The Scalar Field of a Stationary Nucleon in a Non-linear Theory

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The properties of a spherically symmetrical non-linear scalar potential are investigated in detail, using the method of Chaplygin. An approximate numerical solution is derived, and the relation between self-energy and charge of a nucleon is obtained.

1. INTRODUCTION

NON-LINEAR generalizations of various fields (in particular the Maxwell field and the meson field) follows inescapably both from experimental facts and from the ideas of modern relativistic quantum field theory[1,2]. The polarization of the vacuum, and the transmutability of elementary particles,necessarily lead to non-linear effects. It is possible that a non-linear approach may explain to some extent the detailed structure of elementary particles. For example, one may imagine that the specific repulsive force between nucleons at very short distances[3] may be deduced theoretically from the non-linear equations of the meson field. Moreover, some versions of non-linear field theory[1,4] lead in the classical approximation to finite self-energies and thus solve the problem of particle stability. These theories predict from classical considerations effects which formerly were deduced from quantum-mechanical arguments, for example,the refraction of light by a constant field, and the scattering of light by light or by the field of a point charge[5-7]. In this connection it is clearly of great interest to transfer non-linear methods from electrodynamics into mesodynamics[2], a domain where all phenomena are microscopic and where it seems that non-linear effects should be more prominent.

The non-linear generalization of a classical theory can be made in various ways. For example, one may use the Lagrangian

$$L_N = \frac{b^2}{4\pi}\left(1 - \sqrt{1 - \frac{L}{b^2}}\right) \qquad (1)$$

(see the work of Born and Infeld[4,8], and also reference 2). Here b is a constant, the maximum possible field-strength, and L and L_N are the Lagrangians of the linear and non-linear theory respectively. This Lagrangian can be used either for vector or scalar theory.

A second method is to look for a non-linear Lagrangian which has the form of a series in powers of the field-invariants[1,2]. To start with, one may be content to keep only the first non-linear term in the series, assuming that the remaining terms will give only small corrections. We shall apply this method to scalar meson theory (the results will apply equally to a free pseudo-scalar field). We shall investigate the meson field equation

$$\Box\,\varphi - (k_0^2 + \lambda\varphi^2)\,\varphi = 0, \qquad (2)$$

which is obtained from the Lagrangian

$$L_{NS} = -\frac{1}{8\pi}\left\{\frac{\partial\varphi}{\partial x^\nu}\frac{\partial\varphi}{\partial x^\nu} + k_0^2\varphi^2 + \frac{\lambda}{2}\varphi^4\right\}.$$

This equation has been studied by several authors; it leads to the appearance of stauration in nuclear forces[9,10]. We consider the constant λ to be arbitrary. The theory of the nucleon-antinucleon vacuum and various other arguments indicate that $\lambda > 0$ if the non-linearity is of field-theoretic origin[11,12]. However, several authors[10,13] have found this type of theory to be unsatisfactory, both in its internal structure and in its relation to experiment.

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Other investigators[14-16] have considered the case $\lambda < 0$, which is also possible a priori, treating the non-linearity as a free invention which may lead to interesting consequences. Such a free non-linearity, not connected with quantum effects, and belonging to a purely classical field, may give effects which go outside the framework of ordinary classical and quantum-mechanical models.

Our discussion here will be purely classical. In the static spherically symmetrical case Eq. (2) takes the form

$$\varphi'' + \frac{2}{r}\varphi' - (k_0^2 + \lambda\varphi^2)\varphi = 0; \tag{3}$$

For brevity we write $w(r, \varphi, \varphi', \varphi'')$ for the left side of Eq. (3).

2. GENERAL PROPERTIES OF THE SOLUTION AND CONDITIONS WHICH IT MUST SATISFY

We first formulate the general conditions which the potential must satisfy. We start from the fact that the total energy of the static field must be bounded. The total energy is given by

$$U = \iiint T_{44}\, d\tau = \frac{1}{2}\int_0^\infty \left(\varphi'^2 + k_0^2\varphi^2 + \frac{\lambda}{2}\varphi^4\right) r^2\, dr. \tag{4}$$

As $r \to \infty$ the solution must tend to a Yukawa potential (this first condition follows from the correspondence principle), and so the integral converges at infinity. From the requirement that the integral converge at zero, we deduce that as $r \to 0$ the leading term in φ must have the form $\varphi \sim cr^\alpha$ with $\alpha > -1/2$. This is the second condition on the solution.

From the form of Eq. (3) one can establish the general behavior of the potential φ at the origin. Observe first that our equation is exactly satisfied by the constant potentials[12,14]

$$\varphi_\pm = \pm k_0/\sqrt{-\lambda};\quad \varphi = 0, \tag{5}$$

and so we may have $\varphi' = 0$ at $r = 0$. We suppose that φ is bounded. Then we deduce

$$\lim_{r\to 0}(r\varphi'' + 2\varphi') = 0.$$

Suppose that in the neighborhood of $r = 0$ there exists a series[17] $\varphi' = \sum_{k=0}^{\infty} C_k r^k$; . Then

$$\lim_{r\to 0}\sum_{k=0}^{\infty} 2C_k r^k = -\lim_{r\to 0}\sum_{k=1}^{\infty} kC_k r^k;$$

from which follows $C_0 = 0$, so that if φ is bounded at zero then $\varphi'(0) = 0$. We shall consider the solutions which are not bounded at zero separately for each sign of λ.

Since Eq. (3) is unchanged when the sign of φ is changed, it is sufficient to treat the case where $\varphi \geq 0$ on some interval of values of r.

3. SOLUTIONS OF EQ. (3) WITH $\lambda > 0$

a) *Monotonicity of the Potential φ and Its First Two Derivatives*

Suppose that φ possesses an extremum. Then Eq. (3) at the extremum becomes

$$\varphi'' = (k_0^2 + \lambda\varphi^2)\varphi > 0,$$

showing that the extremum is a minimum. Since the equation has no singular points except $r = 0$, it follows from the first condition (the correspondence principle) that such a solution is to be rejected. Thus, any solution which tends to zero at infinity is monotonic.

Consider the condition for an inflection. At a point of inflection Eq. (3) becomes

$$2\varphi' = r(k_0^2 + \lambda\varphi^2)\varphi > 0,$$

which contradicts the monotone property of φ($\varphi' < 0$ for $r < \infty$). Therefore φ' is also monotonic.

Differentiating Eq. (3) once and substituting for $(k_0^2 + \lambda\varphi^2)$ from Eq. (3), we find

$$r\varphi''' + 2\varphi'' = r\left(k_0^2 + 3\lambda\varphi^2 + \frac{2}{r^2}\right)\varphi' < 0.$$

At an extremum of the second derivative we should have $\varphi'' < 0$, which contradicts the absence of inflections, because $\varphi' < 0$ implies that $\varphi'' > 0$ for sufficiently large r and hence $\varphi'' > 0$ everywhere.

In this way we have proved that a solution φ of Eq. (3) for $\lambda > 0$ which vanishes at infinity is monotonic together with its first two derivatives. From this it follows, by the earlier argument about the behavior at zero, that $\varphi(0) = \infty$.

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b) *Behavior of φ in the Neighborhood of Zero*

We here apply a method due to Chaplygin[18] for the approximate integration of differential equations, only we proceed rather differently from Chaplygin. The method gives, in a very simple and exact way, qualitative information about the solutions of various non-linear equations. In spite of the obvious deficiencies of the method, which were emphasized by Chaplygin, we believe that it has not yet received the attention it deserves. According to the theorem which Chaplygin proved, if y is a solution of the equation

$$y'' - f(x, y, y') = 0,$$

and z is a comparison function satisfying the inequality

$$z'' - f(x, z, z') > 0.$$

in the neighborhood of a given point x_0, with $z_0 = y_0$, $z_0' = y_0'$ at $x = x_0$, then $z > y$ in the neighborhood of x_0. A lower bound for y is obtained in the same way. Chaplygin's method is presumably related in some way to variational methods.

The theorem is valid provided that the equation has no singular points in the region under consideration. In our case we have to investigate $\varphi(r)$ as r tends to zero. Taking for a comparison function the hyperbola

$$y = \frac{C_1}{r} + C_2, \; y' = -\frac{C_1}{r^2},$$

which touches the graph of the potential φ at $r = r_0$, we find according to Chaplygin's method

$$w(r, y, y', y'') = -(k_0^2 + \lambda y^2)\, y < 0,$$

and therefore $\varphi > y$ in the neighborhood of the point r_0 where the two curves touch. As we move the point r_0 closer to zero, we obtain a sequence of hyperbolas which descend more and more steeply (by the monotonicity property) with increasing r. Consider two consecutive hyperbolas; they intersect in a single point which lies between their points of contact. Therefore, the region in which y_{n-1} and φ do not intersect can be extended to the left to include the region in which $\varphi > y_n$. In this process the successive positions of r_0 cannot have a condensation-point other than zero, since otherwise we could apply Chaplygin's theorem at the condensation-point.

Thus it is proved that for $\lambda > 0$ the potential φ increases at the origin not slower than hyperbolically, and hence the requirement of a finite total field-energy is violated.

We make now a further estimate of the behavior of φ at the origin. Writing $x = k_0 r$ and $u = \sqrt{\lambda}\, r\varphi$, Eq. (3) takes the simpler form[15]

$$u'' = u + (u^3 / x^2). \tag{3a}$$

If now we suppose u bounded at zero (from the discussion by Chaplygin's method we know that $u_{x=0} \neq 0$), then for small x we have approximately $u'' = -C/x^2$ and hence $u = -C \ln x + C_1 x + C_2$, which contradicts our hypothesis. If we take for comparison a function $y = C_1 r^{\alpha} + C_2$ with $\alpha < -1$, then the graph of the solution of Eq. (3) lies below the graph of y in the neighborhood of the point where the two curves touch. Therefore, the departure of the potential φ from the Yukawa form in the neighborhood of the origin is expressed by a factor which diverges not more rapidly than logarithmically.

4. DISCUSSION OF EQ. (3) WITH $\lambda < 0$

a) *Existence of Non-monotone Solutions, and Behavior of the Potential at the Origin*

For convenience we write $\mu = -\lambda$. Then Eq. (3) becomes

$$\varphi'' + \frac{2}{r}\varphi' - (k_0^2 - \mu\varphi^2)\,\varphi = 0. \tag{6}$$

When $\varphi' < 0$, $\varphi''' > 0$, the inequality

$$\varphi > k_0 / \sqrt{\mu}. \tag{7}$$

is a necessary condition for either a maximum or an inflection.

To define the behavior of the potential at the origin, we consider first the acceptability of a potential which increases to infinity at the origin, in agreement with the second condition of Section 2. Taking a comparison function of the form $y = C_1 r^{\alpha} + C_2$ with $\alpha = -1/2$, it is easy to prove by Chaplygin's method that the potential increases faster than the comparison function at the origin. This is proved most conveniently by using an equation analogous to Eq. (3a) and making r tend to zero. Therefore, it is sufficient to confine our attention to the case $\varphi'(0) = 0$, which implies that the potential is finite in the whole space.

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b) *Condition for an Inflection when* $\varphi' > 0$, $\varphi''' < 0$

We apply Chaplygin's theorem, taking as comparison function the straight line which touches the curve φ at a point of inflection ρ, namely

$$y = C_1 r + C_2;\ y' = C_1 > 0.$$

Then

$$w(r, y, y', y'') = (2C_1/r) - (k_0^2 - \mu y^2)\, y = z.$$

Assuming that $\varphi' > 0$, $\varphi''' < 0$, the condition for an inflection is

$$z > 0 \quad \text{for} \quad r_+ > \rho; \quad z < 0 \quad \text{for} \quad r_- < \rho. \tag{8}$$

From this it follows at once that $k_0^2 > \mu y^2$. We now find a lower bound for φ. Eq. (8) implies

$$(k_0^2 - \mu y_-^2)\, y_- > (k_0^2 - \mu y_0^2)\, y_0 > (k_0^2 - \mu y_+^2)\, y_+. \tag{9}$$

Here y_0 is the potential at the point of inflection. We substitute in Eq. (9) $y_+ = C_1(\rho + \delta) + C_2 = y_0 + C_1\delta$, $y_0 = C_1\rho + C_2$, and then take the limit $\delta \to 0$, which gives

$$-3\mu y_0^2 + k_0^2 < 0;$$

We obtain the same result from the other half of Eq. (9) by taking $y_- = y_0 - C_1\delta$. Therefore, a necessary condition for such a point of inflection is

$$\frac{k_0}{\sqrt{\mu}} > y_0 > \frac{k_0}{\sqrt{3\mu}}. \tag{10}$$

a) *Investigation of the Solutions of Eq. (6) in a Phase Plane*

When the solutions of Eq. (6) are considered in a phase plane[14], the potential φ becomes the x-coordinate of the moving point, and the distance r from the origin becomes the "time" t. With these notations the equation becomes

$$\ddot{x} = -\frac{2}{t}\dot{x} + (k_0^2 + \lambda x^2)\, x = -\frac{2}{t}\dot{x} - \frac{dv}{dx}.$$

On the right stands a "force", which can be formally separated into the gradient of a potential plus a term which depends on the velocity and on the time. If the term $(2/t)\dot{x}$ were absent, the motion of the point would be conservative and its total energy K would be given by

$$K = \frac{\dot{x}^2}{2} + v(x) = \frac{\dot{x}^2}{2} - \frac{1}{2}\left(k_0^2 + \frac{\lambda}{2}x^2\right)x^2;\ \lambda < 0.$$

so that

$$dK/dt = [\ddot{x} - (k_0^2 + \lambda x^2)\, x]\, \dot{x} = 0.$$

Evidently the system will tend toward this state as $t \to \infty$.

The total energy of the point in its non-conservative motion will satisfy

$$\frac{dK'}{dt} = -\frac{3\dot{x}^2}{t} < 0,$$

as can be seen by considering a generalized potential which depends on x, $\dot{x}$ and t. In Fig. 1 the continuous lines are lines of constant energy for the conservative system. In non-conservative motions the energy will steadily decrease, and the orbits will cut continuous lines, as shown by the dotted lines in Fig. 1. Three families of solutions are obtained. Two of them represent functions which oscillate in space and tend to the special constant solutions (stationary points); these do not tend to zero at infinity. The third family is completely different. It is in the first place discrete since the point $\varphi = 0$, $\varphi' = 0$ is a saddlepoint in the phase plane and does not have stability against small changes of the potential at the origin. Secondly, the solutions of this family tend to zero at infinity. These solutions may be spoken of[15] as "particles"; a particular one which is without nodes has been derived in reference 14.

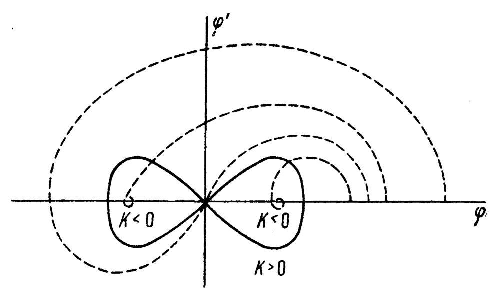

FIG. 1

Summarizing our qualitative investigation, we may say that Chaplygin's method enables us to obtain detailed information about the behavior of a non-linear potential, which is especially important for constructing solutions which change sign as r increases. Solutions with this property are of interest, because they correspond to particles with a repulsive interaction. We have also analyzed the reasons why divergent potentials cannot be used in a theory which requires a finite total field-energy (the divergence is never weak enough). It has been proposed[9] that the divergences in the theory with $\lambda > 0$ and point charges will be weakened because the extended potential has a screening effect on the charges. This proposal is incorrect, although the potential in that case is repulsive.

5. APPLICATION OF THE RITZ METHOD TO FIND AN APPROXIMATE SOLUTION IN EQ. (6)

Eq. (6) is obtained by varying the functional

$$I=\int_0^\infty\left(\varphi'^2+k_0^2\varphi^2-\frac{\mu}{2}\varphi^4\right)r^2dr. \qquad (11)$$

We substitute an approximate solution defined by joining together the functions

$$\varphi_1=a-br^2, \qquad r<\rho;$$
$$\varphi_2=g\left(\frac{e^{-k_0r}}{r}\right), \qquad r>\rho. \qquad (12)$$

The function and its first derivative must be continuous at the join ρ, which is taken as the parameter to be varied. Thus the important condition $\varphi'(0)=0$ is maintained. From the variation of ρ we obtain a relation between ρ and μg^2. With the notation $\epsilon = k_0\rho$, the relation is

$$\mu g^2 = \frac{63+126\varepsilon+111\varepsilon^2+54\varepsilon^3+13\varepsilon^4+2\varepsilon^5}{45{,}7+77{,}7\varepsilon+55{,}6\varepsilon^2+19{,}6\varepsilon^3+3{,}45\varepsilon^4+0{,}2\varepsilon^5}. \qquad (13)$$

Since we cannot a priori attach any definite meaning to the quantity μ or even to the charge g, we proceed to a general discussion of Eq. (13). It is curious that an unbounded solution ($\epsilon = 0$) is obtained already when $g^2 = 1.378$, and not when g^2 tends to zero. Also, if we keep the value of φ at zero bounded for each μ, then g increases without limit as $\mu \to 0$. A similar picture (g decreasing) is obtained when we make $\mu \to \infty$. We find

$$10^{16}\,\mathrm{CGSU} < \mu < 10^{18}\,\mathrm{CGSU},$$

if g is confined within the physically interesting range $5e \le g \le 20e$. For other values of μ (for example greater values) one would have to look for a solution with one or more nodes.

The condition (7) for the possible occurrence of a maximum is easily shown to be satisfied for all physically interesting ϵ ($\epsilon < 1$). This fact is important, because the non-linearity must become significant somewhere "inside" the particle.

6. SELF-ENERGY OF A NUCLEON AT REST

To find the self-energy of a nucleon at rest, we substitute the approximate solution into Eq. (4). Expressing the coefficients a and b in terms of the other parameters by means of the continuity conditions, and carrying out the integrations, we find

$$u=u_1+u_2, \qquad (14)$$

where

$$u_1=(g^2k_0/210)(21\varepsilon^{-1}+42+72\varepsilon+18\varepsilon^2)e^{-2\varepsilon}$$
$$-(k_0\mu g^4/11080)(20523\varepsilon^{-1}+16096 \qquad (15)$$
$$+5664\varepsilon+960\varepsilon^2+64\varepsilon^3)e^{-4\varepsilon},$$
$$u_2=(g^2k_0/2)(1+\varepsilon^{-1})e^{-2\varepsilon}$$
$$-k_0\mu g^4\left\{0{,}25\varepsilon^{-1}e^{-4\varepsilon}+\int_{4\varepsilon}^\infty\frac{e^{-x}}{x}dx\right\}.$$

We substitute into Eq. (14) various values of ϵ, with the values of μg^2 which correspond to them according to Eq. 13), and thus obtain the results given in Table 1.

The ratio g/e is obtained here by comparing the expression for the self-energy of a nucleon of mass $1836.5m_e$ and unit quasi-electric charge with the approximate values which we have calculated.

ε	0.0	0.01	0.1	0.3	0.5	0.7	1.0	2.0
μg^2	1,378	1,41	1,735	2,46	4,35	6,83	13,56	126.9
u/k_0g^2		4,0	1,48	0.74	0,368	0,226	0,126	
g/e		15.3	25.1	36.5	50.3	64.2	85.8	
$10^3\mu e^2$		6,02	2.76	1,95	1,72	1 66	1.84	

It is supposed that the whole of the nucleon mass is of pi-mesonic origin. If only a part of the mass is due to the meson field, the quantity g_i/e will be smaller.

We see that the results come close to the generally accepted values. It is interesting to observe that as ϵ varies μ remains roughly constant and has the order of magnitude 10^{16} CGSU The other quantities also vary over relatively narrow ranges, so that the agreements which are obtained do not seem to be due to chance.

In conclusion I express my deep gratitude to Professor D. D. Ivanenko, who set the theme of this work and gave much valuable advice, and also to Professors A. N. Tikhonov and V. V. Lebedev who took part in discussions.

Note added in proof.- Early in 1955 there appeared a paper by Fornaguera[19], in which non-linear equations are discussed for $\lambda > 0$ without imposing the condition that the total energy of a particle be finite. This paper includes some interesting observations about the many-particle interactions arising from non-linear equations. The results are also valid for the original case ($\lambda > 0$) of our equation, although the interactions are then degenerate. Fornaguera's methods are elegant, and his work supplements the present paper so far as equations of the Schiff type are concerned.

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