The Gauge Transformation of the Green's Function for Charged Particles

L. D. LANDAU AND I. M. KHALATNIKOV
Institute for Physical Problems, Academy of Sciences, USSR
(Submitted to JETP editor February 5, 1955)
J. Exper. Theoret. Phys. USSR 29, 89-93 (1955)

A gauge transformation is carried out on the Green's function and the vertex operator for charged particles interacting with an electromagnetic field.

GAUGE invariance arises in the field theory of charged particles interacting with an electromagnetic field. Given a gauge transformation of the potential of the electromagnetic field

$$A_\mu \to A_\mu + \partial\varphi / \partial x_\mu \tag{1}$$

(ϕ is an arbitrary operator function), the ψ-function of the particle is transformed as follows:

$$\psi \to \psi e^{ie\varphi} \tag{2}$$

(e is the charge, $\hbar$ and c are taken to be unity).

We shall attempt to determine here how the Green's function for the particles will change under such a gauge transformation. The Green's function for particles is well-known to be

$$G(xx') = \langle (\psi(x)\overline{\psi}(x'))_+ \rangle. \tag{3}$$

The brackets denote a vacuum expectation value.

The Green's function $G(xx')$ will change under the gauge transformation (1). From Eq. (2), it can be written in the form:

$$G(xx') = G_0(xx')\langle (e^{ie\varphi(x)} e^{-ie\varphi(x')})_+ \rangle, \tag{4}$$

where $G_0(xx')$ stands for the Green's function for the particular case when the longitudinal (in the four-dimensional sense) part of the photon's Green's function is equal to zero[1]. The Fourier components of the Green's function for photons

$$D_{\mu\nu}(xx') = i\langle (A_\mu(x) A_\nu(x'))_+ \rangle \tag{5}$$

can be written in the general case in the form[1]

$$D_{\mu\nu}(k) = 4\pi d_t \frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + 4\pi d_l \frac{k_\mu k_\nu}{k^4}. \tag{6}$$

The terms containing d_t and d_l represent respectively the transverse and longitudinal parts of the function $D_{\mu\nu}$. Now let us compute the vacuum expectation value of the chronological product. The longitudinal part of Green's function turns out to be the original function. It does not depend upon interaction with the field. Thereby the problem reduces to calculating the vacuum expectation value of the expression $e^{ie\phi(x)} e^{-ie\phi(x')}$, where the operators ϕ represent a free field. Expanding the free field ϕ into plane waves, we have

$$\varphi = \sum_k \varphi_k = \sum \lambda(k^2)(a_k e^{ikx} + a_k^+ e^{-ikx}). \tag{7}$$

a_k and a_k^+ represent respectively creation and annihilation operators for longitudinal photons; $\lambda(k^2)$, the amplitude which characterizes the contribution of the photons, by the four-dimensional wave vector k. We also include in $\lambda(k^2)$ a normalization factor. In view of the smallness of this factor (it contains the reciprocal volume) we shall, in what follows, leave out terms of higher power than λ^2. Expressions of the type $e^{ie\phi}$ can be expanded by means of Eq. (7) into the infinite product

$$e^{ie\varphi(x)} = \prod_k e^{ie\varphi_k}, \quad e^{-ie\varphi(x')} = \prod_l e^{-ie\varphi'_l}. \tag{7'}$$

The operators a_k and a_k^+ corresponding to different values of the wave vectors commute with each other. Let us now take advantage of this circumstance and take into account the fact that the vacuum expectation values of terms of first order in the operator a_k (or ϕ_k) are zero. The vacuum expectation values of the products $\phi_k \phi'_l$, in accordance with what we said above, can differ from zero only if $k = l$. With the help of Eq. (7′) we obtain for the vacuum expectation value of the expression $e^{ie\phi(x)} e^{-ie\phi(x')}$.

$$\langle (e^{ie\varphi(x)} e^{-ie\varphi(x')})_+ \rangle = \left\langle \left(\prod_k \left(1 + ie\varphi_k - \frac{e^2}{2}\varphi_k^2\right) \prod_l \left(1 - ie\varphi'_l - \frac{e^2}{2}\varphi_l'^2\right)\right)_+ \right\rangle \tag{8}$$

[1] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR 95, 773 (1954)

$$= \prod_k (1 - \frac{e^2}{2} \langle \varphi_k^2 \rangle - \frac{e^2}{2} \langle \varphi_k'^2 \rangle + e^2 \langle (\varphi_k \varphi_k') \rangle$$

$$= \exp \left\{ -\frac{1}{2} e^2 \sum_k \langle \varphi_k^2 \rangle + \langle \varphi_k'^2 \rangle - 2 \langle (\varphi_k \varphi_k')_+ \rangle \right\}.$$

As we have already noted, the vacuum expectation value of the product of the two operators a_k and a_k^+ is always zero except when $k = l$. Thereby the Green's function for the ϕ field becomes

$$\Delta_F(xx') = i \langle (\varphi(x) \varphi(x'))_+ \rangle = i \langle (\varphi_k \varphi_k')_+ \rangle. \quad (9)$$

Considering Eq. (9), we rewrite Eq. (8) in the form

$$\langle (e^{ie\varphi(x)} e^{-ie\varphi(x')})_+ \rangle = \exp \{ ie^2 (\Delta_F(0) - \Delta_F(xx') \}. \quad (10)$$

Now let us apply the results we have obtained and write the final formula for the Green's function $G(xx')$ for charged particles; from Eqs. (4) and (10) we have

$$G(xx') = G_0(xx') \times \exp \{ ie^2 (\Delta_F(0) - \Delta_F(xx') \}. \quad (11)$$

Formula (11) links the Green's function for charged particles $G(xx')$ with its value $G_0(xx')$ computed under the assumption that the longitudinal part of the photon D- function is equal to zero. The longitudinal part of the Green's function for photons is connected with the Δ- function of the ϕ field, as evidenced by the expression:

$$D_{F\mu\nu}(xx') = \frac{\partial^2}{\partial x_\mu \partial x_\nu'} \Delta_F(xx'). \quad (12)$$

In accordance with Eq. (6), the expression for the Fourier-components becomes

$$\Delta_F(k) = 4\pi \frac{d_l(k)}{k^4}. \quad (13)$$

It is not generally possible to write in Fourier components form the Green's function as it appears in formula (11). It is possible, however, to obtain a formula for the change in the Fourier components of the Green's function for particles, for an infinitesimal gauge transformation of the potential. Performing a variation on Eq. (11) we obtain

$$\delta G(xx') = ie^2 G(xx') (\delta \Delta_F(0) - \delta \Delta_F(xx')). \quad (14)$$

We rewrite this expression in Fourier components; with the help of Eq. (13), we find

$$\delta G(p) = \frac{ie^2}{\pi} \int \frac{\delta d_l(k)}{k^4} \{ G(p) - G(p-k) \} d^4k. \quad (15)$$

Let us apply the resulting formula to the case of spin ½ particles. The unperturbed Green's function of the free particle is given by

$$G_0(p) = 1 / (\hat{p} - m), \quad \hat{p} = \gamma_\mu p_\mu.$$

Let $d_l(k)$ be a slowly varying function of the argument (k^2), so that the condition $\frac{e^2}{\pi} d_l(k) \gg 1$ is satisfied. Let us substitute into the right side of Eq. (15) the unperturbed value of the function $G(p)$. For slow variations of the function $d_l(k)$ in Eq. (15), significant contributions to the integral come from the region $k^2 \gg p^2$. Because of this, the term $G(p-k)$ may be neglected with respect to $G(p)$ on the right side of Eq. (15). Equation (15) is then satisfied by the Green's function

$$G(p) = \frac{\beta(p^2)}{(\hat{p} - m)} \quad (16)$$

where $\beta(p^2)$ is a slowly varying function of its argument.

For the case when $p^2 \gg m^2$, from Eq. (15) and from the expression

$$d^4k = \frac{i}{4} (-k^2) d(-k^2),$$

we obtain

$$\delta\beta(p^2) = -\frac{e^2}{4\pi} \beta(p^2) \int_{-p^2}^{\infty} \frac{\delta d_l}{-k^2} d(-k^2). \quad (17)$$

Let us denote $\beta(p^2)$ when $d_l = 0$ by the expression $\beta_t(p^2)$. Calculations show that in the case of spin ½, we have $\beta_t(p^2) = 1$ [1].

From Eq. (17) we obtain for a finite gauge transformation the well-known formula

$$\beta(p^2) = \exp \left\{ -\frac{e^2}{4\pi} \int_{-p^2}^{\infty} d_l \frac{d(-k^2)}{-k^2} \right\}. \quad (18)$$

The problem for spin zero can be solved in an analogous fashion. In this case the Green's function is written in the form

$$G(p) = \frac{\beta(p^2)}{p^2 - m^2}. \quad (19)$$

Calculations show that for slowly varying d_l, the quantity $\beta(p^2)$ similarly turns out to be a slowly varying function. For a finite gauge transformation we obtain, analogously to Eq. (18),

$$\beta(p^2) = \beta_t(p^2) \exp\left\{-\frac{e^2}{4\pi}\int_{-p^2}^{\infty} d_l \frac{d(-k^2)}{-k^2}\right\}, \quad (20)$$

where $\beta_t(p^2)$ denotes $\beta(p^2)$ when $d_l = 0$.

Let us now turn to the gauge transformation of the vertex operator $\Gamma_\mu(xx'; \xi)$. We shall start from the vacuum expectation value of the chronological product

$$\langle(\psi(x) A_\mu(\xi) \overline{\psi}(x'))_+\rangle. \quad (21)$$

The vertex operator $\Gamma_\mu(xx'; \xi)$ is linked to the function $B_\mu(xx'; \xi)$ by the integral

$$B_\mu(xx'; \xi) = e^2 \int G(xx'') \Gamma_\nu(x''x'''; \xi') G(x'''x') \times D_{\nu\mu}(\xi'\xi)\, d^4x''\, d^4x'''\, d^4\xi'. \quad (22)$$

Under gauge transformation, the function $B_\mu(xx'; \xi)$ changes as follows:

$$B_\mu(xx'; \xi) \rightarrow B_{0\mu}(xx'; \xi) \langle(e^{ie\varphi(x)} \cdot e^{-ie\varphi(x')})_+\rangle + G_0(xx') \left\langle\left(e^{ie\varphi(x)} \frac{\partial\varphi(\xi)}{\partial\xi_\mu} e^{-ie\varphi(x')}\right)_+\right\rangle. \quad (23)$$

Here $B_{0\mu}(xx'; \xi)$ denotes the function B_μ when the longitudinal part of the photon D-function is equal to zero. The vacuum expectation value of the product which appears in the term containing $B_{0\mu}$ in Eq. (23) is calculated from Eq. (10). As for the factor which appears with $G_0(xx')$, simple calculations involving Eqs. (7) and (8) yield for it

$$\left\langle\left(e^{ie\varphi(x)} \frac{\partial\varphi(\xi)}{\partial\xi_\mu} e^{-ie\varphi(x)}\right)_+\right\rangle = \exp\{ie^2[\Delta_F(0) - \Delta_F(xx')]\} \times e \frac{\partial}{\partial\xi_\mu}(\Delta_F(x\xi) - \Delta_F(\xi x')). \quad (24)$$

Substituting Eq. (24) in Eq. (23) and taking Eq. (11) into account, we finally obtain

$$B_\mu(xx'; \xi) = B_{0\mu}(xx'; \xi) \exp\{ie^2(\Delta_F(0) - \Delta_F(xx'))\} + B_{1\mu}(xx'; \xi), \quad (25)$$

$$B_{1\mu}(xx'; \xi) = eG(xx') \frac{\partial}{\partial\xi_\mu}(\Delta_F(x\xi) - \Delta_F(\xi x')). \quad (26)$$

Let us clarify the connection between each of the terms in Eq. (25) and the Green's functions for particles and photons. First we show that the second term in Eq. (25) coincides exactly with that part of Eq. (22) which corresponds to the longitudinal part of the photon function $D^l_{\mu\nu}$;

$$B_{1\mu}(xx'; \xi) = e \int G(xx'') \Gamma_\nu(x''x'''; \xi') \times G(x'''x') D^l_{\nu\mu}(\xi'\xi)\, d^4x''\, d^4x'''\, d^4\xi'. \quad (27)$$

This expression can be obtained rigorously by going to Fourier components. Using Eq. (26), the Fourier components of $B_{1\mu}(xx'; \xi)$ become

$$B_{1\mu}(p, p-k; k) = e(G(p) - G(p-k))\, k_\mu \Delta_F(k). \quad (28)$$

The Fourier component of the right side of Eq. (27) is evidently given by [see Eq. (6)]:

$$eG(p) \Gamma_\nu(p, p-k; k)\, G(p-k) \frac{k_\nu k_\mu}{k^4} 4\pi d_l(k). \quad (29)$$

We shall now make use of the famous generalized theorem of Ward[2]

$$k_\nu \Gamma_\nu(p, p-k; k) = -(G^{-1}(p) - G^{-1}(p-k)). \quad (30)$$

Expression (29) then becomes:

$$e(G(p) - G(p-k)) \frac{k_\mu}{k^4} 4\pi d_l(k). \quad (31)$$

Comparison of Eqs. (31) and (28), in view of Eq. (13), verifies the correctness of Eq. (27). If Eq. (27) is proved, then it follows from Eqs. (22) and (25) that

$$B_{0\mu}(xx'; \xi) \exp\{ie^2[\Delta_F(0) - \Delta_F(xx')]\} = e \int G(xx'') \Gamma_\nu(x''x'''; \xi') G(x'''x') \times D^t_{\nu\mu}(\xi'\xi)\, d^4x''\, d^4x'''\, d^4\xi' \quad (32)$$

($D^t_{\mu\nu}$ is the transverse part of the D-function).

Rewriting this expression for an infinitesimal gauge transformation and going to Fourier components, we find

$$ie^2 \int \{G(p) \Gamma_\mu(p, p-k; k) G(p-k) - G(p-r) \Gamma_\mu(p-r, p-r-k; k) \times G(p-k-r)\} \frac{\delta d_l(r)}{r^4} d^4r = G(p)\, \delta\Gamma_\mu(p, p-k; k)\, G(p-k) \quad (33)$$

[2] H. Green, Proc. Phys. Soc. 66, 837 (1953)

$$+ie^2\int\{[G(p)-G(p-r)]$$
$$\times\Gamma_\mu(p,p-k;k)G(p-k)$$
$$+G(p)\Gamma_\mu(p,p-k;k)[G(p-k)$$
$$-G(p-k-r)]\}\frac{\delta d_l(r)}{r^4}d^4r.$$

Solving for $\delta\Gamma_\mu$ we finally obtain

$$G(p)\delta\Gamma_\mu(p,p-k;k)G(p-k)$$
$$=-ie^2\int\{G(p)\Gamma_\mu(p,p-k;k)(G(p-k) \quad (34)$$
$$-G(p-k-r))+G(p-r)$$
$$\times[\Gamma_\mu(p-r,p-r-k;k)G(p-k-r)$$
$$-\Gamma_\mu(p,p-k;k)G(p-k)]\}\frac{\delta d_l(r)}{r^4}d^4r.$$

For the case of spin ½ particles, the change in the vertex operator $\Gamma_\mu(p, p-k; k)$ under infinitesimal gauge transformation ($p^2 \gg m^2$) can be found by a method similar to the one that was used for the Green's function. If $d_l(r)$ is a slowly varying function in Eq. (34), then all the terms on the right side except the first can be neglected (upon integration in the significant region of large r^2). After this it is easily found (assuming that $(p-k)^2$, p^2 and k^2 are all of the same order of magnitude) that

$$\delta\Gamma_\mu(p,p-k;k) \quad (35)$$
$$=-ie^2\Gamma_\mu(p,p-k;k)\int_{-p^2}^{\infty}\frac{\delta d_l(r)}{-r^2}d(-r^2).$$

This result is found to be in conformity with Ward's theorem.

In conclusion we wish to extend our thanks to A. A. Abrikosov for taking part in discussions with us.

Translated by M. A. Melkanoff
159