The Quantum Theory of Ferromagnetism

S. V. VONSOVSKII, K. B. VLASOV AND E. A. TUROV
Institute of Metal Physics, Ural Branch of the Academy of Sciences, USSR
(Submitted to JETP editor June 25, 1954)
J. Exper. Theoret. Phys. USSR 29, 37-50 (July, 1955)

On the basis of a many-electron quantum-mechanical model of a crystal, a calculation is made of the magnetic interaction of the electrons of a ferromagnetic. The energy spectrum of the system is calculated for the case of low temperatures. Taking account of the magnetic interaction terms in the original Hamiltonian of the system leads to the appearance in the energy both of components of quasi-classical magnetic type and of anisotropic (magnetic) exchange terms. The latter may have considerable importance, for example, in the calculation of relaxation phenomena in ferromagnetics and antiferromagnetics. Also calculated is the energy of the system in the approximation of energetic centers of gravity (the case of high temperatures). An expression is obtained for the free energy as a function of the magnitude of the magnetization and its orientation in the crystal (the magnetic anisotropy energy) for temperatures close to the Curie point.

1. INTRODUCTION

AS is well known (see, for example reference 1, Part III), the ferromagnetic state of matter is determined by electric (exchange) interactions, connected with the dependence of the electric energy on the spin state of the microsystem; this dependence is a consequence of the basic postulates of quantum mechanics. The magnetic interaction between the electrons of ferromagnetic bodies is several orders of magnitude smaller than the electric interaction. Nevertheless, a whole set of important phenomena in ferromagnetics – for example, their purely magnetic properties, their galvanomagnetic properties, relaxation phenomena, and so on – are basically distinguished by magnetic interaction, despite its relative weakness. A specific property of magnetic interaction is its anisotropic (tensorial) character. There exist quite a large variety of types of magnetic interaction: spin-spin interaction between electrons, and also between electrons and ions; spin-orbit interaction between electrons and ions; orbit-orbit interaction between electrons; etc. In a rigorous formulation of the problem, the calculation of magnetic interaction should be performed within the framework of relativistic quantum mechanics. However, the absence of a fully developed relativistic many-electron theory forces us to perform this calculation by the methods of nonrelativistic theory, in which we simply take into account the existence of the electron spin and, using the methods of the correspondence principle, write down the magnetic part of the Hamiltonian operator of a system of electrons.

The energy operator can be decomposed into two parts – an additive part, into which enter the kinetic energy terms and the interaction with external fields, and a binary part, which takes account of interactions of electrons in pairs. In a calculation of the eigenvalues of the energy operator of a system of electrons, along with quasiclassical (or quasinucleonic) terms, which take account of electric and magnetic interactions of "smeared" electrons, there are also obtained terms special to quantum mechanics. Here we can mention in particular the so-called exchange terms, transport energy, excitation energy, etc. It is important to realize that such separation of the energy into components is quite arbitrary; it is merely a result of approximate treatment of the problem of many interacting particles. None the less, in many cases such a separation is entirely permissible.

Until recently, in the treatment of magnetic interaction in problems of the quantum theory of a solid body, no notice has been taken of the fact that the contribution to the energy from this interaction is not exhausted by quasicoulomb terms or by exchange (magnetic) terms alone. In particular, Holstein and Primakoff[2] , for example, give a calculation only of the quasiclassical terms in the magnetic part of the energy of a ferromagnetic. In contrast, the work of Tiablikov[3] takes account only of the magnetic exchange interaction tensor and not of the quasiclassical terms in the energy. Meanwhile, in connection with the study of the phenomenon of ferromagnetic resonance and of

[1] S. V. Vonsovskii, *Contemporary Study in Magnetism*, Moscow, 1953

[2] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940)

[3] S. V. Tiablikov, J. Exper. Theoret. Phys. USSR **20**, 661 (1950)

relaxation phenomena in magnetic materials, there has recently arisen an urgent need for a more detailed and systematic investigation of magnetic interaction in these bodies. The present work sets as its aim the laying of a foundation for such an investigation. By starting from the most general possible form of the energy operator of a system of interacting electrons, there is accomplished a separation of the quasi-Coulomb and exchange parts both of the electric and of the magnetic interaction. Specially studied are the case of low temperatures (Section 3), where it is possible to apply the method of elementary excitations (quasiparticles), and the case of high temperatures (Section 4), where it is permissible to apply the method of the energetic center of gravity.

2. THE ENERGY OPERATOR IN THE SECOND-QUANTIZATION REPRESENTATION

We consider a system consisting of N similar, positively charged ions, located at the sites of a crystal lattice. We shall neglect the motion of the ions, treating them as immovable sources of a periodic potential field. In addition, the system contains N electrons, whose motion is taken into account. The energy operator of the system, in coordinate representation, is equal to the sum of an additive part (the kinetic energy and the interaction of the electrons with the external "fields")

$$\sum_{k=1}^{N} \hat{H}_1(\mathbf{q}_k, \nabla_{\mathbf{q}_k}, \hat{\mathbf{s}}_k) = -\frac{\hbar^2}{2m}\sum_{k=1}^{N}\Delta_{\mathbf{q}_k} + \sum_{k,\mathbf{n}=1}^{N} G_{\mathbf{n}}^{(l)}(\mathbf{q}_k) + \sum_{k,\mathbf{n}=1}^{N} G_{\mathbf{n}}^{(m)}(\mathbf{q}_k, \Delta_{\mathbf{q}_k}, \hat{\mathbf{s}}_k) \tag{2.1}$$

and of a binary part (interaction in pairs)

$$\sum_{k<l=1}^{N} \hat{\Phi}((\mathbf{q}_k - \mathbf{q}_l);\ \nabla_{\mathbf{q}_k}, \nabla_{\mathbf{q}_l};\ \hat{\mathbf{s}}_k, \hat{\mathbf{s}}_l). \tag{2.2}$$

Here $\mathbf{q}_k$ is the radius-vector of the kth electron; $2\pi\hbar$ is Planck's constant; m is the electron mass; $\frac{\hbar}{i}\nabla_{\mathbf{q}_k}$, $-\frac{\hbar}{2m}\Delta_{\mathbf{q}_k}$ and $\hat{\mathbf{s}}_k$ are respectively the momentum, kinetic energy, and spin operators of the kth electron; $G_{\mathbf{n}}^{(l)}(\mathbf{q}_k)$ and $G_{\mathbf{n}}^{(m)}(\mathbf{q}_k, \nabla_{\mathbf{q}_k}, \hat{\mathbf{s}}_k)$ are respectively the electrostatic and the magnetic (spin-orbit) interaction operators of the kth electron and the ion on site $\mathbf{n}$. Thus the energy operator of the system has the form

$$\hat{H} = \sum_{k=1}^{N} \hat{H}_1(\mathbf{q}_k, \nabla_{\mathbf{q}_k}, \hat{\mathbf{s}}_k) + \sum_{k<l=1}^{N} \hat{\Phi}((|\mathbf{q}_k - \mathbf{q}_l|);\ \nabla_{\mathbf{q}_k}, \nabla_{\mathbf{q}_l};\ \hat{\mathbf{s}}_k, \hat{\mathbf{s}}_l). \tag{2.3}$$

In the second-quantization representation, (2.3) is written in the form

$$\hat{H} = \sum_{\nu,\nu'} L(\nu,\nu')\hat{a}_\nu^+\hat{a}_{\nu'} + \frac{1}{2}\sum_{\nu_1\nu_2\nu_1'\nu_2'} F(\nu_1,\nu_2;\nu_1',\nu_2')\hat{a}_{\nu_1}^+\hat{a}_{\nu_2}^+\hat{a}_{\nu_2'}\hat{a}_{\nu_1'}, \tag{2.4}$$

where

$$L(\nu,\nu') = \int \theta_\nu^*(\mathbf{q},\mathbf{s})\hat{H}_1(\mathbf{q},\nabla_{\mathbf{q}},\hat{\mathbf{s}})\theta_{\nu'}(\mathbf{q},\mathbf{s})\,d\mathbf{q}d\mathbf{s}, \tag{2.5}$$

$$F(\nu_1,\nu_2;\nu_1',\nu_2') = \int \theta_{\nu_1}^*(\mathbf{q},\mathbf{s})\theta_{\nu_2}^*(\mathbf{q}',\mathbf{s}')\hat{\Phi}((\mathbf{q}-\mathbf{q}');\ \nabla_{\mathbf{q}},\nabla_{\mathbf{q}'};\hat{\mathbf{s}},\hat{\mathbf{s}}') \theta_{\nu_1'}(\mathbf{q},\mathbf{s})\theta_{\nu_2'}(\mathbf{q}',\mathbf{s}')\,d\mathbf{q}d\mathbf{q}'d\mathbf{s}d\mathbf{s}' \tag{2.6}$$

are matrix elements, defined by means of a complete system of orthonormal one-particle functions $\theta_\nu(\mathbf{q},\mathbf{s})$; the index ν stands for a whole set of quantum numbers, characterizing a given individual state of an electron. $\hat{a}_\nu^+$ and $\hat{a}_\nu$ are the Fermi operators (annihilation and creation operators) of second quantization; they act on functions of the occupation numbers N_ν of individual states. Below, these states will be numbered by means of the radius vector $\mathbf{n}$ of the site near which a given ion is located, together with the spin quantum number $\sigma = \pm\frac{1}{2}$ ($\nu \equiv \mathbf{n}, \sigma$). The wave functions $\theta_\nu(\mathbf{q},\mathbf{s})$ have the form of "atomic" (localized) orthonormal functions $\phi(\mathbf{q}_k - \mathbf{R}_\mathbf{n}) = \phi_\mathbf{n}(\mathbf{q}_k)$ ($\mathbf{R}_\mathbf{n}$ is the radius vector of the ion at site $\mathbf{n}$), multiplied* by the spin function $c_\sigma(\mathbf{s}_k)$:

$$\theta_{\mathbf{n},\sigma}(\mathbf{q},\mathbf{s}) = \varphi_\mathbf{n}(\mathbf{q})c_\sigma(\mathbf{s}). \tag{2.7}$$

Since no consideration will be given below to phenomena connected with transport of electric charge (electric conductivity and the like), it is possible to use a quasihomopolar approximation[4], which is tantamount to the requirement

$$\sum_\sigma \hat{a}_{\mathbf{n}\sigma}^+\hat{a}_{\mathbf{n}\sigma} = \sum_\sigma \hat{N}_{\mathbf{n}\sigma} = 1 \quad (\text{for all } \mathbf{n}), \tag{2.8}$$

* Here use is made of the smallness of the magnetic interaction in the atom as compared with the electric.

[4] N. N. Bogoliubov, *Lectures on Quantum Statistics*, Kiev, 1949 (in Ukrainian).

and also $\Delta = \infty$

where Δ is the energy of formation of a polar state (the energy gap for formation of an excess-deficiency doublet). The Fermi operators have the properties[4]

$$\hat{a}^{+}_{\mathfrak{n}\sigma} f(\ldots N_{\mathfrak{n}\sigma}\ldots) = \pm\delta(N_{\mathfrak{n}\sigma}) f(\ldots, N_{\mathfrak{n}\sigma}+1, \ldots), \quad \hat{a}_{\mathfrak{n}\sigma} f(\ldots N_{\mathfrak{n}\sigma}\ldots) = \pm\delta(1-N_{\mathfrak{n}\sigma}) f(\ldots, N_{\mathfrak{n}\sigma}-1, \ldots), \tag{2.9}$$

where $\delta(x)$ is the Kronecker symbol. By virtue of Eq. (2.9) it is proper to retain in the operator (2.4) only those terms that, acting on functions of the occupation numbers, produce no change in the total number of electrons near each lattice site, as is required by condition (2.8). This gives instead of Eq. (2.4)

$$\hat{H} = \sum_{\mathfrak{n};\sigma,\sigma'} L(\mathfrak{n},\sigma;\mathfrak{n},\sigma')\hat{a}^{+}_{\mathfrak{n}\sigma}\hat{a}_{\mathfrak{n}\sigma'} + \frac{1}{2}\sum_{\substack{\mathfrak{n}_1\neq\mathfrak{n}_2\\(\sigma_1,\sigma_2;\sigma_1',\sigma_2')}} F(\mathfrak{n}_1\sigma_1;\mathfrak{n}_2\sigma_2;\mathfrak{n}_1\sigma_1';\mathfrak{n}_2\sigma_2') \times \hat{a}^{+}_{\mathfrak{n}_1\sigma_1}\hat{a}^{+}_{\mathfrak{n}_2\sigma_2}\hat{a}_{\mathfrak{n}_2\sigma_2'}\hat{a}_{\mathfrak{n}_1\sigma_1'} + \frac{1}{2}\sum_{\substack{\mathfrak{n}_1\neq\mathfrak{n}_2\\(\sigma_1,\sigma_2;\sigma_1',\sigma_2')}} F(\mathfrak{n}_1,\sigma_1;\mathfrak{n}_2,\sigma_2;\mathfrak{n}_2\sigma_1';\mathfrak{n}_1\sigma_2')\hat{a}^{+}_{\mathfrak{n}_1\sigma_1}\hat{a}^{+}_{\mathfrak{n}_2\sigma_2}\hat{a}_{\mathfrak{n}_1\sigma_2'}\hat{a}_{\mathfrak{n}_2\sigma_1'}. \tag{2.10}$$

By virtue of Eqs. (2.1) and (2.2), the matrix elements (2.5) and (2.6) entering into Eq. (2.10) can be put into the form

$$\begin{aligned} &L(\mathfrak{n},\sigma;\mathfrak{n},\sigma') \\ &= \int \varphi^{*}_{\mathfrak{n}}(\mathbf{q})\left[-\frac{\hbar^2}{2m}\Delta_{\mathbf{q}} + G^{(l)}_{\mathfrak{n}}(\mathbf{q})\right]\varphi_{\mathfrak{n}}(\mathbf{q})\,d\mathbf{q}\,\delta_{\sigma\sigma'} \\ &+ \int \varphi^{*}_{\mathfrak{n}}(\mathbf{q})\Big[\sum_{\mathfrak{n}'(\neq\mathfrak{n})} G^{(l)}_{\mathfrak{n}'}(\mathbf{q})\Big]\varphi_{\mathfrak{n}}(\mathbf{q})\,d\mathbf{q}\,\delta_{\sigma\sigma'} \\ &+ \int \varphi^{*}_{\mathfrak{n}}(\mathbf{q})\,c^{*}_{\sigma}(\mathbf{s}) \\ &\times\Big[\sum_{\mathfrak{n}'} G^{(m)}_{\mathfrak{n}'}(\mathbf{q},\nabla_{\mathbf{q}},\hat{\mathbf{s}})\Big]\varphi_{\mathfrak{n}}(\mathbf{q})\,c_{\sigma'}(\mathbf{s})\,d\mathbf{q}\,d\mathbf{s} \\ &= (E_0 + C)\,\delta_{\sigma\sigma'} + L^{(m)}(\mathfrak{n}\sigma,\mathfrak{n}\sigma'), \end{aligned} \tag{2.11}$$

$$\begin{aligned} &F(\mathfrak{n}_1\sigma_1,\mathfrak{n}_2\sigma_2;\mathfrak{n}_1'\sigma_1',\mathfrak{n}_2'\sigma_2') \\ &= \int \varphi^{*}_{\mathfrak{n}_1}(\mathbf{q})\,c^{*}_{\sigma_1}(\mathbf{s})\,\varphi^{*}_{\mathfrak{n}_2}(\mathbf{q}')\,c^{*}_{\sigma_2}(\mathbf{s}') \\ &\times\hat{\Phi}\big((\mathbf{q}-\mathbf{q}');\nabla_{\mathbf{q}},\nabla_{\mathbf{q}'};\hat{\mathbf{s}},\hat{\mathbf{s}}'\big)\,\varphi_{\mathfrak{n}_1'}(\mathbf{q}) \\ &\times\ c_{\sigma_1'}(\mathbf{s})\,\varphi_{\mathfrak{n}_2'}(\mathbf{q}')\,c_{\sigma_2'}(\mathbf{s}')\,d\mathbf{q}\,d\mathbf{q}'\,d\mathbf{s}\,d\mathbf{s}'. \end{aligned} \tag{2.12}$$

In case the function $\phi_{\mathfrak{n}}(\mathbf{q})$ is exactly an atomic function, the quantity E_0 can be regarded as the atomic energy of the given state $\phi_{\mathfrak{n}}(\mathbf{q})$, and C can be regarded as a quasiclassical energy of electrostatic interaction of the electron of one of the sites with the ions of all the others. The quantity $L^{(m)}$ represents a perturbation of the neutral atomic level, due to spin-orbit interaction. If the functions $\phi_{\mathfrak{n}}(\mathbf{q})$ are not atomic functions, these quantities do not have so simple and graphic a meaning.

By taking account of Eq. (2.11), and also of the commutation relations

$$\hat{a}^{+}_{f}\hat{a}_{f'} + \hat{a}_{f'}\hat{a}^{+}_{f} = \delta_{ff'}; \quad \hat{a}^{+}_{f}\hat{a}^{+}_{f'} + \hat{a}^{+}_{f'}\hat{a}^{+}_{f} = \hat{a}_{f}\hat{a}_{f'} + \hat{a}_{f'}\hat{a}_{f} = 0; \tag{2.13}$$

and of the equations derivable from them[4]

$$\hat{a}^{+}_{\mathfrak{n}_1\sigma_1}\hat{a}^{+}_{\mathfrak{n}_2\sigma_2}\hat{a}_{\mathfrak{n}_2\sigma_2'}\hat{a}_{\mathfrak{n}_1\sigma_1'} = \hat{a}^{+}_{\mathfrak{n}_1\sigma_1}\hat{a}_{\mathfrak{n}_1\sigma_1'}\hat{a}^{+}_{\mathfrak{n}_2\sigma_2}\hat{a}_{\mathfrak{n}_2\sigma_2'}, \quad \hat{a}^{+}_{\mathfrak{n}_1\sigma_1}\hat{a}^{+}_{\mathfrak{n}_2\sigma_2}\hat{a}_{\mathfrak{n}_1\sigma_2'}\hat{a}_{\mathfrak{n}_2\sigma_1'} = -\hat{a}^{+}_{\mathfrak{n}_1\sigma_1}\hat{a}_{\mathfrak{n}_1\sigma_2'}\hat{a}^{+}_{\mathfrak{n}_2\sigma_1}\hat{a}_{\mathfrak{n}_2\sigma_1'},$$

Eq. (2.10) can be written in the form

$$\begin{aligned} \hat{H} &= N(E_0 + C) + \sum_{\mathfrak{n};\,\sigma,\,\sigma'} L^{(m)}(\mathfrak{n},\sigma;\mathfrak{n},\sigma')\,\hat{a}^{+}_{\mathfrak{n}\sigma}a_{\mathfrak{n}\sigma'} \\ &+ \frac{1}{2}\sum_{\substack{\mathfrak{n}_1\neq\mathfrak{n}_2\\(\sigma_1,\sigma_2;\sigma_1',\sigma_2')}} [F(\mathfrak{n}_1\sigma_1,\mathfrak{n}_2\sigma_2;\mathfrak{n}_1\sigma_1',\mathfrak{n}_2\sigma_2') \\ &- F(\mathfrak{n}_1\sigma_1,\mathfrak{n}_2\sigma_2;\mathfrak{n}_2\sigma_2',\mathfrak{n}_1\sigma_1')]\,\hat{a}^{+}_{\mathfrak{n}_1\sigma_1}\hat{a}_{\mathfrak{n}_1\sigma_1'}\hat{a}^{+}_{\mathfrak{n}_2\sigma_2}\hat{a}_{\mathfrak{n}_2\sigma_2'}. \end{aligned} \tag{2.14}$$

The matrix elements of the first term in the double sum (over $\mathfrak{n}_1$ and $\mathfrak{n}_2$) describe the quasiclassical electric and magnetic interaction of the electrons; those of the second term describe the exchange interaction (both electric and magnetic). The older of magnitude of these terms is the same; therefore it is not legitimate to consider them separately, as was done in references 2 and 3. The problem of the present work, as was mentioned above, is the systematic simultaneous calculation of these terms in the energy operator of the system.

3. THE ENERGETIC SPECTRUM OF A SYSTEM IN THE CASE OF MAGNETIC QUASI-SATURATION (THE LOW TEMPERATURE REGION)

The determination of the eigenvalues of the operator (2. 14) in the general case leads to insuperable mathematical difficulties. Therefore we must limit ourselves to the investigation of special cases, in which the operator (2.14) can be reduced to diagonal form by separating additive terms in it, corresponding to one or another elementary excitation. The eigenvalues of the energy operator of these elementary excitations (or quasiparticles)[5], having the form of the energy operators of oscillators of some "field", can be written down at once in explicit form, without solving the wave equation.

Such a computation program can be easily carried out if it turns out to be possible to introduce some "neutral" state of the system with respect to the given type of excitation, and to limit oneself thereafter to the consideration only of small departures of the motion of the particles of the system being studied from this "neutral background of motion". In the case being considered, that of a ferromagnetic, it is necessary to take as "neutral" state a state of spin saturation, and the small departures from it will have the form of elementary excitations or quasiparticles, which are called spin waves or ferromagnons. In a first approximation, these ferromagnons can be treated as an ideal gas, obeying symmetric quantum statistics. It must be emphasized that the ferromagnons represent collective degrees of freedom in the motion of a complex system of interacting electrons, located in an ionic crystal lattice. In this treatment no attention is paid to individual degrees of freedom of electrons. The diagonalization of the operator (2.14) is accomplished by application of successive unitary transformation operations. Before starting on this, it is necessary in Eq. (2.14) to perform an explicit summation over spin variables, and also to go over from the Fermi amplitudes $\hat{a}_{\mathbf{n}\sigma}$ to the Bose operator $\hat{\mathbf{b}}_{\mathbf{n}}$ in accordance with the known relations[4]

$$\hat{a}^{+}_{\mathbf{n},\,-1/2}\hat{a}_{\mathbf{n},\,1/2} = (1-\hat{b}^{+}_{\mathbf{n}}\hat{b}_{\mathbf{n}})^{1/2}\hat{b}_{\mathbf{n}},\quad \hat{a}^{+}_{\mathbf{n},\,1/2}\hat{a}_{\mathbf{n},\,-1/2} = \hat{b}^{+}_{\mathbf{n}}(1-\hat{b}^{+}_{\mathbf{n}}\hat{b}_{\mathbf{n}})^{1/2}, \tag{3.1}$$

$$\hat{a}^{+}_{\mathbf{n},\,-1/2}\hat{a}_{\mathbf{n},\,-1/2} = 1-\hat{b}^{+}_{\mathbf{n}}\hat{b}_{\mathbf{n}},\quad \hat{a}^{+}_{\mathbf{n},\,1/2}\hat{a}_{\mathbf{n},\,1/2} = \hat{b}^{+}_{\mathbf{n}}\hat{b}_{\mathbf{n}}.$$

The operators $\hat{\mathbf{b}}_{\mathbf{n}}$ satisfy the commutation relation

$$\hat{b}_{\mathbf{n}}\hat{b}^{+}_{\mathbf{n}'} - \hat{b}^{+}_{\mathbf{n}'}\hat{b}_{\mathbf{n}} = \delta_{\mathbf{n}\mathbf{n}'}. \tag{3.2}$$

If the system is in a state close to magnetic saturation (low temperatures), then in the operator (2.14), after the substitution (3.1), it is possible to expand the operator functions $(1-\hat{\mathbf{b}}^{+}_{\mathbf{n}}\hat{\mathbf{b}}_{\mathbf{n}})^{1/2}$ as in power series in $\hat{\mathbf{b}}^{+}_{\mathbf{n}}\hat{\mathbf{b}}$** and to keep only terms containing products of no more than four operators $\hat{\mathbf{b}}_{\mathbf{n}}$. Then instead of Eq. (2.14) we get

$$\begin{aligned}\hat{H} = N(E_0+C) &+ \sum_{\mathbf{n}}[L^{(m)}(\mathbf{n}^{+},\mathbf{n}^{+})\hat{b}^{+}_{\mathbf{n}}\hat{b}_{\mathbf{n}} + L^{(m)}(\mathbf{n}^{-},\mathbf{n}^{-})(1-\hat{b}^{+}_{\mathbf{n}}\hat{b}_{\mathbf{n}}) \\ &+ L^{(m)}(\mathbf{n}^{+},\mathbf{n}^{-})(\hat{b}^{+}_{\mathbf{n}} - \hat{b}^{+}_{\mathbf{n}}\hat{b}^{+}_{\mathbf{n}}\hat{b}_{\mathbf{n}}) + L^{(m)}(\mathbf{n}^{-},\mathbf{n}^{+})(\hat{b}_{\mathbf{n}} - b^{+}_{\mathbf{n}}\hat{b}_{\mathbf{n}}\hat{b}_{\mathbf{n}})] \\ &+ \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^{-},\mathbf{n}_2^{+};\mathbf{n}_1^{+},\mathbf{n}_2^{+}) - F(\mathbf{n}_1^{-},\mathbf{n}_2^{+};\mathbf{n}_2^{+},\mathbf{n}_1^{+})]\hat{b}_{\mathbf{n}_1}\hat{b}^{+}_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2} \\ &+ \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^{+},\mathbf{n}_2^{-};\mathbf{n}_1^{+},\mathbf{n}_2^{+}) - F(\mathbf{n}_1^{+},\mathbf{n}_2^{-};\mathbf{n}_2^{+},\mathbf{n}_1^{+})]\hat{b}^{+}_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_2} \\ &+ \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^{+},\mathbf{n}_2^{+};\mathbf{n}_1^{-},\mathbf{n}_2^{+}) - F(\mathbf{n}_1^{+},\mathbf{n}_2^{+};\mathbf{n}_2^{+},\mathbf{n}_1^{-})]\hat{b}^{+}_{\mathbf{n}_1}\hat{b}^{+}_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2} \\ &+ \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^{+},\mathbf{n}_2^{+};\mathbf{n}_1^{+},\mathbf{n}_2^{-}) - F(\mathbf{n}_1^{+},\mathbf{n}_2^{+};\mathbf{n}_2^{-},\mathbf{n}_1^{+})]\hat{b}^{+}_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}\hat{b}^{+}_{\mathbf{n}_2} \\ &+ \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^{-},\mathbf{n}_2^{-};\mathbf{n}_1^{+},\mathbf{n}_2^{+}) - F(\mathbf{n}_1^{-},\mathbf{n}_2^{-};\mathbf{n}_2^{+},\mathbf{n}_1^{+})]\hat{b}_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_2} \\ &+ \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^{+},\mathbf{n}_2^{+};\mathbf{n}_1^{-},\mathbf{n}_2^{-}) - F(\mathbf{n}_1^{+},\mathbf{n}_2^{+};\mathbf{n}_2^{-},\mathbf{n}_1^{-})]\hat{b}^{+}_{\mathbf{n}_1}\hat{b}^{+}_{\mathbf{n}_2}\end{aligned} \tag{3.3}$$

[5] S. V. Vonsovskii, *Collection Dedicated to the Memory of S. I. Vavilov*, Publishing House of the Academy of Sciences, USSR, 1952, p. 363

** As is shown by Kubo[6], in this case we have

$$(1-\hat{b}^{+}_{\mathbf{n}}\hat{b}_{\mathbf{n}})^{1/2} = 1-\hat{b}^{+}_{\mathbf{n}}\hat{b}_{\mathbf{n}}.$$

[6] R. Kubo, Phys. Rev. **87**, 568 (1952)

$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^+,\mathbf{n}_2^-;\mathbf{n}_1^+,\mathbf{n}_2^-)-F(\mathbf{n}_1^+,\mathbf{n}_2^-;\mathbf{n}_2^-,\mathbf{n}_1^+)]\,\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}$$
$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-,\mathbf{n}_2^+;\mathbf{n}_1^-,\mathbf{n}_2^+)-F(\mathbf{n}_1^-,\mathbf{n}_2^+;\mathbf{n}_2^+,\mathbf{n}_1^-)]\,\hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2}$$
$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^+,\mathbf{n}_2^-;\mathbf{n}_1^-,\mathbf{n}_2^+)-F(\mathbf{n}_1^+,\mathbf{n}_2^-;\mathbf{n}_2^+,\mathbf{n}_1^-)]\,\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_2}$$
$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-,\mathbf{n}_2^+;\mathbf{n}_1^+,\mathbf{n}_2^-)-F(\mathbf{n}_1^-,\mathbf{n}_2^+;\mathbf{n}_2^-,\mathbf{n}_1^+)]\,\hat{b}_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2}$$
$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-,\mathbf{n}_2^-;\mathbf{n}_1^-,\mathbf{n}_2^-)-F(\mathbf{n}_1^-,\mathbf{n}_2^-;\mathbf{n}_2^-,\mathbf{n}_1^-)]\,(1-\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}-\hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2})$$
$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^+,\mathbf{n}_2^-;\mathbf{n}_1^-,\mathbf{n}_2^-)$$
$$-F(\mathbf{n}_1^+,\mathbf{n}_2^-;\mathbf{n}_2^-,\mathbf{n}_1^-)]\,(\hat{b}^+_{\mathbf{n}_1}-\hat{b}^+_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}-\hat{b}^+_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2})$$
$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-,\mathbf{n}_2^+;\mathbf{n}_1^-,\mathbf{n}_2^-)$$
$$-F(\mathbf{n}_1^-,\mathbf{n}_2^+;\mathbf{n}_2^-,\mathbf{n}_1^-)]\,(\hat{b}^+_{\mathbf{n}_2}-\hat{b}^+_{\mathbf{n}_2}\hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2}-\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2})$$
$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-,\mathbf{n}_2^-;\mathbf{n}_1^+,\mathbf{n}_2^-)$$
$$-F(\mathbf{n}_1^-,\mathbf{n}_2^-;\mathbf{n}_2^-,\mathbf{n}_1^+)]\,(\hat{b}_{\mathbf{n}_1}-\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}-\hat{b}_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2})$$
$$+\frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-,\mathbf{n}_2^-;\mathbf{n}_1^-,\mathbf{n}_2^+)$$
$$-F(\mathbf{n}_1^-,\mathbf{n}_2^-;\mathbf{n}_2^+,\mathbf{n}_1^-)]\,(\hat{b}_{\mathbf{n}_2}-\hat{b}^+_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2}\hat{b}_{\mathbf{n}_2}-\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_2}).$$

Here an abbreviated form of notation has been adopted, for example

$$F(\mathbf{n}_1^+,\mathbf{n}_2^-;\mathbf{n}_1^-,\mathbf{n}_2^+)$$
$$=F(\mathbf{n}_1,{}^1/_2;\mathbf{n}_2,-{}^1/_2;\mathbf{n}_1,-{}^1/_2;\mathbf{n}_2,+{}^1/_2)$$

etc.

Among the terms in Eq. (3.3), besides terms quadratic in the Bose operators, there are terms in single operators, and also terms with triple products. It is easy to show that the linear terms give an additive "correction" to the energy of the system. To show this, we perform a unitary transformation from the operators $\hat{b}_{\mathbf{n}}$ and $\hat{b}^+_{\mathbf{n}}$ to their Fourier components by means of the formulas

$$\hat{b}_{\mathbf{n}}=N^{-1/2}\sum_{\nu}\hat{b}_{g_\nu}\exp\{i\mathbf{g}_\nu\mathbf{R}_{\mathbf{n}}\},\qquad(3.4)$$
$$\hat{b}^+_{\mathbf{n}}=N^{-1/2}\sum_{\nu}b^+_{g_\nu}\exp\{-i\mathbf{g}_\nu\mathbf{R}_{\mathbf{n}}\}.$$

Then for the linear terms in the additive part of Eq. (3.3), we have, for instance,

$$\sum_{\mathbf{n}}L^{(m)}(\mathbf{n}^+,\mathbf{n}^-)\,\hat{b}^+_{\mathbf{n}}$$
$$=N^{-1/2}L^{(m)}(0^+,0^-)\sum_{\nu}\hat{b}^+_{g_\nu}\sum_{\mathbf{n}}\exp\{-i\mathbf{g}_\nu\mathbf{R}_{\mathbf{n}}\}$$
$$=N^{-1/2}L^{(m)}(0^+,0^-)\sum_{\nu}\hat{b}^+_{g_\nu}\delta(\mathbf{g}_\nu)$$
$$=N^{-1/2}L^{(m)}(0^+,0^-)\,\hat{b}^+_0$$

(here the fact has been used that, with neglect of surface effects, the matrix elements $L^{(m)}(\mathbf{n}^+,\mathbf{n}^-)$ are independent of the lattice-site number $\mathbf{n}$). Similarly for the terms in the binary part of Eq. (3.3) we have, for instance,

$$\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[F(\mathbf{n}_1^-,\mathbf{n}_2^-;\mathbf{n}_1^+,\mathbf{n}_2^-)$$
$$-F(\mathbf{n}_1^-,\mathbf{n}_2^-;\mathbf{n}_2^-,\mathbf{n}_1^+)]\,\hat{b}_{\mathbf{n}_1}$$
$$=\frac{1}{2}\sum_{\mathbf{n}_1,\mathbf{n}_2}f(\mathbf{n}_1-\mathbf{n}_2)\sum_{\nu}\hat{b}_{g_\nu}\exp\{i\mathbf{g}_\nu\mathbf{R}_{\mathbf{n}}\};$$

we make the substitution of summation variables $\mathbf{n}=\mathbf{n}_1-\mathbf{n}_2$, which gives

$$\frac{1}{2}\sum_{\mathbf{n},\nu}f(\mathbf{n})\,\hat{b}_{g_\nu}\sum_{\mathbf{n}_1}\exp\{i\mathbf{g}_\nu\mathbf{R}_{\mathbf{n}}\}$$
$$=\frac{1}{2}\sum_{\mathbf{n},\nu}f(\mathbf{n})\,\hat{b}_{g_\nu}\delta(\mathbf{g}_\nu)=\frac{1}{2}\sum_{\mathbf{n}}f(\mathbf{n})\,\hat{b}_0.$$

Thus in Eq. (3.3), besides the additive part, there are quadratic and third-degree terms in the operators $\hat{b}$. On collecting similar terms and splitting Eq. (3.3) into two parts, we get

$$\hat{H} = \hat{H}_0 + \hat{H}', \tag{3.5}$$

where

$$\hat{H}_0 = U_0 + \sum_{\mathbf{n}} [L^{(m)}(0^+, 0^+) - L^{(m)}(0^-, 0^-)]\, \hat{b}^+_{\mathbf{n}} \hat{b}_{\mathbf{n}} \tag{3.6}$$

$$+ \frac{1}{2} \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^+) - F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^+, \mathbf{n}_1^+)]\, \hat{b}_{\mathbf{n}_1} \hat{b}_{\mathbf{n}_2}$$

$$+ \frac{1}{2} \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_1^-, \mathbf{n}_2^-) - F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_2^-, \mathbf{n}_1^-)]\, \hat{b}^+_{\mathbf{n}_1} \hat{b}^+_{\mathbf{n}_2}$$

$$+ \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^-] - F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^+) - F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^-) + F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^-)]\, \hat{b}^+_{\mathbf{n}_1} \hat{b}_{\mathbf{n}_1}$$

$$+ \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^+) - F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^+, \mathbf{n}_1^-)]\, \hat{b}^+_{\mathbf{n}_1} \hat{b}_{\mathbf{n}_2} + 2\mathfrak{M}\mathcal{H} \sum_{\mathbf{n}} \hat{b}^+_{\mathbf{n}} \hat{b}_{\mathbf{n}}.$$

Also taken into account here is the energy with respect to the external magnetic field H;

$$\mathfrak{M} = e\hbar / 2mc, \tag{3.7}$$

U_0 is the additive part of the energy that does not depend on the spin distribution, and

$$\hat{H}' = \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [L^{(m)}(0^+, 0^-)\delta_{\mathbf{n}_1\mathbf{n}_2} + F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^-) - F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^-)]\, \hat{b}^+_{\mathbf{n}_1} \hat{b}^+_{\mathbf{n}_1} \hat{b}_{\mathbf{n}_1} \tag{3.8}$$

$$- \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [L^{(m)}(0^-, 0^+)\, \delta_{\mathbf{n}_1\mathbf{n}_2} + F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^-) - F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^+)]\, \hat{b}^+_{\mathbf{n}_1} \hat{b}_{\mathbf{n}_1} \hat{b}_{\mathbf{n}_1}$$

$$+ \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_1^+, \mathbf{n}_2^-) - F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_2^-, \mathbf{n}_1^+) - F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_1^-, \mathbf{n}_2^-) + F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_1^-, \mathbf{n}_2^-)]\, \hat{b}^+_{\mathbf{n}_1} \hat{b}^+_{\mathbf{n}_2} \hat{b}_{\mathbf{n}_1}$$

$$+ \sum_{\mathbf{n}_1 \neq \mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_1^+, \mathbf{n}_2^+) - F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_1^+, \mathbf{n}_2^+) - F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_2^+, \mathbf{n}_1^+) - F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^+) + F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^+, \mathbf{n}_1^+)]\, \hat{b}_{\mathbf{n}_1} \hat{b}^+_{\mathbf{n}_2} \hat{b}_{\mathbf{n}_2}.$$

From Eq. (2.12) it is evident that

$$F(\mathbf{n}_1\sigma_1, \mathbf{n}_2\sigma_2; \mathbf{n}_1\sigma_1', \mathbf{n}_2\sigma_2') = F[0\sigma_1, (\mathbf{n}_2 - \mathbf{n}_1)\sigma_2; 0\sigma_1', (\mathbf{n}_2 - \mathbf{n}_1)\sigma_2'],$$

$$F(\mathbf{n}_1\sigma_1, \mathbf{n}_2\sigma_2; \mathbf{n}_2\sigma_2', \mathbf{n}_1\sigma_1') = F[0\sigma_1, (\mathbf{n}_2 - \mathbf{n}_1)\sigma_2; (\mathbf{n}_2 - \mathbf{n}_1)\sigma_2', 0\sigma_1'],$$

that is, the matrix elements of F depend only on the distance between lattice sites, $\mathbf{R}_{\mathbf{n}_2} - \mathbf{R}_{\mathbf{n}_1}$. From the requirement that the operator (2.2) be Hermitian, it follows further that in Eq. (3.6) the coefficients of $\hat{b}_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_2}$ and $\hat{b}^+_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2}$, and in Eq. (3.8) the coefficients of $\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_2}$ and $\hat{b}_{\mathbf{n}_1}\hat{b}^+_{\mathbf{n}_1}\hat{b}_{\mathbf{n}_2}$, are complex conjugates.

Using these properties of the matrix elements, we perform the Fourier transformation (3.4) in Eqs. (3.6) and (3.8):

$$\hat{H}_0 = U_0 + \sum_{\nu} [A_0 + A_1(\mathbf{g}_\nu)]\, \hat{b}^+_{\mathbf{g}_\nu} \hat{b}_{\mathbf{g}_\nu} + \frac{1}{2} \sum_{\nu} B(\mathbf{g}_\nu)\, \hat{b}_{-\mathbf{g}_\nu} \hat{b}_{\mathbf{g}_\nu} + \frac{1}{2} \sum_{\nu} B^*(\mathbf{g}_\nu)\, \hat{b}^+_{-\mathbf{g}_\nu} \hat{b}^+_{\mathbf{g}_\nu}, \tag{3.9}$$

$$\hat{H}' = \sum_{\nu,\nu'} [C_0 + C_1(\mathbf{g}_\nu)]\, \hat{b}^+_{\mathbf{g}_\nu} \hat{b}_{\mathbf{g}_\nu + \mathbf{g}_{\nu'}} \hat{b}^+_{\mathbf{g}_{\nu'}} + \sum_{\nu,\nu'} [C_0^* + C_1^*(\mathbf{g}_\nu)]\, \hat{b}^+_{\mathbf{g}_\nu} \hat{b}_{\mathbf{g}_\nu - \mathbf{g}_{\nu'}} \hat{b}_{\mathbf{g}_{\nu'}}, \tag{3.10}$$

where the following symbols have been introduced:

$$A_0 = L^{(m)}(0^+, 0^+) - L^{(m)}(0^-, 0^-) + 2\mathfrak{M}\mathcal{H} + \sum_{\mathbf{n}} [F(0^+, \mathbf{n}^-; 0^+, \mathbf{n}^-) - F(0^+, \mathbf{n}^-; \mathbf{n}^-, 0^+) - F(0^-, \mathbf{n}^-; 0^-, \mathbf{n}^-) + F(0^-, \mathbf{n}^-; \mathbf{n}^-, 0^-)]; \tag{3.11}$$

$$A_1(\mathbf{g}_\nu) = \sum_{\mathbf{n}} [F(0^+, \mathbf{n}^-; 0^-, \mathbf{n}^+) - F(0^+, \mathbf{n}^-; \mathbf{n}^+, 0^-)] \exp\{i\mathbf{g}_\nu \mathbf{R}_{\mathbf{n}}\};$$

$$B(\mathbf{g}_\nu) = \sum_{\mathbf{n}} [F(0^-, \mathbf{n}^-; 0^+, \mathbf{n}^+) - F(0^-, \mathbf{n}^-; \mathbf{n}^+, 0^+)] \exp\{i\mathbf{g}_\nu \mathbf{R}_\mathbf{n}\};$$

$$C_0 = -N^{-1/2} \sum_{\mathbf{n}} [L^{(m)}(0^+, 0^-) + F(0^+, \mathbf{n}^-; 0^-, \mathbf{n}^-) - F(0^+, \mathbf{n}^-; \mathbf{n}^-, 0^-)];$$

$$C_1(\mathbf{g}_\nu) = N^{-1/2} \sum_{\mathbf{n}} [F(0^+, \mathbf{n}^+; 0^+, \mathbf{n}^-) - F(0^+, \mathbf{n}^+; \mathbf{n}^-, 0^+) - F(0^-, \mathbf{n}^+; 0^-, \mathbf{n}^-) + F(0^-, \mathbf{n}^+; \mathbf{n}^-, 0^-)] \exp\{-i\mathbf{g}_\nu \mathbf{R}_\mathbf{n}\},$$

and $B^*(\mathbf{g})$ and $C^*(\mathbf{g})$ are the complex conjugates of the coefficients $B(\mathbf{g})$ and $C(\mathbf{g})$ respectively. The operator $\hat{H}_0$ describes the energy of elementary excitations of the system near magnetic spin saturation, and the operator $\hat{H}'$ may be regarded as a perturbation energy, which takes account in kinetic phenomena of the "collision" processes between quasiparticles-ferromagnons.

To reduce the operator $\hat{H}_0$ to diagonal form, it is necessary[2,4] to perform the double unitary transformation

$$\hat{b}_\mathbf{g} = 2^{-1/2} e^{i\varphi_\mathbf{g}} (\hat{d}_\mathbf{g} + \hat{d}_{-\mathbf{g}}), \qquad b_{-\mathbf{g}} = 2^{-1/2} e^{i\varphi_{-\mathbf{g}}} (\hat{d}_\mathbf{g} - \hat{d}_{-\mathbf{g}}), \tag{3.12}$$

$$\hat{d}_\mathbf{g} = l^+ \hat{c}_\mathbf{g} + l^- \hat{c}^+_\mathbf{g}, \quad d_{-\mathbf{g}} = l^+ \hat{c}_{-\mathbf{g}} - l^- \hat{c}^+_{-\mathbf{g}},$$

where

$$l^\pm = \left[\frac{1}{2} \frac{A_\mathbf{g} \pm (A^2_\mathbf{g} - |B(\mathbf{g})|^2)^{1/2}}{(A^2_\mathbf{g} - |B(\mathbf{g})|^2)^{1/2}}\right]^{1/2}, \qquad A_\mathbf{g} = A_0 + A_1(\mathbf{g}). \tag{3.13}$$

As a result of laborious but elementary transformations, with use of the commutation relations (3.2), we find

$$\hat{H}_0 = U_0 + \sum_{\mathbf{g}_\nu} [(A^2_{\mathbf{g}_\nu} - |B(\mathbf{g}_\nu)|^2)^{1/2} \hat{c}^+_{\mathbf{g}_\nu} \hat{c}_{\mathbf{g}_\nu} + \tfrac{1}{2}(A^2_{\mathbf{g}_\nu} - |B(\mathbf{g}_\nu)|^2)^{1/2} - \tfrac{1}{2} A_{\mathbf{g}_\nu}], \tag{3.14}$$

where $\hat{c}^+_{\mathbf{g}_\nu} \hat{c}_{\mathbf{g}_\nu} = \hat{N}_{\mathbf{g}_\nu}$ is the ferromagnon number operator in the state with quasimomentum $\mathbf{g}_\nu$; it has the eigenvalues 0,1,2,3,... The expression (3.14) represents the energy levels of the system of interacting electrons in the state of quasisaturation in the first approximation. The operator $\hat{H}'$, as was shown above, can be regarded as a small perturbation; it determines interactions between ferromagnons in the form of collisions between them, leading to a change of the quasimomenta of these quasiparticles or to their creation and annihilation. It is important to note that the result obtained is general for an arbitrary concrete form of the interaction between ferromagnetic electrons. Therefore this result can serve our original purpose, for example, the study of the phenomenon of magnetic anisotropy, of the dependence of spontaneous magnetization on the external field, of relaxation processes, etc.

As an example we consider the case when electric and magnetic spin-spin interaction is present. In this case the binary-interaction operator (2.2) has the form

$$\hat{\Phi}((\mathbf{q} - \mathbf{q}'); \nabla_\mathbf{q}; \nabla'_\mathbf{q}; \hat{\mathbf{s}}, \hat{\mathbf{s}}') = V(|\mathbf{q} - \mathbf{q}'|) + \frac{e^2}{m^2c^2} |\mathbf{q} - \mathbf{q}'|^{-5} \times [|\mathbf{q} - \mathbf{q}'|^2 (\hat{\mathbf{s}}\hat{\mathbf{s}}') - 3(\hat{\mathbf{s}}(\mathbf{q} - \mathbf{q}'))(\hat{\mathbf{s}}'(\mathbf{q} - \mathbf{q}'))], \tag{3.15}$$

and the matrix elements (2.12) will be expressed by the integrals

$$C_\mathbf{n} = \int V(|\mathbf{q} - \mathbf{q}'|) |\varphi(\mathbf{q})|^2 |\varphi(\mathbf{q}' - \mathbf{R}_\mathbf{n})|^2 \, d\mathbf{q}\, d\mathbf{q}'; \tag{3.16}$$

$$E_\mathbf{n} = \beta \int |\varphi(\mathbf{q})|^2 |\varphi(\mathbf{q}' - \mathbf{R}_\mathbf{n})|^2 |\mathbf{q} - \mathbf{q}'|^{-3} \left[1 - 3\frac{(\mathbf{q} - \mathbf{q}')_z^2}{|\mathbf{q} - \mathbf{q}'|^2}\right] d\mathbf{q}\, d\mathbf{q}';$$

$$E^{(i,l)}_\mathbf{n} = 3\beta \int |\varphi(\mathbf{q})|^2 |\varphi(\mathbf{q}' - \mathbf{R}_\mathbf{n})|^2 |\mathbf{q} - \mathbf{q}'|^{-5} (\mathbf{q} - \mathbf{q}')_i (\mathbf{q} - \mathbf{q}')_l \, d\mathbf{q}\, d\mathbf{q}';$$

$$I_\mathbf{n} = \int \varphi^*(\mathbf{q}) \varphi^*(\mathbf{q}' - \mathbf{R}_\mathbf{n}) V(|\mathbf{q} - \mathbf{q}'|) \varphi(\mathbf{q}') \varphi(\mathbf{q} - \mathbf{R}_\mathbf{n}) \, d\mathbf{q}\, d\mathbf{q}';$$

$$J_\mathbf{n} = \beta \int \varphi^*(\mathbf{q}) \varphi^*(\mathbf{q}' - \mathbf{R}_\mathbf{n}) |\mathbf{q} - \mathbf{q}'|^{-3} \left[1 - 3\frac{(\mathbf{q} - \mathbf{q}')_z^2}{|\mathbf{q} - \mathbf{q}'|^2}\right] \varphi(\mathbf{q}') \varphi(\mathbf{q} - \mathbf{R}_\mathbf{n}) \, d\mathbf{q}\, d\mathbf{q}';$$

$$J^{(i,l)}_\mathbf{n} = 3\beta \int \varphi^*(\mathbf{q}) \varphi^*(\mathbf{q}' - \mathbf{R}_\mathbf{n}) |\mathbf{q} - \mathbf{q}'|^{-5} (\mathbf{q} - \mathbf{q}')_i (\mathbf{q} - \mathbf{q}')_l \times \varphi(\mathbf{q}') \varphi(\mathbf{q} - \mathbf{R}_\mathbf{n}) \, d\mathbf{q}\, d\mathbf{q}'$$

$$(i, l = x, y, z), \quad \beta = e^2\hbar^2 / 4m^2c^2 = \mathfrak{M}^2.$$

Using formulas (2.12), (3.15), (3.11), and (3.16), we get

$$A_g = A'_g + A''_g, \tag{3.17}$$

where

$$A'_g = -3\sum_n E_n + \sum_n E_n(1 - e^{igR_n}) + 2\mathfrak{M}\mathcal{H}; \tag{3.18}$$

$$A''_g = \sum_n I_n(1 - e^{igR_n}) + \sum_n J_n(2 + e^{igR_n}); \tag{3.19}$$

$$B_g = B'_g + B''_g;$$

here

$$B'_g = -\sum_n [E_n^{xx} - E_n^{yy} + 2iE_n^{xy}]\, e^{igR_n}; \tag{3.20}$$

$$B''_g = \sum_n [J_n^{xx} - J_n^{yy} + 2iJ_n^{xy}]\, e^{igR_n}; \tag{3.21}$$

$$C_g = C'_g + C''_g,$$

where

$$C'_g = -\frac{2}{\sqrt{N}}\sum_n [E_n^{xz} + iE_n^{yz}]\, e^{igR_n},$$

$$C''_g = \frac{2}{\sqrt{N}}\sum_n [J_n^{xz} + iJ_n^{yz}]\, e^{igR_n}. \tag{3.22}$$

Substituting (3.17) and (3.19) in (3.14), we find the energy of the system of ferromagnons with account taken of magnetic spin-spin interaction. In external form, Eq. (3.14) agrees with the corresponding expression in reference 2. However, there is here an essential difference in the form of the quantities A_g and B_g. The first terms in these quantities [cf. (3.17) and (3.19)] agree exactly with the coefficients A_g and B_g of the cited work, if one treats these terms quasiclassically, considering that the electrons are "bound" to their own lattice sites. These coefficients are expressed by the integrals E_n and $E_n^{(i,l)}$ in Eq. (3.16), which decrease comparatively slowly with the distance between sites, R_n (Coulomb interaction). By virtue of this last fact, in the calculation of the coefficients A'_g and B'_g the summation in the right members of the first lines of formulas (3.18) and (3.20) must be extended over the whole volume of the crystal (as was done in reference 2). Therefore in the specified case of long-range interactions, it is not permissible to limit oneself to the nearest-neighbor approximation, However, in Eqs. (3.17) and (3.19) there also enter the terms A''_g and B''_g, which – with the exception of a term representing the electrostatic exchange interaction, $\sum_n I_n(1 - e^{igR_n})$ – were not taken into account in any way in reference 2. The terms with the integrals J_n and $J_n^{(i,l)}$ represent the anisotropic magnetic exchange interaction of the electrons. They may play an essential role in the consideration, for instance, of the phenomenon of magnetic anisotropy – as is clearly evident from the work of Tiablikov [3]. We mention only that the integrals decrease rapidly with the distance between sites, and therefore in the calculation one can limit oneself to the consideration solely of integrals for nearest-neighbor sites. This fact was utilized in reference 3. However, there no account at all was taken of terms of quasiclassical type, for which the nearest-neighbor approximation is not legitimate.

The nondiagonal part of the energy operator (3.10) offers the possibility, in a more general and complete form than in the work of Akhiezer [7], of studying the process of collision both between ferromagnons and between ferromagnons and phonons; for here also collisions are taken into account that are determined by magnetic interactions of the exchange type (cf. the terms containing C''_g). In the series of researches reviewed in reference 8 the hypothesis was advanced that this same magnetic anisotropy exchange interaction (sometimes called pseudodipole interaction) was also basically responsible for relaxation processes in ferromagnetics. As far as we know, the literature contains no deduction of this interaction from the general equations of the problem of interacting electrons, such as has been obtained in the present work.

4. THE ENERGY OF A SYSTEM NEAR THE CURIE TEMPERATURE

In the region of high temperatures the method of quasiparticles, at any rate in the form in which it was applied in Section 3, is no longer suitable for calculation of the energy of the system under consideration – a system of interacting electrons in a ferromagnetic. In this case, however, it is possible to get some approximate information about the energy spectrum of the system by using the method of the energetic center of gravity, i.e., by calculation of mean values of energy. It is this procedure that is essentially involved in the familiar method of the molecular field in the theory of the ferromagnetism.

To find the energetic center of gravity, we separate the diagonal terms in formula (2.14) ($\sigma_1 = \sigma'_1$, $\sigma_2 = \sigma'_2$) and replace the operators

---

[7] A. I. Akhiezer, J. Phys. (USSR) **10**, 217 (1946)

[8] C. Kittel and E. Abrahams, Revs. Mod.Phys. **25**, 233 (1953)

$\hat{a}^+_{\mathbf{n}\sigma}\hat{a}_{\mathbf{n}\sigma} = \hat{N}_{\mathbf{n}\sigma}$ by their mean values

$$\langle \hat{a}^+_{\mathbf{n},\,1/2}\,\hat{a}_{\mathbf{n},\,1/2}\rangle_{\text{cp}} = \langle \hat{N}_{\mathbf{n},\,1/2}\rangle_{\text{cp}} = N^{-1}\sum_{\mathbf{n}} N_{\mathbf{n},\,1/2} = 1/2\,(1-m), \tag{4.1}$$

$$\langle \hat{a}^+_{\mathbf{n},\,-1/2}\,\hat{a}_{\mathbf{n},\,-1/2}\rangle_{\text{cp}} = \langle \hat{N}_{\mathbf{n},\,-1/2}\rangle_{\text{cp}} = N^{-1}\sum_{\mathbf{n}} N_{\mathbf{n},\,-1/2} = 1/2\,(1+m), \tag{4.2}$$

where

$$m = N^{-1}\sum_{\mathbf{n}}[N_{\mathbf{n},\,-1/2} - N_{\mathbf{n},\,1/2}] \tag{4.3}$$

is the relative magnetization in electronic magnetons $\mathfrak{M}$, corresponding to one atom. Then instead of (2.14) we get for the energy

$$\begin{aligned} E = N(E_0 + C) &+ 1/2\,(1-m)\sum_{\mathbf{n}} L^{(m)}(\mathbf{n}^+, \mathbf{n}^+) + 1/2\,(1+m)\sum_{\mathbf{n}} L^{(m)}(\mathbf{n}^-, \mathbf{n}^-) \\ &+ 1/8\,(1-m)^2 \sum_{\mathbf{n}_1\neq\mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_1^+, \mathbf{n}_2^+) - F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_2^+, \mathbf{n}_1^+)] \\ &+ 1/8\,(1+m)^2 \sum_{\mathbf{n}_1\neq\mathbf{n}_2} [F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^-) - F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^-)] \\ &+ 1/8\,(1-m^2) \sum_{\mathbf{n}_1\neq\mathbf{n}_2} [F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^-) - F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^+) \\ &\quad + F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_1^-, \mathbf{n}_2^+) - F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_2^+, \mathbf{n}_1^-)]. \end{aligned} \tag{4.4}$$

At first glance it may appear that there are in (4.4) terms varying linearly with m in seeming violation of the symmetry property. However, this is not so; for the terms linear in m represent the contribution to the mean energy of the system from spin-orbit interaction, which can be written , for instance, in the form[9]

$$\sum_{i=1}^{N} \xi(\mathbf{q}_i)\,\hat{\mathbf{L}}_i\,\hat{\mathbf{s}}_i,$$

and therefore these terms in Eq. (4.4) must depend not simply on the direction of the magnetization $\mathbf{m}$, but on the relative orientation of the spin and orbital magnetic moments. If there is an antiferromagnetic distribution of orbital moments, then these terms are equal to zero.

If we consider only the electrostatic and magnetic spin-spin interaction, then, using the notation (3.16), we find

$$\begin{aligned} F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_1^+, \mathbf{n}_2^+) = F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_1^-, \mathbf{n}_2^-) &= E_{\mathbf{n}_1-\mathbf{n}_2} + C_{\mathbf{n}_1-\mathbf{n}_2}, \\ F(\mathbf{n}_1^+, \mathbf{n}_2^+; \mathbf{n}_2^+, \mathbf{n}_1^+) = F(\mathbf{n}_1^-, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^-) &= I_{\mathbf{n}_1-\mathbf{n}_2} + J_{\mathbf{n}_1-\mathbf{n}_2}, \\ F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_1^+, \mathbf{n}_2^-) = F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_1^-, \mathbf{n}_2^+) &= -E_{\mathbf{n}_1-\mathbf{n}_2} + C_{\mathbf{n}_1-\mathbf{n}_2}, \\ F(\mathbf{n}_1^+, \mathbf{n}_2^-; \mathbf{n}_2^-, \mathbf{n}_1^+) = F(\mathbf{n}_1^-, \mathbf{n}_2^+; \mathbf{n}_2^+, \mathbf{n}_1^-) &= -J_{\mathbf{n}_1-\mathbf{n}_2}, \\ L^{(m)}(\mathbf{n}^+, \mathbf{n}^+) = L^{(m)}(\mathbf{n}^-, \mathbf{n}^-) &= 0. \end{aligned} \tag{4.5}$$

On substituting Eq. (4.5) in Eq. (4.4), we find that

$$E = E^{(0)} - \frac{m^2}{4}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[I_{\mathbf{n}_1-\mathbf{n}_2} - 2\,(E_{\mathbf{n}_1-\mathbf{n}_2} - J_{\mathbf{n}_1-\mathbf{n}_2})], \tag{4.6}$$

where

$$E^{(0)} = N(E_0 + C) + \frac{1}{2}\sum_{\mathbf{n}_1\neq\mathbf{n}_2}[C_{\mathbf{n}_1-\mathbf{n}_2} - 1/2\,I_{\mathbf{n}_1-\mathbf{n}_2}].$$

From formula (4.6) it is evident that the energetic center of gravity varies quadratically with the magnetization m, just as in the case of the usual exchange model, in which no account is taken of the magnetic interaction (cf. for instance, reference 10). But now, instead of electrostatic exchange alone, Eq. (4.6) takes account also of the quasi-Coulomb interaction $E_{\mathbf{n}}$, and the magnetic exchange interaction $J_{\mathbf{n}}$.

After the energetic center of gravity has been found, it is possible to calculate the partition function and the free energy of the crystal, and as

[9] E. Condon and G. Shortley, *The Theory of Atomic Spectra*, Cambridge University Press, 1935

[10] S. V. Vonsovskii and Ia. S. Shur, *Ferromagnetism*, State Technical Publishers, Moscow-Leningrad, 1948

a result to determine the temperature dependence of the spontaneous magnetization, and also the magnetic anisotropy constant. Through the magnetic interaction integrals there enters a dependence of the energy on the angles between the crystal axes and the spontaneous magnetization. In the approximation under consideration, the diagonal terms, in the case of cubic lattices, contribute by virtue of the symmetry properties only a constant term, independent of the angles. Therefore it is worthwhile to consider the case of a hexagonal lattice, in which the magnetic anisotropy is a first-order effect.

In the above formulas, the z axis was selected as the axis of spatial quantization of spins. Here it is worthwhile to choose other axes, related to the principal directions in the crystal. We point axis 1 along the hexagonal axis [0001]; axes 2 and 3 are to lie in the basal plane at an angle of 120° to each other. It is easy to show that in the chosen system of coordinates the expression (4.6) for the energy takes the form

$$E(m,\varphi) = \text{const} - \frac{m^2}{N} I + \frac{2m^2}{N} P(1-\cos^2\varphi); \tag{4.7}$$

here ϕ is the angle between the vector m and the hexagonal axis,

$$I = \sum_{n'} I_{n-n'};\quad P = E_1 - J_1 - \frac{1}{4}(E_2 - J_2), \tag{4.8}$$

$$E_1 = \sum_{n'} \beta \int \frac{(q-q')_1^2}{|q-q'|^5} |\varphi_n(q)|^2 |\varphi_{n'}(q')|^2 \, dq\, dq',$$

$$J_1 = \sum_{n'} \beta \int \varphi_n^*(q)\, \varphi_{n'}^*(q') \frac{(q-q')_1^2}{|q-q'|^5} \varphi_n(q')\, \varphi_{n'}(q)\, dq\, dq',$$

$$E_2 = \sum_{n'} \beta \int \frac{(q-q')_2^2}{|q-q'|^5} |\varphi_n(q)|^2 |\varphi_{n'}(q')|^2 \, dq\, dq' = \sum_{n'} \beta \int \frac{(q-q')_3^2}{|q-q'|^5} |\varphi_n(q)|^2 |\varphi_{n'}(q')|^2 \, dq\, dq'$$
$$= 2\sum_{n'} \beta \int \frac{(q-q')_2 (q-q')_3}{|q-q'|^5} |\varphi_n(q)|^2 |\varphi_{n'}(q')|^2 \, dq\, dq',$$

$$J_2 = \sum_{n'} \beta \int \varphi_n^*(q)\, \varphi_{n'}^*(q') \frac{(q-q')_2^2}{|q-q'|^5} \varphi_n(q')\, \varphi_{n'}(q)\, dq\, dq' = \sum_{n'} \beta \int \varphi_n^*(q)\, \varphi_{n'}^*(q') \frac{(q-q')_3^2}{|q-q'|^5} \varphi_n(q')\, \varphi_{n'}(q)\, dq\, dq'$$
$$= 2\sum_{n'} \beta \int \varphi_n^*(q)\, \varphi_{n'}^*(q') \frac{(q-q')_2 (q-q')_3}{|q-q'|^5} \varphi_n(q')\, \varphi_{n'}(q)\, dq\, dq'$$

Standard calculations of the partition function (cf. § 19, par. 3 of reference 10) give for the free energy of the crystal in the absence of an external field

$$F = -NkT \times \ln \operatorname{ch}\left\{\frac{1}{2\mathfrak{M} NkT}\left[(I+4P) - 6P\sin^2\varphi\right] M\right\} + \frac{1}{4N\mathfrak{M}^2}\left[(I+4P) - 6P\sin^2\varphi\right] M^2, \tag{4.9}$$

where $M = N\mathfrak{M}m$ is the magnetization of the crystal. From Eq. (4.9) it is evident that the free energy of a hexagonal crystal, in the first approximation (with respect to magnetic interaction), is a quadratic function of the sine of the angle between the magnetization and the hexagonal axis, in full accord with thermodynamic theory [cf. for instance, reference 10, formula (38,29)]. From the conditions of thermodynamic equilibrium

$$\partial F/\partial M = 0 \qquad \text{and} \qquad \partial F/\partial\varphi = 0$$

it is possible to find the well-known dependence of magnetization on temperature and on orientation in the crystal. In the temperature region near the Curie point Θ, by taking advantage of the smallness of the argument of the hyperbolic cosine in the first term on the right in (4.9), we find, after expansion in powers of that argument,

$$F = \text{const} + {}^3/_2\, NP\left[\frac{I+4P}{kT} - 1\right] m^2 \sin^2\varphi.$$

Here all terms that do not contain angles have been segregated in "const". Thus the magnetic anisotropy constant of a hexagonal crystal near the Curie point is equal to

$$K = {}^3/_2\, NP\left[\frac{I+4P}{kT} - 1\right] m^2 \approx {}^3/_2\, NP\left(\frac{I}{kT} - 1\right) m^2. \tag{4.10}$$

The exchange energy is equal in order of magnitude to $k\Theta$, therefore the factor in brackets in Eq. (4.10) is equal in order of magnitude to $(\Theta/T - 1)$, i.e., it is of order of magnitude unity. For some relations between the quantities I and P, the factor $[(I + 4P)/kT - 1]$ can change sign with temperature. At the Curie point of constant K approaches zero because of the factor m^2. The factor $(3/2)\,NPm^2$ is of the order of magnitude 10^6 to 10^7 erg cm^{-3} (since $m \sim 1$, $N \sim 10^{22} cm^{-3}$, $P \sim 10^{-6}$ erg); this should also be the magnitude of the magnetic anisotropy constant, for instance, in a hexagonal monocrystal of cobalt at room temperature. More detailed quantitative calculations, in this energetic center of gravity approximation, would hardly be worthwhile in view of the roughness of the approximation.

5. CONCLUSIONS

1. A systematic general scheme has been developed for the quantum-mechanical many-electron treatment of magnetic interaction in crystals; it can serve as a point of departure for the study of all effects in ferromagnetic and antiferromagnetic crystals that are determined by this type of interaction between electrons.

2. The case of low temperatures has been specially investigated; here it is possible to use the method of quasiparticles. The diagonal part of the energy operator of the system has been determined and its eigenvalues found. An expression has been found for the energy of the perturbation that takes account of collision processes between ferromagnons.

3. The case of high temperatures has been considered in the energetic center-of-gravity approximation, and some of its concrete applications in the theory of ferromagnetism have been given.

Translated by W. F. Brown, Jr.
154