equations are invariant under different kinds of spinor transformations. In connection with this, in each case different tensors are considered primary. This difference is closely connected with the fact that, in the earlier work[2], $\hbar$ is considered a pseudoscalar, not merely a scalar, as in Dirac theory, and that the operators of four-"momentum" have entirely different forms.

The different character of the two systems of differential equations is especially explicit in the transition to the nonrelativistic limit. From the point of view of the earlier work, we deal only with one real spinor, $\psi_{(1)}$. It is very characteristic that the current vector used in nonrelativistic quantum mechanics turns out to be not part of a vector, but of a tensor. Its components are proportional to $T^{(1)}_{4k}$ [see reference 1, Eq. (54) and Zaitsev[14], Eq. (59)].

As for the Dirac equation, in the transition to the equations of nonrelativistic mechanics, the situation is entirely different. In the nonrelativistic limit, ψ is still expressed in terms of two real spinors $\psi_{(1)}$ and $\psi_{(2)}$. The components of the current vector are found from the components of $P_{(+)}$ after making use of the Dirac equation and eliminating some of the terms (see Pauli[5])

[14] G. A. Zaitsev, J. Exper. Theoret. Phys. USSR **25**, 653 (1953)

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171

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The Statistics of Charge-Conserving Systems and Its Application to the Theory of Multiple Production

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The quantum statistics of systems with a variable number of non-interacting particles is generalized to the case of an aggregate of oppositely charged particles, which obey the law of charge conservation. Formulas which differ from the corresponding formulas of ordinary quantum statistics are derived for the total number of particles and the total energy. The results obtained are applied to the theory of multiple production of mesons. The following questions are studied: the dependence of the energy on the relative proportions of neutral and charged mesons, the formation of nucleon-antinucleon pairs, and the relation between the yield and the primary energy. The theory is compared with the available experimental data.

1. INTRODUCTION

IN the statistical treatment of the phenomenon of multiple production of particles at high energies, proposed by Fermi[1], the total number of particles, the total energy of the system, and also the relation between the numbers of particles of different sorts in the "thermodynamic" approximation are calculated by the usual quantum statistical formulas for an ideal Bose or Fermi gas with a variable number of particles. However, in this case, it is more appropriate to use formulas which take into account the conservation of charge(electronic, nuclear, etc). This is particularly important when we consider processes with a low yield. Thus, after generalizing ordinary quantum statistics to the case of charge-conserving systems, a more detailed examination of processes of multiple production in the framework of the "thermodynamic" approximation is possible.

We make this generalization in the present paper, and as a result obtain new formulas for the total number of particles and the total energy, which we relate to the corresponding formulas of ordinary statistics. The results obtained are used to explain several matters pertaining to the theory of multiple production of particles.

[1] E. Fermi, *Elementary Particles*, New Haven, 1951

2. CALCULATION OF THE PARTITION FUNCTION, THE AVERAGE NUMBER OF PARTICLES, AND THE AVERAGE ENERGY OF CHARGE CONSERVING SYSTEMS

We shall consider an ideal gas, consisting of like but oppositely charged particles. The number of positive and negative particles can change as a result of pair production, but the difference between n^+, the number of positive particles, and n^-, the number of negative particles, i.e., the total charge ν of the system, remains constant as a result of charge conservation

$$\nu = n^+ - n^- = \text{const.} \tag{1}$$

If the system has non-degenerate energy levels E_k, then, designating by n_k^+ and n_k^- the corresponding occupation numbers, we can write the partition function of the system in the form[2]

$$Z_1 = \sum_{\substack{n_1^+ \\ n_1^-}} \sum_{\substack{n_2^+ \\ n_2^-}} \ldots \prod_k q_k^{n_k^+ + n_k^-} \delta\left[\nu, \sum_k (n_k^+ - n_k^-)\right], \tag{2}$$

where $q_k = e^{-E_k/\Theta}$, $\Theta = kT$, and $\delta[a,b]$ is the Kronecker symbol

$$\delta[a, b] = \begin{cases} 1, & \text{if} \quad a = b. \\ 0, & \text{if} \quad a \neq b. \end{cases} \tag{3}$$

The product $\prod_k$ and the summation $\sum_k$ extend over all k from 1 to ∞; the summation over n_k^+ and n_k^- extends from 0 to ∞ in the case of Bose statistics, and from 0 to 1 in the case of Fermi statistics.

As is well-known, the Kronecker symbol can be represented in the form

$$\delta[a, b] = \frac{1}{2\pi} \int_0^{2\pi} e^{-i(a-b)\varphi} d\varphi. \tag{4}$$

In this way the partition function can be written as the following integral:

$$Z = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu\varphi} \sum_{\substack{n_1^+ \\ n_1^-}} \sum_{\substack{n_2^+ \\ n_2^-}} \ldots \prod_k q_k^{n_k^+ + n_k^-} e^{i(n_k^+ - n_k^-)\varphi} d\varphi$$
$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu\varphi} \prod_k \sum_{\substack{n_k^+ \\ n_k^-}} q_k^{n_k^+ + n_k^-} e^{i(n_k^+ - n_k^-)\varphi} d\varphi. \tag{5}$$

[2] M. A. Leontovich, *Statistical Physics*, GTTI, Moscow, 1944

The sums in the integrand are easily calculated for both Bose and Fermi statistics. Indeed

$$\sum_{\substack{n_k^+ \\ n_k^-}} q_k^{n_k^+ + n_k^-} e^{i(n_k^+ - n_k^-)\varphi} = \sum_{n_k^+} (q_k e^{i\varphi})^{n_k^+} \sum_{n_k^-} (q_k e^{-i\varphi})^{n_k^-}$$
$$= (1 \pm q_k e^{i\varphi})^{\pm 1} (1 \pm q_k e^{-i\varphi})^{\pm 1} \tag{6}$$

Here, as in what follows, the upper sign (+ in the example given) applies to the case of Fermi statistics, and the lower sign (– in the example given) to the case of Bose statistics.

Using Eq. (6), we can write the partition function (5) in the form

$$Z = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu\varphi + \Phi(\varphi)} d\varphi, \tag{7}$$

where

$$\Phi(\varphi) = \pm \sum_k \ln(1 \pm q_k e^{i\varphi}) \pm \sum_k \ln(1 \pm q_k e^{-i\varphi}). \tag{7'}$$

Having determined Z, we can calculate the average occupation numbers by the well-known formula

$$\bar{n}_k = \overline{n_k^+} + \overline{n_k^-} = q_k \frac{\partial}{\partial q_k} (\ln Z). \tag{8}$$

Using Eqs. (7), (7′), and (8), we obtain

$$\bar{n}_k = \frac{1}{2\pi Z} \int_0^{2\pi} \left(\frac{1}{1 \pm \exp\left\{\frac{E_k}{\Theta} + i\varphi\right\}} + \frac{1}{1 \pm \exp\left\{\frac{E_k}{\Theta} - i\varphi\right\}} \right) e^{-i\nu\varphi + \Phi(\varphi)} d\varphi. \tag{9}$$

Knowing Z and $\bar{n}_k$, we can calculate all the more important thermodynamic quantities.

To calculate Z and $\bar{n}_k$, we must find an approximate expression for $\Phi(\varphi)$. If in the energy interval dE there are, on the average, $dG(E)$ energy levels, $\Phi(\varphi)$ can be approximated as

$$\Phi(\varphi) = \pm \int_0^\infty \left[\ln\left(1 \pm \exp\left\{-\frac{E}{\Theta} + i\varphi\right\}\right) + \ln\left(1 \pm \exp\left\{-\frac{E}{\Theta} - i\varphi\right\}\right) \right] dG(E). \tag{10}$$

Since the difference between the statistics we are considering and statistics with a fixed number of particles can be appreciable only for energies large enough to produce pairs, we restrict ourselves to considering a relativistic gas. For an ideal relativistic gas,

$$dG(E) = \frac{b\Omega}{2\pi^2 c^3 \hbar^3} E^2 dE, \tag{11}$$

where Ω is the volume in which the gas is contained, and b the number of possible states of different polarization. Substituting Eq. (11) in Eq. (10) and integrating, we obtain

$$\Phi(\varphi) = \frac{b\Omega}{2\pi^2 c^3 \hbar^3} 4\,\Theta^3 \sum_{k=1}^{\infty} (\mp 1)^{k-1} \frac{\cos k\varphi}{k^4}. \tag{12}$$

In view of the very rapid convergence of the series obtained, we take only its first term, and obtain as an approximation for $\Phi(\varphi)$ the expression

$$\Phi(\varphi) = \rho \cos\varphi, \text{ where } \rho = 2\,b\,\Omega\Theta^3/\pi^2 c^3 \hbar^3. \tag{13}$$

This approximation corresponds to the transition to Boltzmann statistics.

Substituting Eq. (13) in Eq. (7), we obtain finally

$$Z = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\nu\varphi + \rho\cos\varphi} d\varphi = I_\nu(\rho), \tag{14}$$

where $I_\nu(\rho)$ is the Bessel function of order ν with imaginary argument.

The average number of particles N and the average energy W of the system can be calculated by the formula

$$N = \int_0^\infty \bar{n}(E)\,dG(E),\quad W = \int_0^\infty E\bar{n}(E)\,dG(E), \tag{15}$$

where $n(E)$ is the average number of particles occupying levels with energy E, i.e., $\bar{n}_k$ in the notation of Eq. (9).

To calculate N, we substitute Eq. (9) in Eq. (15) use the expression (12), and integrate first over E, whence we obtain

$$N = \frac{\rho}{2\pi Z}\int_0^{2\pi} \sum_{k=1}^{\infty} (\mp 1)^{k-1} \frac{\cos k\varphi}{k^3} \tag{16}$$

$$\times \exp\left\{-i\nu\varphi + \rho \sum_{k=1}^{\infty} (\mp 1)^{k-1} \frac{\cos k\varphi}{k^4}\right\} d\varphi.$$

Just as in the calculation of Eq. (14), we restrict ourselves to the first terms of the series in the integrand, and obtain

$$N = \frac{\rho}{I_\nu(\rho)} \frac{dI_\nu(\rho)}{d\rho}. \tag{17}$$

By analogous calculations, we also obtain an expression for the average energy

$$W = 3\Theta N. \tag{18}$$

For conciseness, the formulas obtained above for an ideal gas, consisting of positive and negative particles, will be hereafter referred to as the formulas of charge statistics.

3. AN ANALYSIS OF THE FORMULAS OF CHARGE STATISTICS

The dimensionless parameter ρ appearing in Eq. (17) has a simple physical interpretation. According to Eq. (13), ρ is (except for a multiplier near unity) the average number of particles of relativistic gas with $2b$ internal degrees of freedom, calculated by the rules of the usual quantum statistics of systems with a variable number of particles. Therefore ρ is a parameter suitable for comparing formulas (17) and (18) of charge statistics with the corresponding formulas of the usual quantum statistics.

From the theory of Bessel functions, we know that

$$\rho \frac{dI_\nu(\rho)}{d\rho} - \nu I_\nu(\rho) = \rho I_{\nu+1}(\rho) \tag{19}$$

whence, by Eq. (17)

$$N = \nu + \rho \frac{I_{\nu+1}(\rho)}{I_\nu(\rho)}. \tag{20}$$

Thus N is the sum of two terms: ν, the minimum number of particles for the given system, equal to the number of excess charges of whatever sign, and the mean number of produced particles

$$N_\pm = \rho \frac{I_{\nu+1}(\rho)}{I_\nu(\rho)} = \rho\lambda_\nu(\rho). \tag{21}$$

The quantity $\lambda_\nu(\varphi)$ denotes the ratio of the number of produced particles, calculated by charge statistics, to the number of particles produced according to the usual statistics. From the asymptotic behavior of the Bessel functions, it follows that $\frac{I_{\nu+1}(\rho)}{I_\nu(\rho)} \to 1$ as $\rho\to\infty$. Thus $\lambda_\nu(\rho)\to 1$ for sufficiently high temperatures (but bounded values of ν), i.e., charge statistics reduces asymptotically to ordinary quantum statistics. On the other hand, for $\rho \to 0$, i.e., for low temperatures, $\lambda_\nu(\rho) \to 0$, and the formulas of charge statistics are considerably different from the formulas of ordinary statistics.

The quantity $\lambda_\nu(\rho)$ can be approximated with good accuracy (especially in regions of small ρ) by the expression

$$\lambda_\nu(\rho) = I_{\nu+1}(\rho)/I_\nu(\rho) \tag{22}$$

$$= 1 - \exp\{-\rho/2(\nu+1)\}.$$

From this formula it is clear that the number of particles produced at a given temperature (i.e., for given ρ) drops sharply as the charge ν is increased, in other words, the charge ν acts as a kind of anticatalyst in the process of particle production. For sufficiently large values of ν, practically no particles are produced, the average number of particles remains constant, and charge statistics reduces to the ordinary statistics of systems with a constant number of particles.

Of particular interest is the case $\nu = 0$. According to (22), for temperatures that are not too high ($\rho < 1$)

$$\lambda_0(\rho) = 1 - e^{-\rho/2} \cong \rho/2. \qquad (23)$$

Thus, for relativistic but not too high temperatures, the average number of charged particles produced is equal to $\rho^2/2$, while the average number of neutral particles produced is equal to $\rho/2$. This leads to different temperature dependences for the average energy of radiation for a gas of neutral particles ($W \sim \Theta^4$) and for a gas of particles with some kind of charge ($W \sim \Theta^7$).

4. APPLICATIONS TO THE FERMI THEORY OF MULTIPLE PRODUCTION OF MESONS AND NUCLEON-ANTINUCLEON PAIRS

According to Fermi's hypothesis[1], π-mesons and nucleon-antinucleon pairs are produced as a result of nucleon-nucleon collisions in some small volume Ω. The number of these particles is calculated by formulas valid for systems in thermodynamic equilibrium, using the expressions of ordinary quantum statistics, and not those of charge statistics, which, as we have seen, give quite different results for the case of low yield. For very large energies, the charge ν is small compared to the number of particles produced, $\lambda_\nu(\rho) \to 1$, and the formulas of charge statistics coincide with the formulas of ordinary statistics; thus, in this region of energies, the application of charge statistics in the framework of the Fermi hypothesis can not lead to new results. Significantly different results are obtained when charge statistics are applied in the low energy region, where the average number of particles produced is comparable to the charge ν of the system.

In Fermi's calculation of the average number of charged π-mesons, the same formulas are used as in the calculation of the number of neutral mesons. However, the charged π-mesons, unlike the neutral π-mesons, obey charge statistics, and therefore to calculate their average number it is necessary to use the formulas (21) and (22), and not the formula $N = \rho$, valid only for π^0-mesons. According to Eq. (22), we obtain for the ratio of the number of neutral π-mesons to the number of charged π-mesons, produced as a result of nucleon-nucleon collisions

$$\alpha = \frac{1}{2(1 - e^{-\rho/2})}, \qquad (24)$$

since the number of neutral π-mesons produced is equal to $\rho/2$. This ratio coincides with that obtained by Fermi only for $\rho \to \infty$. For small values of ρ, it is clear that α increases, which agrees with the experimental evidence[3,4]*

Because of the conservation of nuclear charge[5], it is also necessary to use the formulas of charge statistics to calculate the number of nucleons and antinucleons produced by nucleon-nucleon collisions. Thus, it follows from Eqs. (21) and (22) that the number of nucleons and antinucleons produced (within the framework of the Fermi hypothesis) by the collision of two nucleons, is given by the formula

$$N_2 = \rho(1 - e^{-\rho/6}), \qquad (25)$$

and not by the formula $N_2 = \rho$, used by Fermi. According to Eq. (25), the number of nucleon-antinucleon pairs produced must be considerably smaller than that given by the Fermi formulas. This number becomes even smaller if the Fermi hypothesis is applied to the collisions of nucleons with nuclei, and it is assumed that the energy is distributed among several nucleons of the nucleus. If we neglect the production of particles in subsequent nucleon-nucleon and meson-nucleon collisions, the number of nucleons and antinucleons produced must be calculated by the formula

$$N_{1+A} = \rho[1 - \exp\{-\rho/2(A+2)\}], \qquad (26)$$

where A is the number of colliding nucleons. In this case, the ratio of the number of nucleon-antinucleon pairs to the number of mesons can be expressed by the formula

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* In the case where the average number of particles produced $N \cong 1$, $\rho/2$ is small, and we may consider that $W = (\rho/2) + (\rho^2/2)$, $\alpha = 1/\rho$. Eliminating ρ, we find that $\alpha \cong 1$.

[3] A. G. Carlson, I. E. Hooper, and D. T. King, Phil. Mag. **41**, 701 (1950)

[4] U. Camerini, P. H. Fowler, W. O. Lock, and H. Muirhead, Phil Mag. **41**, 413 (1950)

[5] E. P. Wigner, Proc. Nat. Acad. Sci. **38**, 449 (1952)

$$\beta = {}^{8}/_{3}\,[1 - \exp\{-\rho/2(A+2)\}], \qquad (27)$$

where ρ is taken for a gas of nucleons and antinucleons ($b = 4$, since the polarization states of both the neutrons and protons are considered). For high energies and small A , this ratio reduces to Fermi's, namely, $\beta = 8/3$, as was to be expected; however, for not too high energies and large A, it is considerably smaller. To calculate β in terms of the given primary energy W, it suffices to know ρ as a function of W. For this, we must use Eq. (18), generalized to the case of the system considered, in analogy to the way we treat the case of nucleon-nucleon collisions below.

We shall derive the dependence on the primary energy $N_{\pm}$, the average number of charged mesons produced in the process of nucleon-nucleon collisions without charge exchange. We designate by γ the total energy of the primary nucleon in the center of mass system, expressed in units of the nucleon rest mass, by $\gamma_0 = 2\gamma^2 - 1$ the same energy in the laboratory system, and by γ_k the energy, incident on the nucleon in the center of mass system and corresponding to the threshold of meson production. The medium energy region is pertinent to charge statistics, where $\gamma_0 < 10$ and, correspondingly, $\gamma < 2.4$. In this case, as experiment shows, the probability of producing nucleon-antinucleon pairs is very small, and we shall neglect it. The production of secondary particles will also be neglected.

Using Eqs. (17) and (18), where $\nu = 0$,

$$W = 2(\gamma - \gamma_k),\ \Theta = 0.1\gamma^{1/4}\rho^{1/4} \quad [\text{see } (13\,]$$

we obtain

$$\rho^{4/3}\left[\lambda_0(\rho) + \frac{1}{2}\right] = 6.7\,\frac{\gamma - \gamma_k}{\gamma^{1/3}}, \qquad (28)$$

$$N_{\pm} = \rho\lambda_0(\rho). \qquad (29)$$

Eliminating ρ from Eqs. (28) and (29) (using graphs) we get the dependence of the number of charge mesons produced on the primary energy. In the energy region considered, this dependence can be written

$$N_{\pm} = k(\gamma - \gamma_k), \qquad (30)$$

where $k \cong 2.67$. If we put $\gamma_k = 1.1$, Eq. (30) gives satisfactory agreement with experiment[6].

[6] J. G. Wilson editor, *Progress in Cosmic Ray Physics*, Vol. 1, New York, 1952

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168