Fluctuations during Collision of High Energy Particles

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On the basis of the Fermi-Landau theory, we calculate the fluctuations in energy and number of particles during collisions of high energy nuclear-active particles.

1. FERMI[1] proposed a statistical theory of the collision of very high energy nucleons. Later the theory was developed and changed fundamentally by Landau[2]. The purpose of the present work is to give a preliminary discussion of the role of fluctuations in such processes.

We should anticipate that fluctuations will be considerable, since the number of particles produced and, especially, the number of particles which carry off a large fraction of the energy will be relatively small. Thus, the calculation of the fluctuation in the number of particles and, even more so, in the energy distribution, is a very real problem. This problem is, however, complicated by the fact that all sorts of undetermined parameters appear in the theory. For example, the final temperature when the breakup of the system begins is not completely determined. At present, therefore, one cannot give a rigorous and consistent treatment of the fluctuations, and we shall limit ourselves to an approximate qualitative discussion.

We shall start from the same assumptions as Landau, i.e., we shall consider the whole system to be some sort of continuous medium, subject to the equation of state of an ideal ultra-relativistic fluid. This system can, however, be divided into elements within each of which there is statistical equilibrium. We shall assume that the entropy of each element is constant, and that the number of particles in it is sufficiently large (unless specifically stated). From these assumptions it follows that the energy distribution of the particles in each element satisfies the law of black-body radiation.

In addition, we shall use classical methods for calculating fluctuations in systems in thermodynamic equilibrium; the influence of dynamical factors associated with the time development of the system will be treated indirectly.

2. In accordance with Landau's theory, we shall consider the motion of an ideal ultra-relativistic fluid, each of whose elements is characterized by a definite four-velocity and temperature. If we assume that exchange of thermal energy between the elements can occur*, then we can calculate the fluctuations in the number of particles, n, and the thermal energy E.

We assume first that there are only Bose particles inside the element. The average number of particles is

$$\bar{n} = \frac{a\Omega}{2\pi^2(\hbar c)^3}\int_0^\infty \frac{\varepsilon^2 e^{-\varepsilon/kT}}{1-e^{-\varepsilon/kT}}\,d\varepsilon = 2.41\,b, \tag{1}$$

$$b = \frac{a\Omega}{2\pi^2}\left(\frac{kT}{\hbar c}\right)^3,$$

where Ω is the volume of the element and a is the number of internal degrees of freedom of the particle (e.g., $a = 3$ for π-mesons).

The average energy $\bar{E}$ concentrated in the volume Ω is

$$\bar{E} = \frac{a\Omega}{2\pi^2(\hbar c)^3}\int_0^\infty \frac{\varepsilon^3 e^{-\varepsilon/kT}}{1-e^{-\varepsilon/kT}}\,d\varepsilon = 6.49\,bkT. \tag{2}$$

The dispersion of the number of particles is

$$D(n) = \frac{a\Omega}{2\pi^2(\hbar c)^3}\int_0^\infty \frac{\varepsilon^2 e^{-\varepsilon/kT}}{(1-e^{-\varepsilon/kT})^2}\,d\varepsilon = 3.29\,b. \tag{3}$$

The dispersion of the energy is

$$D(E) = \frac{a\Omega}{2\pi^2(\hbar c)^3}\int_0^\infty \frac{\varepsilon^4 e^{-\varepsilon/kT}}{(1-e^{-\varepsilon/kT})^2}\,d\varepsilon \tag{4}$$

$$= 25.97\,b\,(kT)^2$$

[1] E. Fermi, Progr. Theor. Phys. 5, 570 (1950)

[2] L. D. Landau, Izv. Akad. Nauk SSSR, Ser. Fiz. 17, 51 (1953)

* We shall refer to this case as isothermal. The case of adiabatic isolation of the elements will be considered separately.

and

$$\overline{(n-\bar{n})(E-\bar{E})} \tag{5}$$

$$= \frac{a\Omega}{2\pi^2(\hbar c)^3}\int_0^\infty \frac{\varepsilon^3 e^{-\varepsilon/kT}(1+e^{-\varepsilon/kT})}{(1-e^{-\varepsilon/kT})^2}\,d\varepsilon = 7.65\,bkT.$$

Similarly, for Fermi particles:

$$\bar{n} = \frac{a\Omega}{2\pi^2(\hbar c)^3}\int_0^\infty \frac{\varepsilon^2 e^{-\varepsilon/kT}}{1+e^{-\varepsilon/kT}}\,d\varepsilon = 1.81\ b \tag{6}$$

(for nucleons, $a = 8$),

$$\bar{E} = \frac{a\Omega}{2\pi^2(\hbar c)^3}\int_0^\infty \frac{\varepsilon^3 e^{-\varepsilon/kT}}{1+e^{-\varepsilon/kT}}\,d\varepsilon = 5.68\,bkT; \tag{7}$$

$$D(n) = \frac{a\Omega}{2\pi^2(\hbar c)^3}\int_0^\infty \frac{\varepsilon^2 e^{-\varepsilon/kT}}{(1+e^{-\varepsilon/kT})^2}\,d\varepsilon = 1.64\,b; \tag{8}$$

$$D(E) = \frac{a\Omega}{2\pi^2(\hbar c)^3}\int_0^\infty \frac{\varepsilon^4 e^{-\varepsilon/kT}}{(1+e^{-\varepsilon/kT})^2}\,d\varepsilon \tag{9}$$

$$= 22.72\,b\,(kT)^2;$$

$$\overline{(n-\bar{n})(E-\bar{E})} \tag{10}$$

$$= \frac{a\Omega}{2\pi^2(\hbar c)^3}\int_0^\infty \frac{\varepsilon^3 e^{-\varepsilon/kT}}{1+e^{-\varepsilon/kT}} = \bar{E}.$$

The joint distribution of particle number and energy is given by the two-dimensional Gauss law:

$$f(n,E) \tag{11}$$

$$= \frac{\sqrt{\Delta}}{2\pi}\exp\{-{}^1/{}_2[c_{11}(n-\bar{n})^2$$

$$+2c_{12}(n-\bar{n})(E-\bar{E})$$

$$+c_{22}(E-\bar{E})^2]\},$$

where

$$c_{11} = \Delta D(E);\ c_{22} = \Delta D(n);$$

$$c_{12} = -\Delta\,\overline{(n-\bar{n})(E-\bar{E})};\ \Delta = c_{11}c_{22} - c_{12}^2.$$

3. According to Landau's theory, there is no heat exchange between different elements (adiabatic motion). Then, if we fix the value of E in Eq. (11), we find for the dispersion of the particle number

$$D'(n) = D(n) - \frac{[\overline{(n-\bar{n})(E-\bar{E})}]^2}{D(E)}, \tag{12}$$

where the quantities $D(n)$, $\overline{(n-\bar{n})(E-\bar{E})}$, $D(E)$ are calculated from formulas (8) - (10), i.e., are calculated for varying energy.

From Eq. (12) we get, for Bose particles,

$$D'(n) = 1.03\,b \tag{13}$$

and for Fermi particles,

$$D'(n) = 0.22\,b. \tag{14}$$

The assumption that the energy is fixed is equivalent to requiring energy conservation. The inclusion of other conservation laws (momentum, angular momentum, charge) does not affect the fluctuations in the number of particles, since the corresponding correlation terms of the type $\overline{(e-\bar{e})(n-\bar{n})}$ are equal to zero (or are close to zero).

4. From Eqs. (1), (3), (6) and (8) it follows that

$$D(n) = \alpha\bar{n}, \tag{15}$$

where $\alpha = 1.37$ for Bose particles and $\alpha = 0.91$ for Fermi particles in the isothermal case, while in the adiabatic case $\alpha = 0.43$ and 0.12, respectively. The relatively small value of the fluctuation for Fermi particles is related to the Pauli principle*. The essential point is that in all cases the value of α does not depend on either the volume of the element or its temperature. Therefore, no matter what the temperature distribution is, we may suppose that the relation (15) also holds for the total number of particles (with the same values of α).

Thus, the fact that the system, according to Landau, is not in thermodynamic equilibrium, does not effect the fluctuations of the number of

* If $kT \sim mc^2$, the difference between the values of α decreases. Improvement of the results for this case requires consideration of the interaction between the particles.

particles in first approximation* .

5. The relation (15) does not depend on either the details of the mechanism of collision or on the energy E_0 of the initial particle. This permits us to express Eq. (15) in another form which is more suitable for comparison with experiment. Let us assume, say, that for energies of the order of 10^{12} - 10^{13} ev, the energy spectrum of the initial particles follows the power law $\frac{\gamma-1}{W_0^{\gamma}} E_0^{\gamma-1}$, where W_0 is some threshold energy determined by the conditions of the experiment. Noting that in the Fermi-Landau theory the average number of particles generated in each elementary act is proportional to $E_0^{1/4}$, and using Eq (15), we easily obtain the relation**

$$\overline{\sigma^2} = \frac{(\gamma - {}^5/_4)^2}{(\gamma - {}^3/_2)(\gamma - 1)} (\bar{\sigma})^2 - \alpha\bar{\sigma}, \qquad (16)$$

where $\bar{\sigma} = \Sigma n_i / N$; $\overline{\sigma^2} = \Sigma n_i^2/N$, n_i is the number of particles in the ith process, and N is the number of processes observed. An essential point is that the value of the threshold energy W_0, which we do not know, does not appear in Eq. (16).

In analyzing experiments in emulsions, we must take into account the fact that only charged particles are seen in the emulsion. An elementary computation shows that this merely results in changing the quantity α to

$$\alpha' = 1 + (\alpha - 1)p, \qquad (17)$$

where p is the probability that a particle of a given type is charged ($p = ½$ for nucleons, for π-mesons, $p = {}^2/_3$). Fluctuations can also be studied by comparing the numbers n_1 and n_2 of prongs inside narrow and wide cones in stars accompanied by small numbers of black and gray tracks (nucleon-nucleon collision). If we assume that the total thermal energy in both cones is the same, then $\bar{n}_1 = \bar{n}_2$, and because of the statistical independence,

$$D(n_1 - n_2) = D(n_1 + n_2) = \alpha(\bar{n}_1 + \bar{n}_2). \qquad (18)$$

Since $\alpha \sim 1$, for not too high energy, both n_1 and n_2 and, in particular, their difference $n_1 - n_2$ can fluctuate quite strongly*. This has to be considered, for example, in those comparisons of the number of particles formed in each act with the total energy liberated, which are made to test the theory. We note that the energy of the initial particle, which is usually determined from the well-known relation**

$$E_0 \sim \cot \vartheta_f \cot \vartheta_{1-f} / 2,$$

also can only be evaluated very roughly, since the number of particles which emerge within a small solid angle fluctuates relatively more strongly than the total number of particles.

6. Now let us consider the energy fluctuation. In our opinion, the most interesting question in interpreting experimental facts is: what is the probability that some one of the secondary particles carries off a considerable fraction of the total energy?

Suppose that some element of the nuclear matter, at temperature T, moves relative to the laboratory coordinate system with velocity $\beta = v/c$. We shall characterize the state of motion by the quantity $\gamma = 1\sqrt{1 - \beta^2}$. We transform to a reference frame fixed in the volume under consideration, and fix our attention on some particle. Its angular distribution is obviously isotropic, while its energy distribution $p(\epsilon)$ is determined by the relation between kT and mc^2. The energy distribution will follow the Planck law if $kT/mc^2 \gg 1$, or will be Maxwellian if $kT/mc^2 \ll 1$. So, after making some simple kinematical transformations (cf, for example, reference 3,) we can calculate the energy distribution of the particle in

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* The last conclusion is valid so long as we can neglect surface effects and specifically, quantum fluctuations, i.e., so long as the wavelength $\lambda \sim \hbar c/kT$ is small compared to the dimensions of the system $R \sim \hbar/\mu c$, which reduces to the condition $\mu c^2/kT < 1$.

** In deriving Eq. (16) it was assumed that the number of secondary particles does not depend on the impact parameter. Including this dependence would have given rise to additional fluctuations. Lack of a sufficiently complete theory of the elementary process, which would also include peripheral collisions, prevents us from making this correction in Eq. (16).

* The first stage of Landau's process (passage of the shock wave) also is statistical. It is therefore not excluded that $\bar{n}_1 \neq \bar{n}_2$ in the individual processes. If the number of particles produced is small, the symmetry of the cones is also disturbed because the colliding particles can be of different types (e.g., collision of proton and neutron). Both of these causes may increase the fluctuation of the difference $n_1 - n_2$.

** θ_f, θ_{1-f} are angles containing the fractions f and $1 - f$ of the total number of shower particles.

[3] I. L. Rozental', Usp. Fiz. Nauk **54**, 405 (1954)

the laboratory coordinate system. It is then clear than in the region $kT < mc^2$ the energy fluctuation must drop sharply with decreasing temperature, since for $kT/mc^2 \to 0$, the particle energy will always be the same value $m\gamma$. Now going over to a quantitative discussion, we first treat the auxiliary problem for particles whose rest mass can be neglected (photons). In addition, we shall impose the condition $\gamma \gg 1$, which is always fulfilled for the narrow cone.

The required energy distribution in the laboratory system has the form[3]

$$P_{\rm lab}(E)\,dE = \frac{dE}{2\gamma}\int\limits_{E/2\gamma} \frac{p(\varepsilon)}{\varepsilon}\,d\varepsilon. \qquad (19)$$

If we measure the energy in units of $2\gamma kT$, i.e., if we introduce the variable $\xi = E/2\gamma kT$, we get the relations

$$u_{\rm B}(\xi)\,d\xi = \frac{d\xi}{2{,}4}\int\limits_{\xi}^{\infty} \frac{x\,dx}{e^x - 1} \qquad (20)$$

for Bose particles

$$u_{\rm F}(\xi)\,d\xi = \frac{d\xi}{1{,}8}\int\limits_{\xi}^{\infty} \frac{x\,dx}{e^x + 1} \qquad (21)$$

for Fermi particles; these do not contain the quantities γ or kT. The corresponding curves are shown in Fig. 1, from which we see that the difference between them is small.

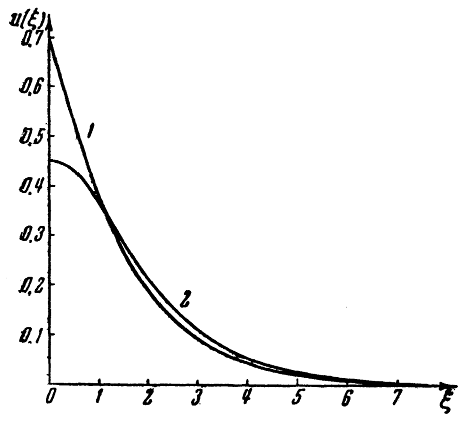

FIG. 1. *1.* $u_{\rm B}(\xi)$, *2.* $u_{\rm F}(\xi)$.

From Eqs. (20) and (21) it also follows that the relative fluctuation of the energy, $\delta = \sqrt{D(E)}/E$, also does not depend on kT and γ. It is very large --- $\delta = 1.1$ for bosons and $\delta = 0.75$ for fermions; thus, in both cases the fluctuations are of the same order of magnitude as the average energy.

7. We now take account of the finite mass of the particles. We shall assume that the average velocity of the thermal motion is smaller than the translational velocity, i.e., $\overline{\gamma_{\rm T}} < \gamma$. This is the case which usually occurs in experiment. Then

$$P_{\rm lab}(E) = \frac{1}{2\gamma}\int\limits_{E/2\gamma}^{\infty} \frac{p(\varepsilon)}{\sqrt{\varepsilon^2 - m^2c^4}}\,d\varepsilon, \qquad (22)$$

$$p(\varepsilon) = \left[\int\limits_{mc^2}^{\infty} \frac{x\sqrt{x^2 - m^2c^4}}{\exp(x/kT) \mp 1}\,dx\right]^{-1} \cdot \frac{\varepsilon\sqrt{\varepsilon^2 - m^2c^4}}{\exp(\varepsilon/kT) \mp 1} \qquad (23)$$

(the minus sign refers to bosons, the plus sign to fermions).

From Eq. (22) it follows that in the case of finite mass, except for an unimportant normalization factor, the distribution functions $v_{\rm B}(\xi)$ and $v_{\rm F}(\xi)$ are related to the functions $u_{\rm B}(\xi)$ and $u_{\rm F}(\xi)$ by

$$v_{\rm B}(\xi) = u_{\rm B}\left[\xi + \left(\frac{mc^2}{2kT}\right)^2\frac{1}{\xi}\right], \qquad (24)$$
$$v_{\rm F}(\xi) = u_{\rm F}\left[\xi + \left(\frac{mc^2}{2kT}\right)^2\frac{1}{\xi}\right].$$

It is easy to show that all these distributions have their maxima at $\xi = \gamma mc^2/2kT$, i.e., at $E = \gamma mc^2$, and that the upper end ($E > \gamma mc^2$) is spread out more for larger values of kT/mc^2. If $kT/mc^2 \ll 1$, the distribution becomes symmetric, and all the possible values of the energy are concentrated in a narrow range around $E = \gamma mc^2$. Figure 2 shows the distributions for a particle of mass μ for the cases $kT = mc^2$ and $kT = \mu c^2$ $\mu = 1/6$ m).

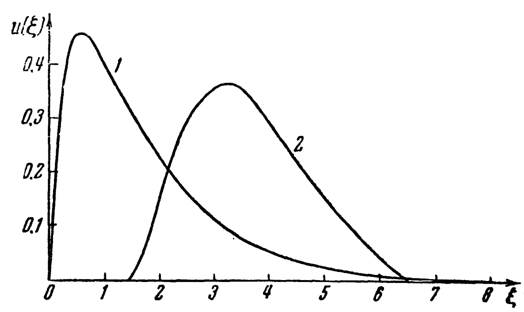

FIG. 2. *1.* $kT = mc^2$, *2.* $kT \sim mc^2/6$.

The relations obtained apply to any volume element filled with nuclear matter, and describe the energy fluctuations of the particles which constitute that element. We must remember the following: all the calculations were made on the assumption that just before the start of the stage of free breakup the quantity γ has a completely definite value for each element (i.e., it does not fluctuate). Strictly speaking, this is not so, because of the occurrence of fluctuations in the dynamical characteristics of the process (pressure fluctuations). However, in first approximation we need not consider pressure fluctuations, since they average out over the course of the whole stage of hydrodynamic expansion.

8. In order to obtain final numerical estimates of the probability that some one particle carries off a definite fraction of the energy of the initial particle, we must average Eq. (24) over the various parts of the volume containing the nuclear matter, using the distribution of energy and number of particles given by Landau's theory. However, such a procedure actually has little meaning, since it is clear beforehand that the region close to the front surface of the volume plays a fundamental role.

Actually, according to Landau's theory, the entropy and the quantity γ are distributed nonuniformly over the volume. Most of the particles have relatively low velocity ($\gamma \sim 1$); at the same time the front region, which contains only a small fraction of the particles, carries the main part of the energy and has a high velocity ($\gamma \gg 1$). Therefore, for an approximate evaluation of the energy fluctuation it is sufficient to consider only the region near the front surface. Unfortunately, the solution obtained by Landau is not sufficiently accurate in just this region, since in getting the solution it was assumed that the quantity γ varies sufficiently slowly, and this condition is not fulfilled in this region. We have, therefore, not used the velocity and energy distribution over this region but have regarded the whole front surface as a single entity. Its entropy and energy were calculated from the difference between the total entropy (or energy) and the entropy of the whole region back of the front surface (where it is permissible to use Landau's solution).

Separating out the region of the front surface in this way,* we determined the number of particles in it and the total energy. For the temperature at breakup we chose the values $kT \sim mc^2$ or $kT \sim \mu c^2$ *. Using the fact that in the system of reference in which the front surface is at rest, the energy distribution of the particles is that of black body radiation, we can calculate the probability $p(\delta)$ that one particle will carry off more than the fraction δ of the total energy (in the lab system). The results are given in the Table.

$p(\delta)$	0.25	0.33	0.40
δ	0.5	0.25	0.10

From the Table we see that these probabilities are very large. We should also remark that they vary slowly with energy of the initial particle in the interval from 10^{14} to 10^{16} ev.

The Table applies to the fluctuations in energy of protons when the temperature at breakup is determined by the proton mass ($kT \sim mc^2$); it also describes the energy fluctuations of mesons if the breakup temperature is $kT \sim \mu c^2$**.

We should emphasize that the number of particles near the front surface is relatively small (this number hardly depends on the energy of the initial particle in the interval from 10^{14} to 10^{16} ev, and is approximately five). Therefore, the application of thermodynamic formulas (black body radiation law) to the energy distribution of the particles in the front surface is not entirely justified. More correct would be the application of statistical laws, for example, in the spirit of Fermi's first paper[1], where a system is considered which consists of a small number of particles. We have, however, not carried out such computations because of the complexity of the resulting expressions, since we expect that the order of magnitude of the fluctuation will not be changed.

So the energy fluctuation is large and it may be essential to consider it for various processes. For example, inclusion of fluctuations may affect the altitude variation of wide atmospheric showers.

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* Although most quantities (temperature, γ, etc.) change markedly when we go from the main mass to the front surface, this separation of the front surface is still not entirely unique. Therefore, the question arises to what extent the result depends on the choice of the boundary which separates the system into a front surface and a residual part. Computations showed that the dependence is slight, so that the element of uncertainty in this choice is small.

* The temperature at breakup, i.e., the temperature at which the particles (mainly π-mesons) cease interacting strongly, was taken to be $\sim \mu c^2/k$ in Landau's work. There is, however, some basis for assuming (for example, if we suppose that the mesons interact with one another via virtual nucleon pairs) that the interaction of two π-mesons is large only when their relative energy is $\sim mc^2$. On this basis, the temperature at breakup will be $\sim mc^2/k$.

** It is also assumed that there are particles of only a single kind in the region of the front surface. If this is not the case, then in calculating the fluctuations of energy we must assign the ratio of the numbers of particles of various types.

Thus, it was shown that the absorption coefficient of wide showers, calculated according to Landau's theory, is somewhat greater than the experimental value[4]. The altitude variation of showers is determined to a large extent by high energy particles, so that the absorption coefficient is decreased if one of the secondary particles takes a large part of the energy of the initial particle.

According to our estimates, inclusion of fluctuations lowers the value of the absorption coefficient of wide showers at high altitude by approximately 10%, which greatly improves the agreement between experiment and theory*.

CONCLUSIONS

1. The fluctuations in number of particles are proportional to the square root of the number of particles and are very large in absolute value. The coefficient of proportionality is essentially different for fermions (nucleons and antinucleons) and bosons (π-mesons).

2. It was shown that the fluctuations in the energy carried off by a single particle are very large. This applies especially to the narrow cone of particles, in which most of the energy is contained in the laboratory system. Marked fluctuations should also be observed in the angular distribution of the energy in an elementary process.

3. The energy fluctuations are important for the interpretation of the altitude variation of wide atmospheric showers. The theoretical absorption coefficient is decreased when we include fluctuations, which improves the agreement of theory with experiment.

The authors thank S. Z. Belen'kii for valuable comments.

* We should remark that the absence of fundamental data concerning nuclear interactions at moderate energies $\sim 10^{10} - 10^{12}$ ev prevents us from making a sufficiently accurate calculation of the altitude variation of wide showers.

[4] G. T. Zatsepin and L .I. Sarycheva, Dokl. Akad. Nauk SSSR 99, 951 (1954)

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