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The Quantum Theory of Fields I

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A simple derivation is carried out of the equations proposed by Schwinger[1]. The resultant equations are discussed and renormalization is obtained in the original integral equations*.

INTRODUCTION

IN connection with the inapplicability of the usual excitation theory for large energies of interacting particles, and also for the calculation of the characteristic properties of the existing theories of quantum fields (the presence of infinities, the asymptotic character of a series of excitation theories, the asymptotic character of the Green's function for high momenta, etc.), the development of more precise methods of solution of the quantum mechanical equations presents great interest. In this connection, equations of the form suggested by Schwinger possess fundamental interest. Inasmuch as we want to solve these equations not only by the methods of ordinary excitation theory, it becomes essential to carry out the renormalization (not according to the excitation theory) in the original equations. Moreover, even within the framework of ordinary excitation theory, it is of interest to obtain a system of equations which does not preserve the infinities. On the other hand, to clarify the physical meaning of quantities which enter into the equations, obtained by Schwinger from a variational principle, and to establish the boundary conditions which must be imposed on the solutions of these equations, it is important to obtain these equations from the ordinary scheme of the theory of quantum fields, without resorting to a variational principle. Such a derivation of the desired equations is given in the present work, and the renormalization is carried out in the integral equations that are obtained.

1. We consider a quantum mechanical system which is a generalization of the ordinary system. Let the Lagrangian L of the system consist of the ordinary Lagrangian L^{or} and the Lagrangian of the interaction with external sources L^{in}.

For definiteness, we consider the case of quantum electrodynamics (the results are simply carried over to the general case of quantum mechanical theory). The initial Lagrangian in this case has the form:

$$L = L^{or} + L^{in}; \tag{1}$$

$$L^{or} = -{}^1/_4 [\bar{\psi}, \gamma_\mu(-i\partial_\mu - eA_\mu)\psi + m\psi] + {}^1/_4 F^2_{\mu\nu} - {}^1/_4 \{F_{\mu\nu}, \partial_\mu A_\nu - \partial_\nu A_\mu\} + \text{Hermitian conjugate}; \tag{1a}$$

$$L^{in} = {}^1/_2 [\bar{\psi}, \eta] + {}^1/_2 [\bar{\eta}, \psi] + J_\mu A_\mu, \tag{1b}$$

where J is the external source of the photon field; η is the anti-commutating external source of the spinor field;

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2\delta_{\mu\nu}; \tag{1c}$$

$$\delta_{11} = \delta_{22} = \delta_{33} = 1; \; \delta_{44} = -1.$$

The Hamiltonian of the system can be written in the form

$$H = H^{or} + H^{in}, \tag{2}$$

where H^{in} has the form

$$H^{in} = -\int \{{}^1/_2 [\bar{\psi}\eta] + {}^1/_2 [\bar{\eta}\psi] + J_\mu A_\mu\} \, d^3x. \tag{3}$$

Conditionally we write H^{in} in the form

$$H^{in} = -\sum_n \int \pi_{f_n}(x, t) f_n(x, t) \, d^3x, \tag{4}$$

where f_n is the source of an nth type field ($J_\mu, \eta, \bar{\eta}$).

In the Heisenberg representation, where the wave function of the system is independent of time, $\Psi(t)$ = const, the equations of motion for the operators of the field have the form

$$\gamma_\mu(-i\partial_\mu - eA_\mu)\psi + m\psi = \eta; \tag{5}$$

* The renormalization was obtained by us in reference 2; in the present paper, a more appropriate derivation of the renormalized system of equations is given.

[1] J. Schwinger, Proc. Nat. Acad. Sci. U. S. 37, 452 (1951)

[2] E. S. Fradkin, J. Exper. Theoret. Phys. USSR **26**, 751 (1954)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \tag{6}$$

$$-\Box A_\mu + \partial_\mu \partial_\nu A_\nu = J_\mu + j_\mu;$$

$$(\partial A_\mu / \partial x_\mu) = \psi = 0, \tag{7}$$

where

$$j_\mu = {}^1/_2\, e [\bar{\psi}, \gamma_\mu \psi]. \tag{8}$$

We transform to a new representation, where the operators of the field do not depend on the external sources (analogous to the representation of the interaction in external interaction) and have the form

$$\gamma_\mu (-i\partial_\mu - e\tilde{A}_\mu)\tilde{\psi} + m\tilde{\psi} = 0; \tag{9}$$

$$-\Box \tilde{A}_\mu + \partial_\mu \partial_\nu \tilde{A}_\nu = \tilde{j}_\mu. \tag{10}$$

The wave function in the new representation changes with time and can be written in the form:

$$\Phi = S(t, -\infty)\Psi_0, \tag{11}$$

where Ψ_0 is the value of the wave function for $t = -\infty$, and S satisfies the equation

$$i\frac{\partial S(t, -\infty)}{\partial t} = \tilde{H}^{\text{in}} S(t, -\infty). \tag{12}$$

We construct the matrix $S(t_1, t_2)$

$$S(t, t') = S(t, -\infty) S^{-1}(t', -\infty). \tag{13}$$

In accordance with Eqs. (12) and (13), we get the equation for $S(t_1, t_2)$ in integral form:

$$S(t, t') = 1 - i\int_{t'}^{t} H^{\text{in}}(t_1) S(t_1, t')\, dt_1 \tag{14}$$

$$= 1 + i\iint_{t'}^{t} \bar{\pi}_{f_n}(t_1, x_1) f_n(x_1, t_1) S(t_1, t')\, d^4x_1.$$

We take the functional derivative of both parts of Eq. (14) with respect to any external source $f_n(\xi)$ at the point ξ, first setting $t' = -\infty$*,

$$\frac{\delta S(t, -\infty)}{\delta f(\xi)} = \begin{cases} i\tilde{\pi}_{f_n}(\xi) S(\xi_0, -\infty) + i\int_{\xi_0}^{t} \tilde{H}^{\text{in}}(t_1) \dfrac{\delta S(t, -\infty)}{\delta f_n(\xi)}\, dt_1, & \text{when } t > \xi_0, \\ 0, & \text{when } t < \xi_0. \end{cases} \tag{15}$$

Multiplying Eq. (14) by $i\pi_{f_n}(\xi_0) S(\xi_0, -\infty)$ on the left, and setting $t' = \xi$, we get

$$iS(t, \xi_0)\tilde{\pi}_{f_n}(\xi_0) S(\xi_0, -\infty) \tag{16}$$

$$= i\tilde{\pi}_{f_n}(\xi) S(\xi_0, -\infty)$$

$$+ i\int_{\xi_0}^{t} \tilde{H}^{\text{in}}(t_1) S(t_1, \xi)\, i\pi_{f_n}(\xi_0) S(\xi, -\infty)\, dt_1.$$

Comparing Eqs. (15) and (16), we obtain

$$\frac{\delta S(t, -\infty)}{\delta f(\xi)} \tag{17}$$

$$= \begin{cases} iS(t, \xi_0)\tilde{\pi}_{f_n}(\xi) S(\xi_0, -\infty), & t > \xi_0; \\ 0, & t < \xi_0. \end{cases}$$

But, as is well-known, an arbitrary operator $\tilde{\pi}$ is connected with the corresponding operator in the Heisenberg representation by the relation

$$S^{-1}(t, -\infty)\tilde{\pi} S(t, -\infty) = \pi(t). \tag{18}$$

We get, finally,

$$\frac{\delta S(t, -\infty)}{\delta_{f_n}(\xi)} \tag{19}$$

$$= \begin{cases} iS(t, -\infty)\pi_{f_n}(\xi), & t > \xi_0; \\ 0, & t < \xi_0. \end{cases}$$

It is easy to show that the following general formula exists for functional differentiation with respect to the external source $f_n(\xi)$:

$$\frac{\delta S(\infty)\tilde{F}(x)}{\delta_{f_n}(\xi)} \tag{20}$$

$$= \frac{\delta}{\delta f_n(\xi)} [S(\infty) S^{-1}(t) F(x) S(t)]$$

$$= iS(\infty) P'_{x_0\xi_0} [F(x)\pi_{f_n}(\xi)],$$

where

$$P'_{x_0\xi_0}[F(x)\pi_{f_n}(\xi)] \tag{21}$$

$$= \begin{cases} F(x)\pi_{f_n}(\xi), & \text{if } x_0 > \xi_0; \\ \pm \pi_{f_n}(\xi) F(x), & \text{if } x_0 < \xi_0. \end{cases}$$

The plus sign is used in Eq. (21) when the source f commutes with the operator $F(x)$, and the minus sign in the opposite case [e.g., $f_n = \eta$, $F(x) = \psi$].

* This method was applied in reference 3 for the case of a photon field, and Eq. (15) was obtained, although the S matrix in reference 3 is different from ours (it considers interactions not only with external sources).

[3] R. Utiyma et al, Progr. in Theor. Phys. **81** (1953); K. Yamaraki, Progr. in Theor. Phys. **7**, 449 (1952)

In particular, we get from Eqs. (19) and (20) the formulas

$$\frac{\delta S(\infty)}{\delta\eta(x)} = -iS(\infty)\bar{\psi}(x); \qquad (22)$$

$$\frac{\delta S(\infty)}{\delta\eta(x)\,\delta J(x')} = +S(\infty)P(\psi(x)A(x'));$$

$$\frac{\delta S(\infty)}{\delta\bar{\eta}(x)} = iS(\infty)\psi(x);$$

$$\frac{\delta S(\infty)}{\delta\bar{\eta}(x)\,\delta J(x')} = -S(\infty)P(\psi(x)A(x')); \qquad (23)$$

$$\frac{\delta S(\infty)}{\delta J_\mu(x)} = iS(\infty)A_\mu(x);$$

$$\frac{\delta S(\infty)}{\delta\bar{\eta}(x)\,\delta\eta(x')} = S(\infty)P'(\psi(x)\bar{\psi}(x'));$$

$$\prod_{i=1}^{n}\frac{\delta}{\delta J(x_n)}\frac{\delta}{\delta\bar{\eta}(x)}S(\infty) \qquad (24)$$

$$= (i)^n S(\infty)P\left[\prod_{i=1}^{n}A(x_n)\psi(x)\right] \text{ etc.}$$

With the aid of the formulas obtained for the function derivatives of $S(\infty)$, we can write down the operator equations (5) to (8) in compact form and get the same functional equation for the determination of the operators $S(\infty)$. For this purpose we multiply Eqs. (5) - (8) on the left by $iS(\infty)$: from Eq. (23), we get

$$\left[\gamma_\mu\left(-i\partial_\mu + ie\frac{\delta}{\delta J_\mu(x)}\right) - m\right]\frac{\delta S(\infty)}{\delta\bar{\eta}(x)} \qquad (25)$$

$$= i\eta(x)S(\infty);$$

$$\left(-\Box\frac{\delta S(\infty)}{\delta J_\mu} + \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\frac{\delta S(\infty)}{\delta J_\nu}\right) \qquad (26)$$

$$= +iJ_\mu(x)S(\infty)$$

$$+\frac{ie}{2}\operatorname{Tr}\gamma_\mu\left[\frac{\delta^2}{\delta\bar{\eta}(x)\,\delta\eta(x+\varepsilon)} + \frac{\delta^2}{\delta\bar{\eta}(x)\,\delta\eta(x-\varepsilon)}\right]S(\infty); \quad \varepsilon\to 0;$$

$$\frac{\partial}{\partial x_\mu}\frac{\delta S(\infty)}{\delta J_\mu(x)}\Psi = 0. \qquad (27)$$

The system of operator equations (25) - (27) must be solved for the following boundary conditions:

$$\left[\frac{\delta S(\infty)}{\delta\bar{\eta}_\alpha(x,t)}, \frac{\delta S(\infty)}{\delta\eta_\beta(x',t)}\right]_{+J=0,\,\eta=0} \qquad (28)$$

$$= -\gamma_0\delta_{\alpha\beta}\delta(x-x');$$

$$\left[\frac{\delta S(\infty)}{\delta\eta(x,t)}, \frac{\delta S(\infty)}{\delta\eta(x',t)}\right]_{+J=0,\,\eta=0}$$

$$= \left[\frac{\delta S(\infty)}{\delta\bar{\eta}(x,t)}, \frac{\delta S}{\delta\bar{\eta}(x',t)}\right]_+ = 0;$$

$$\left[\frac{\delta S(\infty)}{\delta J_\mu(x,t)}, \frac{\partial}{\partial t}\frac{\delta S(\infty)}{\delta J_\nu(x',t)}\right]_{J=0,\,\eta=0} \qquad (29)$$

$$= \delta_{\mu\nu}\delta(x-x');$$

$$\left[\frac{\delta S(\infty)}{\delta J_\mu(x,t)}, \frac{\delta S}{\delta\eta(x',t)}\right]_-$$

$$= \left[\frac{\delta S(\infty)}{\delta J_\mu(x,t)}, \frac{\delta S}{\delta\bar{\eta}(x',t)}\right]_- = 0;$$

$$\Psi^* S^*(\infty)S(\infty)\Psi = 1.$$

From Eqs.(22) - (24) it is easy to prove that if the matrix element of transition from $S(\infty)$ is found in the vacuum state for $t = -\infty$ and for $y = +\infty$, then we find the matrix elements of the transitions in the successive functional differentiations with respect to the external sources. Therefore, instead of solving the operator equations (25) - (28), it suffices to solve the system of equations obtained from these equations for the matrix element of transition vacuum-vacuum*:

$$(\Phi_0^*(+\infty)\Phi_0(-\infty)) \qquad (30)$$

$$= (\Psi_0^*(-\infty)S(\infty)\Psi_0(-\infty)) = Z.$$

From Eqs. (25) - (27) and (30), we obtain

$$\left[\gamma_\mu\left(-i\partial_\mu + ie\frac{\delta}{\delta J_\mu}\right) + m\right]\frac{\delta Z}{\delta\bar{\eta}(x)} = i\eta(x)Z; \qquad (31)$$

$$\left(+i\partial_\mu + ie\frac{\delta}{\delta J_\mu}\right)\frac{\delta Z}{\delta\eta}\gamma_\mu \qquad (31a)$$

$$+ m\frac{\delta Z}{\delta\eta} = -i\bar{\eta}(x)Z;$$

$$-\Box\frac{\delta Z}{\delta J_\mu} + \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\frac{\delta Z}{\delta J_\nu} = iJ_\mu(x)Z \qquad (32)$$

$$+\lim_{\varepsilon\to 0}\frac{ie}{2}\operatorname{Tr}\gamma_\mu\left[\frac{\delta^2 Z}{\delta\eta(x)\,\delta\eta(x-\varepsilon)} + \frac{\delta^2 Z}{\delta\eta(x-\varepsilon)\,\delta\eta(x)}\right];$$

* By functional differentiation with respect to external sources [see Eqs. (22) - (24)], and by then setting $\eta = J = 0$, we can obtain all the matrix elements for the effects, with consideration of all radiative corrections, from Eq. (30).

$$\frac{\partial}{\partial x_\mu}\frac{\delta Z}{\delta J_\mu}=0. \qquad (33)$$

From Eqs. (30) - (33) it is easy to obtain the system of equations suggested by Schwinger[1]. For this purpose, we consider that the matrix element of an arbitrary operator $F(x)$ (in the notation of Schwinger) is

$$\langle F(x)\rangle=\frac{\Psi_0^* S(\infty)F(x)\Psi_0}{\Psi_0^* S(\infty)\Psi_0}, \qquad (34)$$

where Ψ_0 describes the vacuum state. In view of this equation, Eqs. (31) - (34) take on the form

$$\gamma_\mu\left(-i\partial_\mu - e\langle A_\mu\rangle + ie\frac{\delta}{\delta J_\mu}\right)\langle\frac{\delta Z}{\delta\eta(x)}\rangle \qquad (35)$$

$$+m\langle\frac{\delta Z}{\delta\bar\eta(x)}\rangle=i\eta(x);$$

$$\left(i\partial_\mu - e\langle A_\mu\rangle + ie\frac{\delta}{\delta J_\mu}\right)\langle\frac{\delta Z}{\delta\eta(x)}\rangle\gamma_\mu \qquad (36)$$

$$+m\langle\frac{\delta Z}{\delta\eta(x)}\rangle=-i\bar\eta(x);$$

$$\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}-\Box\delta_{\mu\nu}\right)\langle\frac{\partial Z}{\partial J_\mu}\rangle \qquad (37)$$

$$=iJ_\mu(x)+\lim_{\varepsilon\to 0}\frac{ie}{2}\,\mathrm{Tr}\,\gamma_\mu$$

$$\langle\left[\frac{\delta^2}{\delta\bar\eta(x-\varepsilon)\,\delta\eta(x)}+\frac{\delta^2}{\delta\bar\eta(x)\,\delta\eta(x-\varepsilon)}\right]Z\rangle;$$

$$\frac{\partial}{\partial x_\mu}\langle\frac{\delta Z}{\delta J_\mu}\rangle=0 \qquad (38)$$

with boundary conditions

$$\langle\frac{\delta Z}{\delta\eta(x)}\rangle=\langle\frac{\delta Z}{\delta\eta(x)}\rangle=\langle\frac{\delta Z}{\delta J_\mu}\rangle=0, \qquad (38a)$$

when

$$J=0;\qquad \eta=0;\qquad \bar\eta=0$$

and with the condition that

$$\left.\frac{\delta^2 Z}{\delta\eta(x)\,\delta\eta(x')}\right|_{\eta=0\ J=0};\qquad \left.\frac{\delta^2 Z}{\delta J_\mu(x)\,\delta J_\nu(x')}\right|_{J=0,\eta=0}$$

take on only positive frequencies $x_0>x_0'$ and only negative frequencies for $x_0<x_0'$.

Following Schwinger, we introduce the following definitions:

$$-i\frac{\delta}{\delta\eta(x')}\langle\frac{\delta Z}{\delta\eta(x)}\rangle\Big|_{\eta=0}=G(x,x')$$

$$=i\,\langle P'(\psi(x),\bar\psi(x'))\rangle;$$

$$-i\frac{\delta}{\delta J_\mu(x')}\langle\frac{\delta Z}{\delta J_\nu(x)}\rangle=D_{\mu\nu}(x,x')$$

$$=i\,[P\langle A_\mu(x)A_\nu(x')\rangle-\langle A_\mu(x)\rangle\langle A_\nu(x')\rangle].$$

G and D are the Green's functions of the electron and photon fields, respectively, From Eqs. (35) - (38) we obtain the following system of equations:

$$\left[\gamma_\mu(-i\partial_\mu - e\langle A_\mu\rangle)+ie\gamma_\mu\frac{\delta}{\delta J_\mu}\right] \qquad (39)$$

$$\times\ G(x,x')+mG(x,x')=\delta(x-x');$$

$$\left[i\partial_\mu' - e\langle A_\mu(x')\rangle + ie\frac{\delta}{\delta J_\mu(x')}\right] \qquad (40)$$

$$\times\ G(x,x')\gamma_\mu - mG(x,x')$$

$$=-\delta(x-x');$$

$$\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}-\Box\delta_{\mu\nu}\right)\langle A_\mu\rangle \qquad (41)$$

$$=J_\mu+\lim_{x'\to x}\mathrm{Tr}\,\gamma_\mu\left[\frac{ie}{2}(G(x,x')+G(x',x))\right];$$

$$\left(-\Box\delta_{\mu\nu}+\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\lambda}\right)D_{\lambda\nu}(x,x_1) \qquad (42)$$

$$=\delta_{\mu\nu}\,\delta(x-x_1)$$

$$+\lim_{x'\to x}\frac{\delta}{\delta J_\nu(x_1)}\mathrm{Tr}\,\gamma_\mu\left[\frac{ei}{2}G(x,x')+G(x',x)\right];$$

$$\frac{\partial}{\partial x_\nu}D_{\mu\nu}=0.$$

Here $G(x,x')$ and $D(x,x')$ contain only positive frequencies for $x_0>x_0'$ and only negative frequencies for $x_0<x_0'$; at the point where $x_0=x_0'$, these functions are defined as the half sum of the quantities for $x_0'=x_0\pm\epsilon$.

We transform to the momentum representation, obtaining

$$B(x_1,x_2,\ldots,x_n) \qquad (43)$$

$$=\frac{1}{(2\pi)^{4(n-1)}}\int\exp\{ip_1x_1-ip_2x_2-\ldots$$

$$-ip_nx_n\}B(p_1,p_2,\ldots,p_n)\,d^4p_1\ldots d^np_n$$

$$(B=A,\ G,\ J,\ D).$$

From Eqs. (39) - (43), we obtain the following system of equations

$$(\hat p+m)G(p,p_1) \qquad (44)$$

$$-e\langle\int\hat A(p-k)\rangle G(k,p_1)\,d^4k$$

$$+\frac{ie}{(2\pi)^4}\int\gamma_\mu \frac{\delta G(p+k,\ p_1)}{\delta J_\mu(k)}d^4k=\delta(p-p_1);$$

$$k^2\langle A_\mu(k)\rangle = J_\mu(k) \tag{45}$$

$$+\frac{ie}{(2\pi)^4}\,\mathrm{Tr}\left\{\int\gamma_\mu G(p+k,\ p)\,d^4p\right\};$$

$$(k^2\delta_{\mu\lambda}-k_\mu k_\lambda)D_{\lambda\nu}(k,\ k_1) \tag{46}$$

$$=\delta_{\mu\nu}\delta(k-k_1)+\frac{ie}{(2\pi)^4}\,\mathrm{Tr}\left\{\int\gamma_\mu\frac{\delta G(p+k,\ p)}{\delta J_\nu(k_1)}\right\}.$$

In what follows, it is advantageous to go over to another form of the equations obtained for the Green's function:

$$(\hat{p}+m)G(p,\ p_1) \tag{47}$$

$$-e\int\langle\hat{A}(p-k)\rangle G(k,\ p_1)\,d^4k$$

$$+\int\sum(p,\ k)\,G(k,\ p_1)\,d^4k=\delta(p-p_1);$$

$$(k^2\delta_{\mu\nu}-k_\mu k_\nu)A_\nu(k) \tag{48}$$

$$=J_\mu(k)+\frac{ie}{(2\pi)^4}\,\mathrm{Tr}\left\{\int\gamma_\mu G(p+k,\ p)\,d^4p\right\};$$

$$(k^2\delta_{\mu\lambda}-k_\mu k_\lambda)D_{\lambda\nu}(k,\ k_1)=\delta_{\mu\nu}\delta(k-k_1)-\int P_{\mu\sigma}(k,\ p)\,D_{\sigma\nu}(p,\ k_1)\,d^4p; \tag{49}$$

where

$$\sum(p,\ k_1)=-\frac{e^2}{(2\pi)^4 i}\int\gamma_\mu\frac{\delta G(p+k,\ k_1)}{\delta J_\mu(k)}G^{-1}(p_1,\ k_1)\,d^4p_1d^4k=$$

$$=-\frac{e^2}{(2\pi)^4 i}\int\gamma_\mu G(p+k,\ p_1)\,\Gamma_\nu(p_1,\ k_1,\ k_2)\,D_{\nu\mu}(k_2,\ k)\,d^4k_2d^4kd^4p_1; \tag{50}$$

$$P_{\mu\nu}(p,\ k_1)=\frac{e^2}{(2\pi)^4 i}\,\mathrm{Tr}\left\{\gamma_\mu\int\frac{\delta G(p+k,\ k)}{\delta e\langle A_\nu(k_1)\rangle}d^4k\right\}=$$

$$=\frac{e^2}{(2\pi)^4 i}\,\mathrm{Tr}\left\{\int\gamma_\mu G(p+k,\ k_2)\,\Gamma_\nu(k_2,\ k_3,\ k_1)\,G(k_3,\ k)\,d^4k\,d^4k_2d^4k_3\right\}; \tag{51}$$

$$\Gamma_\mu(p,\ k,\ k_1)=-\frac{\delta G^{-1}(p,\ k)}{\delta e\langle A_\mu(k_1)\rangle}=\gamma_\mu\delta(p-k-k_1)-\frac{\delta\Sigma(p,\ k)}{\delta e\langle A_\mu(k_1)\rangle}\,. \tag{52}$$

From this set of equations we can, in particular, obtain an infinite set of coupled equations by means of successive differentiation of Eqs. (47) - (52) with respect to $A_\mu(k)$.

In the frequent case when the external sources are absent ($J=0$; $\eta=0$), this sytem of coupled equations has the form[4]:

$$\{\hat{p}+m+\varepsilon^{(0)}(p)\}\,G^0(p)=1; \tag{53}$$

$$\{(k^2\delta_{\mu\rho}-k_\mu k_\rho)+P^0_{\mu\rho}(k)\}\,D^0_{\rho\nu}(k)=1; \tag{54}$$

$$\varepsilon^{(n)}_{\mu_1,\ \dots,\ \mu_n}\left(p,\ p-\sum_{m=1}^{n}s_m,\ s_1,\ \dots,\ s_n\right) \tag{55}$$

$$\times\ -\frac{e^2}{(2\pi)^4 i}\int\gamma_\mu\sum_{m,\ k=0}^{n}{}^{*}\,G^{(m)}$$

$$\times\left(p+k,\ p+k-\sum_{r=1}^{m}s_{i_r},\ \dots,\ s_{i_m}\right)$$

$$\times\ \Gamma^{(k)}_{\nu\mu_{i_{m+1}}\dots\mu_{i_{m+k}}}\left(p+k-\sum_{r=1}^{m}s_{i_r},\ p-\sum_{r=1}^{n}s_r,\right.$$

$$\left.k+\sum_{r=m+k+1}^{n}s_{i_r},\ s_{i_{m+1}},\ \dots,\ s_{i_{m+k}}\right)\times D^{(n-m-k)}$$

$$\times\left(k+\sum_{r=m+k+1}^{n}s_{i_r},\ k,\ s_{i_{m+k+1}},\ \dots,\ s_{i_n}\right)d^4k;$$

$$\Gamma^{(n)}_{\mu,\ \mu_1\dots\mu_n}\left(p+k,\ p-\sum_{m=1}^{n}s_m,\ k,\ s_1\dots s_n\right) \tag{56}$$

$$=\gamma_\mu\delta_{n0}-\varepsilon^{(n+1)}_{\mu,\ \mu_1\dots\mu_n}$$

$$\times\left(p+k,\ p-\sum_{m=1}^{k}s_m,\ k,\ s,\ \dots,\ s_m\right);$$

$$G^{(n)}_{\mu_1,\dots,\mu_n}\left(p,\ p-\sum_{r=1}^{n}s_r,\ s_1\dots s_n\right) \tag{57}$$

$$=\sum_{m=0}^{n}{}^{*}\,G^{(m)}_{\mu_{i_1}\dots\mu_{i_m}}\left(p,\ p-\sum s_{i_r},\ s_{i_1}\dots s_{i_m}\right)$$

$$\times\Gamma^{(n-m-1)}_{\mu_{i_{m+1}}\mu_{i_n}}\left(p-\sum_{r=1}^{m}s_{i_r},\right.$$

$$\left.p-\sum_{r=1}^{n}s_r;\ s_{i_{m+1}}\dots s_{i_n}\right)G^{(0)}\left(p-\sum_{r=1}^{n}s_r\right);$$

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[4] B. L. Ioffe, Compt. Rend., Acad. Sci., USSR 95, 761 (1954)

$$D^{(n)}_{\rho\nu\mu,\ \mu_n}\left(p,\ p-\sum_{r=1}^{n} s_r,\ s_1\ldots s_n\right) \tag{58}$$

$$=-\sum^{*} D^{(m)}_{\mu_{i_1}\ldots\mu_{i_m}}\left(p,\ p-\sum_{n=1}^{m} s_{i_r},\quad s_{i_1}\ldots s_{i_m}\right)$$

$$\times P^{(n-m)}_{\rho\nu\mu_{i_{m+1}}\ldots\mu_{i_n}}\left(p-\sum_{r=1}^{m} s_{i_r},\ p-\sum_{r=1}^{n} s_r,\ s_{i_{m+1}}\ldots s_{i_n}\right)D^{(0)}\left(p-\sum_{r=1}^{n} s_r\right);$$

$$P^{(n)}_{\nu\rho\mu_1\ldots\mu_n}\left(p,\ p-\sum s_{i_n},\ s_1\ldots s_{i_n}\right)= \tag{59}$$

$$=\frac{e^2}{i\,(2\pi)^4}\operatorname{Tr}\left\{\int\gamma_\nu\sum^{*} G^{(m)}_{\mu_{i_1}\ldots\mu_{i_m}}\left(p+k,\ p+k-\sum_{r=1}^{m} s_{i_r},\ s_{i_1}\ldots s_{i_m}\right)\right.$$

$$\times\Gamma^{(k)}_{\rho\mu_{i_m}\ldots\mu_{i_{m+k}}}\left(p+k-\sum_{r=1}^{n} s_{i_r},\ k+\sum_{r=1}^{n} s_{i_r},\ p-\sum_{r=1}^{n} s_r,\ s_{i_{m+1}}\cdot s_{i_{m+k}}\right)$$

$$\times G_{\mu,\ m+k+1}\left(k+\sum_{r=m+k+1}^{n} s_{i_r},\ k,\ s_{i_{m+k+1}}\ldots s_{i_n}\right)d^4k,$$

where $\sum\limits_{r=1}^{m}{}^{*} f(s_{i_r})$ denotes that all the variables i_r take on values from $r=1$ to $r=m$, but such that

$$i_1\neq i_r\neq\ldots\neq i_m.$$

In drawing up such a program of renormalization, we transform to new variables G', $D'_{\mu\nu}$, Γ'_μ, A'_μ, J'_μ, e', ψ', η', $\bar\psi'$, $\bar\eta'$ which are related to the former variables in the following way:

$$G'=\frac{G}{Z_2};\qquad D'_{\mu\nu}=\frac{D_{\mu\nu}}{Z_3}; \tag{60}$$

$$\Gamma'_\mu=Z_1\Gamma_\mu;$$

$$\langle A'_\mu\rangle=\frac{\langle A_\mu\rangle}{Z_3^{1/2}};\ \psi'=\frac{\psi}{Z_2^{1/2}},$$

$$\bar\psi'=\frac{\bar\psi}{Z_2^{1/2}};\ \eta'=Z_2^{1/2}\eta;\ \bar\eta'=Z_2^{1/2}\bar\eta;\ J'_\mu=Z_3^{1/2}J_\mu.$$

The renormalization reduces to a choice of definite values for the constants Z_1, Z_2, Z_3 and to the substitution for m and e the experimentally observed values for these quantities. This leads to the impositon of the following conditions on the solution of the renormalized equations in the absence of external sources ($J=0$, $\eta=0$).

1) The Green's function for the electron must have a first order pole at the momentum $p=-m_{exp}$, where m_{exp} is the experimental value of the electronic mass. The constant Z_2 must be so chosen that it would follow from the equation for G' that

$$G'(p)\to\frac{1}{\hat p+m_{exp}},\quad\text{when}\quad \hat p\to -m_{exp}. \tag{61}$$

2) The Green's function of the photon has a pole for momentum $k^2=0$. The constant Z_3 is so chosen that it would follow from the equation for G' that

$$D'_{\mu\mu}(k)\to 1/k^2\qquad\text{for}\qquad (k^2)=0. \tag{62}$$

3) It follows from the relativistic invariance of the theory that for $\hat p=\hat p_0=-m_{exp}$ the quantity $\Gamma_\mu(p^0,p^0,0)$ is proportional to γ_μ.

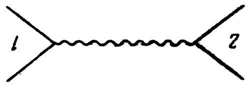
We choose the constant Z_1 so that it would follow from the equation for Γ that

$$\Gamma'(p^0,\ p^0,\ 0)=\gamma_\mu. \tag{63}$$

For these conditions it is not difficult to prove that $e'=Z_1^{-1}Z_2Z_3^{1/2}e$ is equal to the experimental charge e_{exp}. Actually, let us consider any experiment with whose help the value of the electric charge is determined (for example, the scattering of electrons of small momenta). This process is described, to a first approximation, by the excitation theory diagram cited. The matrix element in this approximation can be written in the form

$$e^2\,(\psi_1^{*(0)}\psi_2^{*(0)}\gamma_\mu D^{(0)}_{\mu\nu}\gamma_\nu\psi_1^{(0)}\psi_2^{(0)}). \tag{64}$$

Calculation of the radiation corrections reduces to the replacement of the zeroth approximation



ψ^0, γ_μ, $D^{(0)}_{\mu\nu}$ by the values of these quantities with account taken of the radiation corrections to them, i.e., ψ, Γ_μ and $D_{\mu\nu}$, respectively. However, as follows from Eqs. (60) - (63), for such momenta of the electrons, when they are almost free ($-\hat{p} \approx m_{exp}$) and quanta of low energy are exchanged (the momentum of the photons $k \approx 0$), consideration of the radiation corrections does not change the type of matrix element of the zeroth approximation, but only leads to a replacing of e by $e' = Z_1^{-1} Z_2 Z_3^{1/2} e$ in Eq. (64) in the exact matrix element $\bar{e}'$. Comparing the predictions of the theory with experiment, we determine the experimental value of the charge which appears in front of the brackets of the form (64) in the exact matrix element, and this constant is e', as we have shown *.

From Eqs. (47) - (52) and (60), we obtain the following system of equations:

$$Z_2(\hat{p} + m)G'(p, p_1) - e_{exp} Z_1 \int \langle \hat{A}(p-k)\rangle G'(k, p_1)\, d^4k + \int \sum{}'(p, k)\, G'(k, p_1)\, d^4k = \delta(p - p_1); \tag{65}$$

$$Z_3\{k^2\delta_{\mu\nu} - k_\mu k_\nu\}\langle A'_\mu(k)\rangle = J'_\mu(k) + \frac{ie_{exp}}{(2\pi)^4} Z_1 \operatorname{Tr}\left\{\int \gamma_\mu G'(p+k, p)\, d^4p\right\}; \tag{66}$$

$$Z_3\{k^2\delta_{\mu\lambda} - k_\mu k_\lambda\} D'_{\lambda\nu}(k, k_2) + \int P'_{\mu\lambda}(k, p)\, D'_{\lambda\nu}(p, k_1)\, d^4p = \delta(k - k')\,\delta_{\mu\nu}; \tag{67}$$

$$P'_{\mu\nu}(p, k_1) = \frac{e^2_{exp}}{(2\pi)^4 i} Z_1 \operatorname{Tr}\left\{\int \gamma_\mu G'(p+k, k_2)\,\Gamma'_\nu(k_2, k_3, k_1) \times G'(k_3, k)\, d^4k_2\, dk_3\, dk;\right. \tag{68}$$

$$\sum{}'(p, k_1) = -\frac{e^2_{exp}}{(2\pi)^4 i} Z_1 \int \gamma_\mu G'(p+k, p_1)\,\Gamma'_\nu(p_1, k_1, k_2)\, D'_{\nu\mu}(k_2 k)\, d^4k_2\, dp_1\, dk; \tag{69}$$

$$\Gamma'_\mu(p, k, k_1) = Z_2\gamma_\mu\delta(p - k - k_1) - \frac{\delta\Sigma'(p, k)}{\delta e'_{exp}\langle A'_\mu(k_1)\rangle}. \tag{70}$$

The constants Z_1, Z_2, Z_3 are determined from the conditions (61) - (63). To find them, it is appropriate to write down the set (65) - (70) for the case $\eta = 0$, $J = 0$:

$$\left[Z_2(\hat{p} + m) + \sum{}'(p)\right] G'(p) = 1; \tag{71}$$

$$[Z_3(k^2\delta_{\mu\rho} - k_\mu k_\rho) + P'_{\mu\rho}(k)]\, D'_{\rho\nu}(k) = \delta_{\mu\nu}; \tag{72}$$

$$\sum{}'(p) = \frac{-e^2_{exp}}{(2\pi)^4 i} \times Z_1 \int \gamma_\mu G'(p+k)\,\Gamma'_\nu(p+k, p, k)\, D'_{\nu\mu}(k)\, d^4k; \tag{73}$$

$$P'_{\mu\nu}(p) = \frac{e^2}{(2\pi)^4 i} Z_1 \times \operatorname{Tr}\left\{\int \gamma_\mu G'(p+k)\,\Gamma_\nu(p+k, k, p)\, G(k)\, d^4k\right\}; \tag{74}$$

$$\Gamma'_\mu(p+k, p, k) = Z_1\gamma_\mu - \sum{}^{(1)}_\mu(p+k, p, k) \text{ and so on,} \tag{75}$$

where

$$\sum{}^{(1)}_\mu = \frac{\delta\Sigma'}{\delta e_{exp}\langle A'_\mu\rangle} \quad \text{for} \quad J = 0. \tag{76}$$

From Eq. (71) we have

$$G'(p) = \frac{1}{Z_2(\hat{p} + m) + \Sigma'(p)} \tag{77}$$

When $-\hat{p} \to m_{exp}$, expanding $\Sigma'(p)$ in a series in $\hat{p} + m_{exp}$ we get .

$$G'(p)|_{-\hat{p}\to m_{exp}} \to \frac{1}{Z_2(\hat{p}+m) + \Sigma'(p^0) + \dfrac{\partial\Sigma'(p^0)}{\partial p^0}(\hat{p}+m_{exp})}. \tag{78}$$

* It is a reflection of this fact that in the renormalized equations (see below) for the almost free electrons ($-\hat{p} \approx m_{exp}$) in the interaction with low energy quanta (momentum $k \approx 0$), there is no radiation correction. As a coupling constant in these equations, we have not e but e'; to which, consequently, it is necessary to add the value of the experimental charge.

On the other hand, in accord with Eq. (61) this same quantity must equal $1/(\hat{p} + m_{exp})$, whence we obtain

$$m = m_{exp} - \frac{\Sigma'(p^0)}{Z_2}; \quad Z_2 = 1 - \frac{\partial \Sigma'(p^0)}{\partial \hat{p}^0}. \quad (79)$$

Analogously, it follows from Eqs. (62) and (72) that*

$$Z_3 = 1 - \frac{1}{2}\frac{\partial^2 P_{\mu\mu}(k^0)}{\partial k_\nu^0 \partial k_\nu^0}\bigg|_{\hat{k}^0=0} \quad (80)$$

$$= 1 - \frac{\partial P_{\mu\mu}(k^0)}{\partial (k^0)^2}\bigg|_{(k^0)^2=0}$$

Here we have considered the circumstance that 1) $P_{\mu\nu}(k^0)$ is equal to zero (the proper mass of the photon) from the gradient invariance of the theory; 2) from the law of conservation of the Dirac current and the gradient invariance, $P_{\mu\nu}(k) = B(k^2) \times (k^2\delta_{\mu\nu} - k_\mu k_\nu)$, where $B(k^2)$ is some function of the square of the momentum and therefore the first term of the expansion in powers of k is equal to zero for $k = 0$.

Finally, from Eqs. (75) and (63), we get

$$Z_1\gamma_\mu = \gamma_\mu + \Sigma_\mu^{(1)}(p^0, p^0, 0); \quad (81a)$$

$$Z_3 = 1 + \frac{\gamma_\mu}{4}\Sigma_\mu^{(1)}(p^0, p^0, 0). \quad (81b)$$

Substituting Eqs. (79) - (81) in the infinite set of coupled equations (53) - (59), we get the following set of renormalized coupled equations (we omit the primes):

$$[\hat{p} + m_{exp} + \varepsilon_1^{(0)}(p)]\, G(p) = 1; \quad (82)$$

$$[k^2\delta_{\mu\rho} - k_\mu k_\rho + P_{1\mu\rho}^{0\,(k)}]\, D_{\rho\mu} = 1; \quad (83)$$

$$\varepsilon_1^{(0)}(p) = \varepsilon^{(0)}(p) - \varepsilon^{(0)}(\hat{p}^0) \quad (84)$$

$$-\frac{\partial \varepsilon^{(0)}(p^0)}{\partial \hat{p}^0}(\hat{p} - m_{exp})|_{-\hat{p}^0 = m_{exp}};$$

$$P_{1\mu\rho}^{(0)}(k) = P_{\mu\nu}(k) - P_{\mu\nu}(0) - k_\lambda \frac{\partial P_{\mu\nu}(k^0)}{\partial k_\lambda^0} \quad (85)$$

$$-\frac{1}{2}k_\lambda k_\rho \frac{\partial P_{\mu\nu}}{\partial k_\lambda^0 \partial k_\rho^0}\bigg|_{(k^0)^2=0};$$

$$\Gamma_\mu^{(0)}(p + k, p, k) = \gamma_\mu \quad (86)$$

$$-\varepsilon^{(1)}(p + k, p, k) + \varepsilon^{(1)}(p^0, p^0, 0);$$

$$\varepsilon_{\mu_1\ldots\mu_n}^{(n)}\Big(p, p - \sum_{m=1}^{n} s_{i_n}, s_1 \ldots s_n\Big) \quad (87)$$

$$= -\frac{e_{exp}^2}{(2\pi)^4 i} Z_1 \int \gamma_\mu \sum_{m,k=0}^{n}{}^{*} G^*\Big(p + k, p + k - \sum_{r=1}^{m} s_{i_r}, s_{i_r} \ldots s_{i_m}\Big) \times \Gamma_{\nu\mu_{i_{m+1}}\ldots\mu_{i_{m+k}}}^{(k)}\Big(p + k - \sum_{r=1}^{m} s_{i_r}, p - \sum_{r=0}^{n} s_{i_r},$$

$$k + \sum_{r=m+k+1}^{n} s_{i_r}, s_{i_{m+1}} \ldots s_{i_{m+h}}\Big) \times D_{\mu\nu}^{(n-m-k)}$$

$$\times\Big(k + \sum_{r=m+n+1}^{n} s_{i_r}, k, s_{i_{m+k+1}} \ldots s_{i_n}\Big) d^4k;$$

$$\Gamma_{\mu,\mu_1\ldots\mu_n}^{(n)}\Big(p + k, p - \sum_{m=1}^{n} s_m, k, s_1 \ldots s_n\Big) \quad (88)$$

$$= \gamma_\mu\delta_{n_0} - \varepsilon_{\mu,\mu_1\ldots\mu_n}^{(n+1)}\Big(p + k, p - \sum_{m=1}^{n} s_m, k, s_1 \ldots s_n\Big) + \varepsilon^{(1)}(p^0, p^0, 0);$$

$$G_{\mu_1\ldots\nu_n}^{(n)}\Big(p, p - \sum_{r=1}^{n} s_r, s_1 \ldots s_n\Big) \quad (89)$$

$$= \sum_{m=0}^{n}{}^{*} G_{\mu_{i_1}\ldots\mu_{i_m}}^{(m)}\Big(p, p - \sum_{r=1}^{n} s_{i_r}, s_{i_1} \ldots s_{i_m}\Big) \times \Gamma_{\mu_{i_{m+1}}\ldots\mu_{i_n}}^{(n-m+1)}\Big(p - \sum_{r=1}^{m} s_{i_r},$$

$$p - \sum_{r=1}^{n} s_r, s_{i_{m+1}} \ldots s_{i_n}\Big) G^{(1)}\Big(p - \sum_{r=1}^{n} s_r\Big);$$

$$D_{\rho\nu\mu_1\ldots\mu_n}^{(n)}\Big(p, p - \sum_{r=1}^{n} s_r, s_1 \ldots s_n\Big) \quad (90)$$

$$= -\sum{}^{*} D_{\mu_{i_1}\ldots\mu_{i_m}}\Big(p, p - \sum_{r=1}^{m} s_{i_r}, s_{i_1} \ldots s_{i_m}\Big) \times P_{\rho\nu\mu_{i_{m+1}}\ldots\mu_{i_n}}^{(n-m-1)}\Big(p - \sum_{r=1}^{m} s_{i_r},$$

$$p - \sum_{r=1}^{n} s_r, s_{i_{m+1}} \ldots s_{i_n}\Big) D^{(0)}\Big(p - \sum_{r=1}^{n} s_r\Big);$$

* In Eq. (78) the summation is not carried out over the repeated indices μ.

$$P^{(n)}_{\rho\nu\mu_1\ldots\mu_n}\Big(p,\ p-\sum_{m=1}^{n} s_{i_n},\ s_1\ldots s_n\Big) \tag{91}$$

$$= -\frac{e^2_{\rm exp}}{(2\pi)^4 i} Z_1 \,\mathrm{Sp}\Big\{\int\gamma_\rho \sum^{*}_{m,\,k=0} G\Big(p+k,\ p+k-\sum_{r=1}^{m} s_{i_r},\ s_{i_1}\ldots s_{i_n}\Big)$$

$$\times\Gamma^{(k)}_{\nu_{i_{m+1}}\ldots\mu_{i_{m+k}}}\Big(p+k-\sum_{r=1}^{m} s_{i_r},\ k+\sum_{r=m+k+1}^{n} s_{i_r},\ p-\sum_{r=1}^{n} s_r,\ s_{i_{n+1}}\ldots s_{i_n}\Big)$$

$$\times G^{(n-m-k)}_{\mu_{i_{m+k+1}}\ldots\mu_{i_n}}\Big(k+\sum_{r=m+k+1}^{n} s_{i_r},\ k,\ s_{i_{m+k+1}}\ldots s_{i_n}\Big)\,d^4k\Big\}.$$

From Eqs. (81) - (91), we obtain the following equation for Z_1*

$$Z_1^{-1} = 1 + \frac{e^2_{\rm exp}}{4(2\pi)^4 i} \tag{92}$$

$$\times\int\gamma_\mu\gamma_\rho\,[G^{(0)}(\hat{p}+\hat{k})\,\Gamma_\rho(\hat{p}^0+\hat{k};\ \hat{p}^0+\hat{k},\ 0)$$

$$\times G^{(0)}(\hat{p}_0+\hat{k}^0)\,\Gamma^0_\nu(\hat{p}^0+\hat{k},\ p^0,\ k)\,D^{(0)}_{\mu\nu}(k)$$

$$+G^0(\hat{p}^0+\hat{k})$$

$$\times\Gamma^{(1)}_{\mu\nu}(\hat{p}^0+\hat{k},\ p^0,\ k,\ 0)\,D^{(0)}_{\mu\nu}(k)]\,d^4k.$$

Substituting the quantities Z_1, Z_2, Z_3 in Eqs. (65) - (70), we obtain the set of renormalized equations produced in reference 2.

For the effective exclusion of overlapping infinities of Z_1, it is necessary to express Γ_μ, $\partial G^{-1}/\partial p_\mu$ and $\partial D^{-1}/\partial k_\mu$ by the corresponding (uncited) graph (the role of the cited graphs is that they reduce the γ_μ which enter in the uncited graph to Γ_μ); the very automaticity also effectively leads to the exclusion of the overlapping infinities. In this case we introduce the following quantities*:

$$R^{(J)}_{\mu\lambda}(p,\ s\,|\,p_1,\ s_1) = \frac{e^2}{(2\pi)^4 i}\int\frac{\delta}{\delta e_{\rm exp}\langle A_\mu(s)\rangle} \tag{93}$$

$$\times[G(p,\ k)\,\Gamma_\nu(k,\ p_1,\ k_1)\,D_{\nu\lambda}(k_1,\ s_1)]^{(J)}d^4k\,d^4k_1 - \frac{e}{(2\pi)^4 i}\int\frac{\delta}{\delta e_{\rm exp}A_\rho(-k_3)}$$

$$\times[G(p,\ k)\,\Gamma_\nu(k,\ k_1,\ k_2)\,D_{\nu\lambda}(k_2,\ s_1)]^{(J)}R^{(J)}_{\mu\rho}(k_1,\ k_3|p_1,s)\,d^4k\,d^4k_1 d^4k_2\,d^4k_3;$$

$$K_{\mu\lambda}(p,\ p-k,\ k) = -\frac{e^2}{(2\pi)^4 i}\frac{\partial}{\partial p_\mu}[G(p)\,\Gamma_\nu(p,\ k)\,D_{\nu\lambda}(k)] \tag{94}$$

$$+\frac{e^2}{(2\pi)^4 i}\int[G(p)\,\Gamma_\rho(p,\ -k_1)\,G(p+k_1)\,\Gamma_\nu(p+k_1,\ k) + G(p)\,\Gamma^{(1)}_{\nu\rho}(p,\ k,-k_1)]$$

$$\times D_{\nu\lambda}(k)\,K_{\mu\rho}(p+k_1-k,\ k_1,\ p-k);$$

$$Y_{\lambda\mu}(p+k,\ k) = \frac{\partial}{\partial k_\mu}[G(p+k)\,\Gamma_\lambda(p+k,\ k)] \tag{95}$$

$$-\int[G(p+k)\,\Gamma_\rho(p+k,\ -k_1)\,G(p+k+k_1)\,\Gamma_\nu(p+k+k_1,\ k)$$

$$+G(p+k)\,\Gamma^{(1)}_{\rho\nu}(p+k,\ k,\ -k_1)]\,D_{\nu\lambda}(k)\,Y_{\lambda\mu}(k_1+p,\ k_1)\,d^4k_1.$$

The set of renormalized equations then takes the following form:

$$\frac{\partial G^{-1}(p)}{\partial p_\mu} = \gamma_\mu + \int\{\Gamma_\lambda(p,\ -k)\,K_{\lambda\mu}(p+k,\ k,\ p) \tag{96}$$

$$-\Gamma_\lambda(p_0,\ -k)\,K_{\lambda\mu}(p_0+k,\ k,\ p_0)\}\,d^4k;$$

$$\frac{\partial D^{-1}(k)}{\partial k_\mu} = 2k_\mu \tag{97}$$

$$+\frac{e_{\rm exp}}{(2\pi)^4 i}\Big\{\Pi_\mu(\hat{k}) - \Pi_\mu(k_0) - k_{0\sigma}\frac{\partial\Pi_\mu(k_0)}{\partial k_{0\sigma}}\Big\};$$

* Formally, in the brackets in Eq. (92), we must still add one term $G^0(\hat{p}^0+\hat{k})\times\Gamma^0_\nu(p^0+k,\ p_0,\ k)\times D^{(1)}_{\nu\mu\rho}(k,\ k,\ 0)$; however, as was shown in the Appendix, this term is equal to zero.

* Here and below the symbol J denotes that the given quantity is taken for the presence of external photon sources; if this symbol is absent, this means that $J=0$.

$$\Gamma_{\mu}^{(J)}(p_1, p_2, p_3) = \gamma_\mu \delta(p_1 - p_2 - p_3) \quad (98)$$
$$- \int \{\Gamma_{\lambda}^{(J)}(p_1 s, -s_1) R^{(J)}(s, s_1|p_2, p_3) \cdot$$
$$- \Gamma_\lambda(p_0, s, -s_2) \dot{R}_{\mu\lambda}(s, s_1|p_2, 0)\} d^4s\, d^4s_1;$$

$$\Pi_\mu(k) = \frac{1}{3} \operatorname{Tr} \left\{\int \Gamma_\lambda(p, -k) \times Y_{\lambda\mu}(p+k, k)\, d^4p\right\}; \quad (99)$$

where

$$D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) D(k^2) + \frac{k_\mu k_\nu}{k^2} d_l(k^2), \quad (100)$$
$$k_0^2 = 0, \quad p_0^2 = m_{\exp}^2.$$

Here $G^{-1}(p)$ and $D^{-1}(k)$ satisfy the following boundary conditions:

$$(p_0 - m)\, G(p_0) = 1, \quad k_0^2 D(k_0^2) = 1. \quad (101)$$

Choosing the set of equations (73) - (76) along uncited graphs (of the same order in charge) for Γ_μ, Π_μ, $\partial G^{-1}(p)/\partial p_\mu$ we obtain a complete set of renormalized equations.

APPENDIX

a) By way of an example of reduction with functional derivatives, we consider the solution of the set of Eqs. (25), (26) in the absence of interaction. Averaging these equations over all states and transforming to the momentum representation, we get

$$(\hat{p} + m) \frac{\delta \langle S(\infty)\rangle}{\delta \eta(p)} = i\eta(p) \langle S(\infty)\rangle; \quad (A1)$$
$$\frac{\delta \langle S(\infty)\rangle}{\delta \eta(p)} (\hat{p} + m) = -i\bar{\eta}(p) \langle S(\infty)\rangle;$$
$$\Box \frac{\delta \langle S(\infty)\rangle}{\delta J_\mu} = iJ_\mu \langle S(\infty)\rangle.$$

The boundary conditions for $J = 0$, $\eta = 0$ have the form

$$\frac{\delta \langle S(\infty)\rangle}{\delta \bar{\eta}(p)} = i\langle \psi(p)\rangle; \quad (A2)$$
$$\frac{\delta \langle S(\infty)\rangle}{\delta \eta(p)} = -i\langle \psi(p)\rangle;$$
$$\frac{\delta \langle S(\infty)\rangle}{\delta J_\mu} = i\langle A_\mu(p)\rangle.$$

It follows from Eqs. (A1) and (A2) that

$$\langle S(\infty)\rangle = \exp\left[i \int \left\{\bar{\eta}(p) \frac{1}{\hat{p} + m} \eta(p) \right.\right. \quad (A3)$$
$$+ \frac{1}{2} J_\mu(p) \frac{1}{p^2} J_\mu(p) + J_\mu(p)\langle A_\mu(p)\rangle$$
$$\left.\left. + \langle \psi(p)\rangle \eta(p) + \eta(p) \langle \psi(p)\rangle \right\} d^4p\right].$$

In particular, when an average is taken over the vacuum-vacuum states, in accordance with Eq. (38a) we get

$$(\Psi_0^*, S(\infty)\Psi_0) = \exp\left[i \int \left\{\bar{\eta}(p) \frac{1}{\hat{p} + m} \eta(p) \right.\right. \quad (A4)$$
$$\left.\left. + \frac{1}{2} J_\mu(p) \frac{1}{p^2} J_\mu(p)\right\} d^4p\right].$$

Finding a solution for S does not present much difficulty, even in the case of the presence of interaction. The solution is carried out by means of the theory of excitation, assuming a solution in a power series in the charge. In this case, Eq. (A4) plays the role of the zeroth approximation.

b) *The Theory of Ferry.* In terms of the functional derivatives with respect to $\langle A_\mu\rangle$, the theorem of Ferry can be formulated in the following manner: in the absence of external sources ($J = 0$, $\eta = 0$), the odd functional derivatives with respect to $\langle A_\mu\rangle$ from the Green's function of the photons $D_{\mu\nu}$ are equal to zero. In fact, taking into account the charge symmetry of the theory, as is not difficult to show, the polarization operator $P_{\nu\mu}$ [see Eq. (51)]can be written in the form:

$$P_{\mu\nu}(p, k_1) \quad (A5)$$
$$= \frac{e^2}{2(2\pi)^4 i} \operatorname{Sp}\left\{\gamma_\mu \int \frac{\delta}{\delta e \langle A_\nu(k_1)\rangle} [G(p+k, k) - \tilde{G}(p+k, k)]\right\} d^4p.$$

where G is the Green's function of the charge-coupled equation. Here G is defined by an equation analogous to the equation for G [see Eq. (47)] only with this difference, that the charges e are taken with opposite sign. If we take the solution for G in the form of a functional series in $\langle A_\mu\rangle$:

$$G(p, k) = \sum_{n=0}^{\infty} \int e^n G^{(n)}_{\mu_1\ldots\mu_n}(p, k, s_1 \ldots s_n) \quad (A6)$$
$$\times \langle A_\mu(s_1)\rangle \ldots \langle A_{\mu_n}(s_n)\rangle d^4s_1 \ldots d^4s_n,$$

then the solution for $\tilde{G}(p, k)$ will have the form

$$\tilde{G}(p, k) = \sum_{n=0}^{\infty} \int (-e)^n G^{(n)}_{\mu_1\ldots\mu_n}(p, k, s_1 \ldots s_n) \quad (A7)$$
$$\times \langle A_{\mu_1}(s_1)\rangle \ldots \langle A_{\mu_n}(s_n)\rangle d^4s_1 \ldots d^4s_n.$$

From Eqs. (A5), (A6) and (A7) (and taking into account that for $J = 0$, $\eta = 0$), we find that the odd functional derivatives of $P_{\mu\nu}$ with respect to $<A_{\mu}>$ are equal to zero for $J = 0$.

Translated by R. T. Beyer
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