}) = \frac{\lambda}{4\pi} R^{\varepsilon\xi}_{++}(\mathbf{p},\ \mathbf{p}_0) + i\frac{\lambda}{4}\int d\mathbf{p}' R^{\varepsilon\xi}_{++}(\mathbf{p},\mathbf{p}')\,f^{++}(\mathbf{p}')\,\delta(E' + \omega' - W) + \frac{\lambda}{4\pi}\sum_{\varepsilon'\xi'}\int d\mathbf{p}'\,\frac{R^{\varepsilon\xi}_{\varepsilon'\xi'}(\mathbf{p},\mathbf{p}')\,f^{\varepsilon'\xi'}(\mathbf{p}')}{W - \varepsilon' E' - \xi'\omega'}. \tag{5.15}$$

To separate the angular variables, we employ the system of invariant functions $L_l^{\pm}$ for the spin and angles, as discussed in reference 14:

$$L_l^+(\mathbf{n},\ \mathbf{n}') = (l+1)\,P_l(\cos\theta) - i\vec{\sigma}[\mathbf{n}\mathbf{n}']\,P_l^1(\cos\theta) \quad \text{for } j = l + {}^1/_2, \tag{5.16}$$
$$L_l^-(\mathbf{n},\ \mathbf{n}') = lP_l(\cos\theta) + i\vec{\sigma}[\mathbf{n}\mathbf{n}']\,P_l^1(\cos\theta) \quad \text{for } j = l - {}^1/_2.$$

Thanks to the transformation (5.8) introduced above, all that is needed to separate the angle variables is to expand the functions $f^{\epsilon\xi}$ and the kernels $R^{\epsilon\xi}_{\epsilon'\xi'}$ in polynomials of $L_l^{\pm}$. Introducing into Eq. (5.15) the expansion of the functions

$$f^{\varepsilon\xi}(\mathbf{p}) = \sum L_l^{\pm}\left(\frac{\mathbf{p}}{p},\frac{\mathbf{p}_0}{p_0}\right) f^{\varepsilon\xi}_{jl}(\mathbf{p}), \tag{5.17}$$

we obtain, after eliminating the angle and spin variables, a system of equations for the scattering amplitudes $f^{\epsilon\xi}_{jl}$, corresponding to the prescribed values of the total and orbital momentum, and depending only on the modulus of p:

$$f^{\varepsilon\xi}_{jl}(p) = \frac{\lambda}{4\pi}\,{}^{jl}R^{\varepsilon\xi}_{++}(p,\ p_0)\left\{1 + i\lambda\,\frac{4\pi^2 p_0 E_0\omega_0}{E_0 + \omega_0}\,f^{++}_{jl}(p_0)\right\} + \lambda\sum_{\varepsilon'\xi'}\int\frac{p'^2 dp'\ {}^{jl}R^{\varepsilon\xi}_{\varepsilon'\xi'}(p,\ p')}{W - \varepsilon' E' - \xi'\omega'}\,f^{\varepsilon'\xi'}_{jl}(p'). \tag{5.18}$$

Here the kernels ${}^{jl}R$ are related to the functions ${}^{jl}S$ and ${}^{jl}T$ by the earlier Eq. (5.10), whereby, unlike Eq. (5.11), we have

[14] I. E. Tamm, Iu. A. Gol'fand and V. Ia. Fainberg, J. Exper. Theoret. Phys. USSR 26, 649 (1954)

$$
\begin{aligned}
{}^{jl}S^{\varepsilon\xi}_{\varepsilon'\xi'} = \varepsilon\varepsilon'\xi \left(\frac{p}{E+M}\right)^{(1-\varepsilon)/2}\left(\frac{p'}{E'+M}\right)^{(1-\varepsilon')/2} & [\varepsilon(W-M)J_{k_1}(E+\varepsilon\varepsilon'E'-\varepsilon W) \\
& +\xi(\varepsilon E+\varepsilon'E'+\xi\omega+\xi'\omega'-M-W)J_{k_1}(\omega+\xi\xi'\omega'-\xi W)] \\
+\xi\left(\frac{p}{E+M}\right)^{(1+\varepsilon)/2}\left(\frac{p'}{E'+M}\right)^{(1+\varepsilon')/2} & [\varepsilon(W+M)J_{k_2}(E+\varepsilon\varepsilon'E'-\varepsilon W) \\
& +\xi(\varepsilon E+\varepsilon'E'+\xi\omega+\xi'\omega'+M-W)J_{k_2}(\omega+\xi\xi'\omega'-\xi W)].
\end{aligned} \tag{5.19}
$$

We do not need the expression for ${}^{jl}T$, and we will not write it down. Equation (5.19) employs the notation

$$
J_k(z) = \frac{1}{2}\int_{-1}^{+1}\frac{P_k(x)\,dx}{E_q(E_q+z)}, \tag{5.20}
$$

$$
E_q = \sqrt{M^2+p^2+p'^2+2pp'x},
$$

where $P_k(x)$ are the Legendre polynomials. For the state $S_{1/2}$ ($j = 1/2$, $l = 0$) it is necessary to put in (5.18) $k_1 = 0$ and $k_2 = 1$; for the $P_{1/2}$ state we have $k_1 = 1$ and $k_2 = 0$, and for the $P_{3/2}$ state we have $k_1 = 1$ and $k_2 = 2$.

Equations (5.18) for the functions f_{jl} contain imaginary coefficients. However, with the aid of the transformation

$$
U^{\varepsilon\xi}_{jl}(p) = f^{\varepsilon\xi}_{jl}(p)\left[1 + i\lambda\frac{4\pi^2 p_0\omega_0 E_0}{E_0+\omega_0}f^{++}_{jl}(p_0)\right]^{-1} \tag{5.21}
$$

it is possible to obtain an equation with real coefficients for the real functions $U^{\epsilon\xi}_{jl}$. These equations differ from Eq. (5.18) only by the substitution of U for f and by the absence of an imaginary component in the brackets.

It is easy to verify that the phase of the scattering of the pi-mesons by nucleons for the waves j, l and I is given by the equation

$$
\operatorname{tg}{}^{I}\delta_{jl} = -\frac{4\pi^2 p_0 E_0\omega_0}{E_0+\omega_0}{}^{I}U^{++}_{jl}(p_0). \tag{5.22}
$$

The numerical values of the coefficients N_1 and N_2 in Eq. (5.10), relating R with S and T, depend on the isotopic spin I of the state, as shown in Eq. (5.5).

Let us note that according to Eq. (5.5) the coefficient N_2 in the function T differs from zero only when $I = 1/2$, and moreover, the coefficients ${}^{jl}T$ themselves, obtained by expanding T in polynomials of $L^{\pm}_{l}$, differ from zero only for the $S_{1/2}$ and $P_{1/2}$ states. This is explained by the fact that the kernel T corresponds to a "chain with absorption" Y_2, i.e., in the intermediate state we have only one nucleon at rest in the center-of-inertia system, and consequently having spins $J = I = 1/2$.

Unlike the kernel S, the isotopic behavior of the kernel T at large momenta is such, that those of Eqs. (5.18) containing the kernel T (that is, the equations for ${}^{1/2}S_{1/2}$ and ${}^{1/2}P_{1/2}$) have no finite solutions. This is due to the fact that iteration of the diagrams Y_1 and Y_2, corresponding to the kernels S and T and therefore taken into account in our integral equations, leads to diagrams containing peaks and overlapped by singularities of the self-energy type. Thus the equations for the states ${}^{1/2}S_{1/2}$ and ${}^{1/2}P_{1/2}$ must be subjected, in addition to renormalization of the self-energy terms, to an additional renormalization that eliminates the above singularities.[11] Since we have not yet performed this additional renormalization, we are restricitng ourselves, like the authors of reference 2, to consideration of those Eqs. (5.18) that do not contain the kernel T. Let us note that all the equations considered in this section agree fully with the results of reference 2, provided we neglect all amplitudes and kernels for which ϵ and ξ differ from + 1.

When the energies W are not too large, only S and P waves play a considerable part in the scattering. At the present time we are engaged in solving numerically the system of Eq. (5.18) for the four states ${}^{3/2}S_{1/2}$, ${}^{3/2}P_{1/2}$, ${}^{3/2}P_{3/2}$ and ${}^{1/2}P_{3/2}$ at various energies (up to the energy corresponding to the kinetic energy of mesons in a laboratory system of the order of 300 mev). The results of the calculation will be published separately.

6. RESULTS OF RENORMALIZATION

Because of lack of space we cannot consider in detail the question of renormalization, and we shall therefore restrict ourselves to a brief statement of the results.

Inclusion of terms Y_n and Y_m in Eq. (5.1) leads to the appearance of additional "finite" (after renormalization) terms in the left halves of Eqs. (5.9), (5.15) and (5.18). As a result, for example, it is necessary to replace $f^{\epsilon\xi}$ in the left half of Eq. (5.18) by $\sum_{\varepsilon'\xi'}\Delta^{\varepsilon\xi}_{\varepsilon'\xi'}f^{\varepsilon'\xi'}$,

$$
\Delta^{\varepsilon\xi}_{\varepsilon'\xi'} = b^{\varepsilon\xi}_{\varepsilon'\xi'}(W-\varepsilon E-\xi\omega)^{-1}, \tag{6.1}
$$

where

$$b^{\varepsilon\xi}_{\varepsilon'\xi'} = \delta_{\varepsilon\varepsilon'}\delta_{\xi\xi'}\Big\{(W - \varepsilon E - \xi\omega)\,[1 + A(\xi)] - \varepsilon\frac{M}{E}B(\xi) + \xi C(\varepsilon)\Big\} + \delta_{\varepsilon\varepsilon'}\delta_{\xi,\,-\xi'}\xi C(\varepsilon) + \delta_{\xi\xi'}\delta_{\varepsilon,\,-\varepsilon'}B(\xi)\frac{p}{E}; \tag{6.2}$$

$$A(\xi) = \frac{3}{2}\lambda\int_0^1 dU(1-U)\left\{\ln\left|\frac{\mu^2(1-U)+M^2U^2}{\mu^2(1-U)+\nu^2(U-U^2)+UM^2}\right| + \frac{2M^2U^2}{\mu^2(1-U)+M^2U^2}\right\}; \tag{6.3}$$

$$B(\xi) = \frac{3}{2}\lambda\int_0^1 dU\,U\ln\left|\frac{\mu^2(1-U)+M^2U^2}{\mu^2(1-U)+\nu^2(U-U^2)+UM^2}\right|, \tag{6.4}$$

$$C(\varepsilon) = -4\lambda\int_0^1 du\,\Big\{[3\rho^2(U-U^2)+M^2]\ln\left|\frac{M^2+\rho^2(U-U^2)}{M^2-\mu^2(U-U^2)}\right| - \frac{(\rho^2+\mu^2)(U-U^2)[M^2-3\mu^2(U-U^2)]}{M^2-\mu^2(U-U^2)}\Big\}, \tag{6.5}$$

$$\nu^2 = p^2 - (W - \xi\omega)^2, \qquad \rho^2 = p^2 - (W - \varepsilon E).^2 \tag{6.6}$$

For large values of $\mathbf{p}$ and for sufficiently small values of λ, the determinant of the matrix $\Delta^{\epsilon\xi}_{\epsilon'\xi'}$ has the following form

$$\Delta \approx \left(1 - \frac{3}{4}\lambda\ln p\right)^2\left(1 - \frac{11}{4}\lambda\ln p\right)^2, \tag{6.7}$$

that is, it can vanish. Δ can also vanish for large values of λ. Here, as λ increases, the value p for which the determinant vanishes decreases. The vanishing of Δ is closely related to the results of the investigations described in reference 15, where it is shown that both in electrodynamics and in meson dynamics the solution of the approximate equations for the Green's function leads to the appearance in the corresponding functions of additional poles that have no direct physical meaning (see also reference 10). All this points out the limited applicability of the approximate equations.

In general, however, taking the higher approximations into account can lead to a radical change in the asymptotic behavior of Green's function, or in our case, to a corresponding substantial change in the behavior of Δ. Let us remark that if we limit ourselves to consideration of the functions U^{++} and U^{-+}, and if we also ignore the polarization of the meson vacuum, we obtain

$$\Delta = \left\{1 + A(+) - \frac{MB(+)}{(W-\omega)^2 - E^2}\right\}^2 + B^2(+)\frac{p^2 - (W-\omega)^2}{[(W-\omega)^2 - E^2]^2}. \tag{6.8}$$

This expression does not vanish if λ is not too large.

[15] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR **95**, 497, 773, 1177 (1954); **96**, 261 (1954); A. A. Abrikosov, A. D. Galanin and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR **97**, 793 (1954)

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