

Fluctuations in Oscillating Systems of the Thomson Type. II

S. M. RYTOV

P. N. Lebedev Institute of Physics, Academy of Sciences, USSR

(Submitted to JETP editor December 2, 1954)

J. Exper. Theoret. Phys. USSR 29, 315-328 (September, 1955)

The theory developed in the preceding paper is applied to a non-isolated system with one degree of freedom. A vacuum-tube oscillator synchronized by a harmonic driving force is considered as an example. The same method is applied in the study of a system with two degrees of freedom — a vacuum-tube oscillator coupled to a high-Q circuit — and in the elucidation of the influence of this method of frequency stabilization on fluctuations of amplitude and phase. The way in which thermal fluctuations are taken into account in this theory is considered.

1. INTRODUCTION

THE application of symbolic differential equations and correlation theory to the question of fluctuations in oscillating systems offers such a simplification of the statistical part of the problem that it becomes feasible to consider more complicated cases than that of the simple isolated system having one degree of freedom¹ (hereafter referred to as I). At the outset we may note certain results obtained below in connection with two such more complicated problems.

A *non-isolated* oscillating system having one degree of freedom is considered first, and the theory for this case is applied to a vacuum-tube oscillator synchronized by an external harmonic emf. The spectrum of the synchronized oscillator is shown to be discrete-continuous, since a strictly sinusoidal synchronizing emf implies the existence of a discrete line. Also, the forced synchronization leads to a situation in which the random excursions of the phase of the oscillations do not increase in accordance with a diffusion law but have, as do the amplitude fluctuations, a stationary spread about that value of the phase which the external emf imposes on the system in the absence of fluctuations. The dependence of the strength of the fluctuations on the amplitude of the synchronizing emf and on the difference between the frequency of this emf and the natural frequency of the oscillatory circuit is established. The ratio of the energies of the discrete line and the continuous part of the spectrum is also obtained.

The second of the problems considered in this paper is that of the fluctuations in a vacuum-tube oscillator stabilized by being coupled to a high-Q circuit (a system with two degrees of freedom). It

is known that in order to apply a method involving an expansion in a small parameter this frequency-stabilization scheme can be treated in the first approximation as the synchronization of an oscillator by an external driving force². It is clear that this treatment is applicable not only to the purely dynamic problem but also, to some extent, to the question of fluctuations in systems of this type; in particular, it is shown that stabilization by means of a high-Q oscillatory circuit reduces the phase diffusion coefficient in second order. Thus, in the first approximation, there occurs only a stationary spread about the value given by the initial conditions. The indicated attenuation of the phase diffusion means that the relative departure of the frequency for some fixed time is one order of magnitude smaller than that which obtains in the absence of stabilization. The analysis is based on deviations which can be attributed to fluctuations and not to the instability of the system parameters. The clear distinction between these effects and other pertinent considerations in regard to the natural and practical line-widths of the oscillator spectrum have been given by Gorelik³.

2. SYNCHRONIZED OSCILLATING SYSTEM

Returning to Eqs. (3.8) of I in which, for the non-isolated system, Φ_{10} and Ψ_{10} now depend on R_0 and φ_0 , the steady-state conditions require the vanishing of R_0 and φ_0 . This requirement yields the equations

$$\Phi_{10}(R_0, \varphi_0) = 0, \quad \Psi_{10}(R_0, \varphi_0) = 0, \quad (2.1)$$

² S. M. Rytov, A. M. Prokhorov and M. E. Zhabotinskii, *J. Exper. Theoret. Phys. USSR* 15, 557, 613 (1945)

³ G. S. Gorelik, *J. Exper. Theoret. Phys. USSR* 20, 351 (1950); *Izv. Akad. Nauk SSSR, Ser. Fiz.* 14, 187 (1950)

¹ S. M. Rytov, *J. Exper. Theoret. Phys. USSR*, 29, 304 (1955); *Soviet Phys.* 2, 217 (1956). Hereafter referred to as I.

which determine the constants R_0 and φ_0 - the radius of the generating circle and the phase shift of the locked oscillations with respect to the synchronizing force. The generating point now fluctuates in a random fashion about the uniformly moving "dynamic" point, and is connected to it in terms of both radial and tangential fluctuations. In the steady state the "scatter region" performs uniform rotation around the generating circle without becoming deformed and, in particular, without spreading along the circle according to a diffusion law (Fig. 1).

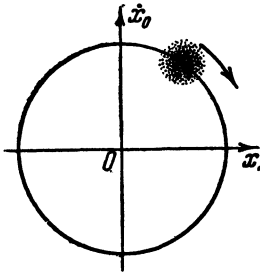


FIG. 1

Since the derivatives of R_0 and φ_0 are zero, Eqs. (3.9) of I assume the form

$$2R_1' + \frac{\partial \Phi_{10}}{\partial R_0} R_1 + \frac{\partial \Phi_{10}}{\partial \varphi_0} \varphi_1 = -A_1(R_0, \varphi_0) + F_{\perp}, \quad (2.2)$$

$$2R_0 \varphi_1' - \frac{\partial \Psi_{10}}{\partial R_0} R_1 - \frac{\partial \Psi_{10}}{\partial \varphi_0} \varphi_1 = B_1(R_0, \varphi_0) + F_{\parallel}.$$

The constants A_1 and B_1 , which arise from the non-linearity distortions, now represent regular correction terms to both R_1 and φ_1 . Designating these regular correction terms by r_1 and ψ_1 , i.e., taking

$$R_1 = r_1 + \rho(\tau), \quad \varphi_1 = \psi_1 + \chi(\tau), \quad (2.3)$$

From Eq. (2.2) we obtain expressions for r_1 and ψ_1 :

$$r_1 = -\frac{q_2 A_1 + p_2 B_1}{p_1 q_2 - p_2 q_1}, \quad \psi_1 = \frac{q_1 A_1 + p_1 B_1}{p_1 q_2 - p_2 q_1}, \quad (2.4)$$

and the following equations for the fluctuations of amplitude ρ and phase χ :

$$\rho' + p_1 \rho + p_2 \chi = F_{\perp}(\tau)/2, \quad (2.5)$$

$$\chi' + q_1 \rho + q_2 \chi = F_{\parallel}(\tau)/2R_0,$$

in which

$$p_1 = \frac{1}{2} \frac{\partial \Phi_{10}}{\partial R_0}, \quad p_2 = \frac{1}{2} \frac{\partial \Phi_{10}}{\partial \varphi_0}, \quad (2.6)$$

$$q_1 = -\frac{1}{2R_0} \frac{\partial \Psi_{10}}{\partial R_0}, \quad q_2 = -\frac{1}{2R_0} \frac{\partial \Psi_{10}}{\partial \varphi_0}.$$

Being interested only in the steady state, we take solutions of Eqs. (2.5) for initial conditions, corresponding to the vanishing of the quantities of interest at $\tau = -\infty$:

$$\rho(\tau) = \frac{1}{2(\lambda_1 - \lambda_2)} \int_{-\infty}^{\tau} \left\{ [(p_1 + \lambda_1) e^{\lambda_2(\tau-\theta)} - (p_1 + \lambda_2) e^{\lambda_1(\tau-\theta)}] F_{\perp}(\theta) \right. \quad (2.7)$$

$$\left. + \frac{p_2}{R_0} [e^{\lambda_2(\tau-\theta)} - e^{\lambda_1(\tau-\theta)}] F_{\parallel}(\theta) \right\} d\theta,$$

$$\chi(\tau) = -\frac{1}{2(\lambda_1 - \lambda_2)}$$

$$\int_{-\infty}^{\tau} \left\{ \frac{(p_1 + \lambda_1)(p_1 + \lambda_2)}{p_2} [e^{\lambda_2(\tau-\theta)} - e^{\lambda_1(\tau-\theta)}] F_{\perp}(\theta) \right.$$

$$\left. + \frac{1}{R_0} [(p_1 + \lambda_2) e^{\lambda_2(\tau-\theta)} \right.$$

$$\left. - (p_1 + \lambda_1) e^{\lambda_1(\tau-\theta)}] F_{\parallel}(\theta) \right\} d\theta.$$

Here λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 + (p_1 + q_2)\lambda + p_1 q_2 - p_2 q_1 = 0, \quad (2.8)$$

having real negative values for the stable mode. Since λ_1 and λ_2 are complex conjugates, the expressions (2.7) are always real.

Omitting rather tedious but elementary calculations based on Eqs. (2.7), (2.8) and formulas (2.6) of I, we present the results of a calculation of the correlation functions for ρ and χ :

$$\overline{\rho(\tau)\rho(\tau')} = \frac{\mu C}{4(\lambda_1^2 - \lambda_2^2)}$$

$$\times \left\{ [(p_1 + \lambda_2)(\lambda_1 - q_2) + \frac{p_2^2}{R_0^2}] \frac{1}{\lambda_1} e^{\lambda_1(\tau'-\tau)} \right. \quad (2.9)$$

$$\left. - [(p_1 + \lambda_1)(\lambda_2 - q_2) + \frac{p_2^2}{R_0^2}] \frac{1}{\lambda_2} e^{\lambda_2(\tau'-\tau)} \right\},$$

$$\overline{\chi(\tau)\chi(\tau')} = \frac{\mu C^*}{4(\lambda_1^2 - \lambda_2^2)} \left\{ \left[q_1^2 + \frac{p_1^2 - \lambda_1^2}{R_0^2} \right] \right.$$

$$\times \frac{1}{\lambda_1} e^{\lambda_1(\tau'-\tau)} - \left[q_1^2 + \frac{p_1^2 - \lambda_2^2}{R_0^2} \right] \frac{1}{\lambda_2} e^{\lambda_2(\tau'-\tau)} \left. \right\}.$$

Whence, if it is assumed that $\tau' = \tau$ and if use is again made of Eq. (2.8), the following expressions are obtained for the mean-square values of ρ and χ :

$$\overline{\rho^2} = \frac{\mu C}{4(p_1 + q_2)} \left\{ 1 + \frac{q_2^2 + (p_2^2 / R_0^2)}{p_1 q_2 - p_2 q_1} \right\}, \quad (2.10)$$

$$\overline{\chi^2} = \frac{\mu C}{4(p_1 + q_2)} \left\{ \frac{q_1^2 + (p_1^2 / R_0^2)}{p_1 q_2 - p_2 q_1} + \frac{1}{R_0^2} \right\}.$$

According to Eq. (2.3) and formulas (3.2) and (3.5) of I the frequency of the fundamental oscillation is given by

$$x_{\text{fund}} = (R_0 + \mu r_1 + \mu \rho) \times \cos(t - \varphi_0 - \mu \psi_1 - \mu \chi).$$

We find the correlation function for x_{fund} neglecting the amplitude correction μr_1 and the amplitude fluctuations $\mu \rho$. Then

$$\begin{aligned} & \overline{x_{\text{fund}}(t) x_{\text{fund}}(t')} \\ &= \frac{R_0^2}{2} \overline{\{\cos [t - t' - \mu (\chi - \chi')] + \cos [t + t' - 2(\varphi_0 + \mu \psi_1) - \mu (\chi + \chi')]\}}. \end{aligned} \quad (2.11)$$

In the case of an isolated system it is assumed that the random phase shift during the time $\tau' - \tau$, i.e., the quantity $u = \chi' - \chi$, has a normal distribution, and that, in the steady state, the sum $v = \chi' + \chi$ is distributed uniformly over the interval $(-\pi, \pi)$ and is not correlated with u . Under these assumptions, the second term in Eq. (2.11) vanishes and the first gives

$\frac{R_0^2}{2} e^{-\mu^2 \overline{u^2} / 2} \times \cos(t - t')$, i.e., the result depends only on $t' - t$, as is to be expected for a stationary process.

In the present case, we do not make the same assumptions with regard to u and v . On the contrary, it is natural to assume on the basis of the central limit theorem, as before, that now the random tangential excursions of the "dynamic" point are also normally distributed as are the radial deviations. Thus, we assume that the two-dimensional distribution $\chi = \chi(t)$ and $\chi' = \chi(t')$ has

the form

$$\begin{aligned} & w(\chi, \chi') d\chi d\chi' \\ &= \frac{d\chi d\chi'}{2\pi\sigma^2 \sqrt{1-r^2}} \exp \left\{ -\frac{\chi^2 + \chi'^2 - 2r\chi\chi'}{2\sigma^2(1-r^2)} \right\}, \end{aligned}$$

where

$$\sigma^2 = \overline{\chi^2}, \quad r = \frac{\overline{\chi(\tau)\chi(\tau')}}{\sigma^2}. \quad (2.12)$$

Then $u = \chi' - \chi$ and $v = \chi' + \chi$ are statistically independent, and are characterized by normal distributions and mean-square values $\overline{u^2} = 2\sigma^2(1-r)$ and $\overline{v^2} = 2\sigma^2(1+r)$. Taking averages in Eq. (2.11), we get

$$\begin{aligned} & \overline{x_{\text{fund}}(t) x_{\text{fund}}(t')} \\ &= \frac{R_0^2}{2} \{ \overline{\cos(t - t') \cos \mu u} \\ &+ \overline{\cos [t + t' - 2(\varphi_0 + \mu \psi_1)] \cos \mu v} \} \\ &= \frac{R_0^2}{2} \{ \overline{\cos(t - t') e^{-\mu^2 \sigma^2 (1-r)}} \\ &+ \overline{\cos [t + t' - 2(\varphi_0 + \mu \psi_1)] e^{-\mu^2 \sigma^2 (1+r)}} \}, \end{aligned} \quad (2.13)$$

i.e., as was to be expected, we get a correlation function corresponding to a non-stationary process.

As is well known, (see for example reference 4, Sec. 11) the average intensity at the output of a filter, into which is fed a non-stationary signal $x_{\text{fund}}(t)$, is determined from the time average of the correlation of $x_{\text{fund}}(t)$. In taking the time average of Eq. (2.13), designated by the wavy line, the second term vanishes so that

$$\begin{aligned} & \overline{x_{\text{fund}}(t) x_{\text{fund}}(t')} \\ &= \frac{R_0^2}{2} e^{-\mu^2 \sigma^2 (1-r)} \cos(t' - t), \end{aligned} \quad (2.14)$$

in which $\sigma^2(1-r) = \overline{\chi^2} - \overline{\chi(\tau)\chi(\tau')}$ and the correlation function and the mean-square value of χ are given by formulas (2.9) and (2.10).

Using (2.14) one can, in the usual manner, determine the spectral density of the oscillations in which we are interested:

⁴ V. I. Bunimovich, *Fluctuation Processes in Radio Receiving Apparatus*, published in "Soviet Radio", Moscow, 1951.

$$\begin{aligned}
g(\Omega) &= \frac{2}{\pi} \int_0^{\infty} \overline{x_{\text{fund}}(t) x_{\text{fund}}(t+\theta)} \cos \Omega \theta d\theta \\
&= \frac{R_0^2}{\pi} \int_0^{\infty} \exp \{ \mu^2 [\overline{\chi(\tau) \chi(\tau + \mu\theta)} - \overline{\chi^2}] \} \cos \theta \cos \Omega \theta d\theta.
\end{aligned}$$

Taking into account the fact that

$$\int_0^{\infty} \cos \theta \cos \Omega \theta d\theta = \frac{\pi}{2} \delta(\Omega - 1),$$

we get for $g(\Omega)$ the expression

$$\begin{aligned}
g(\Omega) &= \frac{R_0^2}{2} e^{-\mu^2 \overline{\chi^2}} \delta(\Omega - 1) \\
&+ \frac{R_0^2}{\pi} e^{-\mu^2 \overline{\chi^2}} \int_0^{\infty} (\exp \{ \mu^2 \overline{\chi(\tau) \chi(\tau + \mu\theta)} \} - 1) \\
&\quad \times \cos \theta \cos \Omega \theta d\theta. \tag{2.15}
\end{aligned}$$

The first term corresponds to a discrete line at the frequency of the synchronizing signal $\Omega = 1$, and the second gives the continuous part of the spectrum. We can now compare the total energy in the discrete line with the total energy of the continuous spectrum. In accordance with the separation of the correlation function in Eq. (2.15), in particular,

$$\begin{aligned}
\overline{x_{\text{fund}}(t) x_{\text{fund}}(t+\theta)} &= \frac{R_0^2}{2} \exp \{ -\mu^2 \overline{\chi^2} \} \\
&\times [1 + (\exp \{ \mu^2 \overline{\chi(\tau) \chi(\tau + \mu\theta)} \} - 1)],
\end{aligned}$$

we have for $\theta = 0$

$$\begin{aligned}
\overline{x_{\text{fund}}^2(t)} &= W_d + W_c \\
&= \frac{R_0^2}{2} \exp \{ -\mu^2 \overline{\chi^2} \} + \frac{R_0^2}{2} (1 - \exp \{ -\mu^2 \overline{\chi^2} \}).
\end{aligned}$$

Thus, the ratio of the noise energy to the energy in the discrete line is

$$W_c / W_d = \exp \{ \mu^2 \overline{\chi^2} \} - 1. \tag{2.16}$$

If

$$\mu^2 \overline{\chi^2} \ll 1, \tag{2.17}$$

i.e., if the continuous spectrum is relatively weak, then its density can be found without difficulty. In this case, from the second term of formula (2.15), taking into account (2.9), we get

$$\begin{aligned}
g_{\text{cont}}(\Omega) &= \frac{\mu^2 R_0^2}{\pi} \int_0^{\infty} \overline{\chi(\tau) \chi(\tau + \mu\theta)} \cos \theta \cos \Omega \theta d\theta \\
&= \frac{\mu^2 C R_0^2}{8\pi} \left(q_1^2 + \frac{p_1^2 + \alpha^2}{R_0^2} \right) \frac{1}{(\alpha^2 + \lambda_1^2)(\alpha^2 + \lambda_2^2)}
\end{aligned}$$

where we have introduced the notation

$$\alpha = (\Omega - 1) / \mu.$$

Later we shall see (Section 3) that at the center of the synchronization band $p_2 = q_1 = 0$. In this case, according to Eq. (2.8), $\lambda_1 = -p_1$, $\lambda_2 = -q_2$ and the expression for the density of the continuous spectrum assumes the form

$$g_{\text{cont}}(\Omega) = \frac{\mu^2 C}{8\pi} \frac{1}{\alpha^2 + q_2^2}, \tag{2.18}$$

i.e., the spectrum is in the form of a resonance curve with respect to the dimensionless parameter α and has a halfwidth q_2 . In terms of the true frequency, this means a halfwidth $\delta\omega = \mu q_2 \omega$. One should keep in mind that formula (2.18) is valid only under the conditions given in Eq. (2.17).

3. VACUUM-TUBE OSCILLATOR SYNCHRONIZED BY A HARMONIC EMF

We now apply the results of the preceding section to the oscillator whose circuit is shown in the Figure in I. In the presence of the synchronizing emf, the right-hand part of Eqs. (2.1) of I is given by Eq. (5.5). In this case Eqs. (2.1) of the preceding section become:

$$R_0(1 - Z) + 2H \sin \varphi_0 = 0, \tag{3.1}$$

$$\Delta R_0 + 2H \cos \varphi_0 = 0 \quad (Z = R_0^2/4).$$

The constant correction terms (2.4) to the radius of the generating circle r_1 , and to the phase shift ψ_1 , are found to be

$$r_1 = - \frac{R_0 Z^2 \Delta}{8[(3Z-1)(Z-1) + \Delta^2]},$$

$$\psi_1 = \frac{Z^2(3Z-1)}{8[(3Z-1)(Z-1) + \Delta^2]},$$

and the coefficients in Eq. (2.5) have the form

$$p_1 = \frac{3Z-1}{2}, \quad p_2 = \frac{\Delta R_0}{2}, \tag{3.2}$$

$$q_1 = - \frac{\Delta}{2R_0}, \quad q_2 = \frac{Z-1}{2}.$$

Using Eq. (3.2) in formula (3.10) to calculate the mean-square values of the fluctuations of amplitude and phase, we get

$$\overline{\rho^2} = \frac{\mu C [(2Z - 1)(Z - 1) + \Delta^2]}{2(2Z - 1)[(3Z - 1)(Z - 1) + \Delta^2]}, \quad (3.3)$$

$$\overline{\chi^2} = \frac{\mu C [(3Z - 1)(2Z - 1) + \Delta^2]}{8Z(2Z - 1)[(3Z - 1)(Z - 1) + \Delta^2]}.$$

According to Eq. (3.1), the equation of the resonance curve in the synchronization case is

$$Z[(Z - 1)^2 + \Delta^2] = H^2, \quad (3.4)$$

so that the right-hand parts of (3.3) are determined (aside from the factor μC) only by the amplitude of the synchronizing force and the deviation Δ . In the theory of synchronization one distinguishes, as is known⁵, between the cases of large and small amplitudes $H(H^2 \geq 8/27)$, which correspond to different types of loss of stability of the synchronized mode. If H is sufficiently small, then Z is close to unity and, according to Eq. (3.4), $Z \approx 1 + H$ at the center of the synchronization band. In this case it follows from Eq. (3.3) that

$$\overline{\rho^2} = \frac{\mu C}{4}, \quad \overline{\chi^2} = \frac{\mu C}{8H} \quad (\Delta = 0, H^2 \ll \frac{8}{27}). \quad (3.5)$$

Thus, at the center of the synchronization band the amplitude fluctuations in this case are the same as those of the isolated oscillator [see Eq. (5.9) of I] and the phase fluctuations which are now also stationary, increase indefinitely with the reduction of H . This increase is connected with the fact that the width of the synchronization band is proportional to H ; at its boundaries the synchronized mode being studied loses stability and $\overline{\rho^2}$ and $\overline{\chi^2}$ become infinite [the second factor in the denominator of Eq. (3.3) approaches zero]. Within the band, the phase shifts associated with the frequency shifts are found to be more sensitive to the narrowing of the band limits than are the amplitude fluctuations.

In the case of small H , which is being considered, the halfwidth of the synchronization band (in terms of the dimensionless deviation Δ) is equal to H . The halfwidth of the continuous spectrum, according to Eq. (2.18) is $q_2 = (Z - 1)/2 \approx H/2$. Thus, the continuous

spectrum is one-half as wide as the synchronization band. This result is valid under the conditions given in Eq. (2.17), which, according to Eq. (3.5), can be put in the form

$$\mu^3 C / 8H \ll 1$$

and consequently cannot be carried over to arbitrarily small values of H .

At the center of the synchronization band ($\Delta = 0$) and for large values of H we may take $Z \gg 1$; hence, from Eq. (3.4) we have $Z \approx H^{2/3}$. In this case Eq. (3.3) gives

$$\overline{\rho^2} = \frac{\mu C}{6H^{1/3}}, \quad \overline{\chi^2} = \frac{\mu C}{8H^{1/3}}, \quad (3.6)$$

i.e., with an increase in the value of H , the magnitude of the phase fluctuations falls off quicker than that of the amplitude fluctuations. For $H \geq 8$, $\mu^2 \overline{\rho^2} / R_0^2$ is reduced by a factor of 24 or more as compared with the isolated oscillator.

At the limits of the synchronization band [where the first factor in the denominator of the expressions in Eq. (3.3) becomes zero] we find that $\overline{\rho^2}$ and $\overline{\chi^2}$ can also increase without limit in the case of large H . As has already been noted, this means that in such cases the choices of the orders of magnitude ρ and χ made at the beginning of the analysis are no longer valid.

Equation (2.16) indicates that with the higher values of H , the ratio of the energy in the continuous spectrum to the energy in the discrete line approaches zero at the center of the synchronization band because

$$\frac{W_c}{W_d} = \mu^2 \overline{\chi^2} = \frac{\mu^3 C}{8H^{1/3}} = \frac{C_0 M L \omega_0^3}{8} \sqrt[3]{\frac{2I \omega_0 S}{V^2 C_0^4}}. \quad (3.7)$$

However, close to the limits of the band, where $\overline{\chi^2} \rightarrow \infty$, the discrete line, whose existence in the oscillator spectrum is dependent on the sinusoidal synchronizing emf, "blends" in with the noise. If the spectrum of this emf is not monochromatic, there will be no discrete line, and the width of the continuous spectrum will be considerably increased. Strictly speaking, the oscillator synchronization force is not sinusoidal but the treatment of a randomly modulated synchronizing force requires special attention.

For large amplitudes of the synchronizing force the halfwidth of the synchronization band is $\sqrt{2}H$ while the halfwidth of the continuous spectrum is $q_2 = (Z - 1)/2 \approx H^{2/3}/2$. Thus the relative halfwidth of the continuous spectrum is $1/(2\sqrt{2}H^{1/3})$. For

⁵ A. A. Andronov and A. A. Vitt, Zh. Prikl. Fiz. 7, 3 (1930)

these results to be valid, the quantity given in Eq. (3.7) must be small compared with unity.

4. SYSTEMS WITH TWO DEGREES OF FREEDOM

In analyzing an oscillator which is frequency-stabilized through coupling to a high-Q system*, one finds typically an order-of-magnitude asymmetry in the right-hand side of the equations of motion². Typical equations for the problem at hand, where we have introduced a fluctuation force of order μ^2 , have the form

$$\begin{aligned} \frac{d^2x}{dt^2} + x &= \mu f\left(x, \frac{dx}{dt}, y, \frac{dy}{dt}, \mu\right) + \mu^2 F(t); \\ \frac{d^2y}{dt^2} + y &= \mu^2 g\left(x, \frac{dx}{dt}, y, \frac{dy}{dt}, \mu\right), \end{aligned} \quad (4.1)$$

where x is the current in the oscillator circuit and y is the current in the stabilization element, which for convenience we will in all cases call the "crystal". Thus, to an accuracy of the order of μ , the crystal represents an independent conservative oscillator. It acts on the oscillator with a force of the first order and synchronizes the oscillator in the region of resonance (with deviations $\sim \mu$). The crystal losses and their influences on the oscillator are taken into account in second order. Furthermore - and is closely related to the question of fluctuations - the random force acting on the oscillator also enters in second order.

The accuracy of the solutions of equations (4.1) is the same as that for the case of a system having one degree of freedom. Since we are interested only in the effects of stabilization on fluctuations in vacuum-tube oscillators, we turn now from the general equations of (4.1) to a concrete frequency-stabilization circuit which has two degrees of freedom.

To facilitate comparison with the example considered earlier, we consider the coupling scheme (Fig. 2) which differs from the circuit of the Figure given in I in the presence of the stabilization loop L_1, C_1, R_1 , which is inductively coupled to the oscillator circuit. The high-Q of the circuit L_1, C_1, R_1 is reflected in the following choices of the parameters: $L_1 \sim 1/\mu$, $C_1 \sim \mu$, $R_1 \sim \mu$. The mutual-inductance coefficient $N \sim \mu$. From these we have:

$$\frac{1}{L_1 C_1} = \omega^2, \quad \frac{R_1}{L_1 \omega} = \mu^2 h, \quad (4.2)$$

$$\frac{N}{L_1} = \mu^2 k_2, \quad \frac{N}{L} = \mu k_1.$$

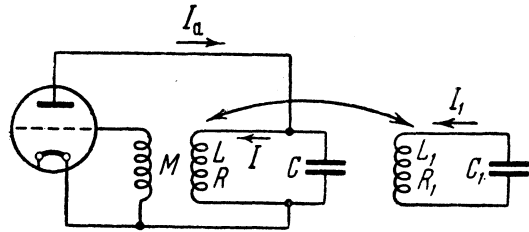


FIG. 2

Through these quantities and the parameters introduced earlier [cf. Eqs. (5.3) and (5.4)] the equations for the circuit of Fig. 2 can be written in the form:

$$\begin{aligned} \frac{d^2x}{dt^2} + x &= \mu \left\{ \frac{dx}{dt} \left[1 - \frac{1}{3} \left(\frac{dx}{dt} \right)^2 \right] \right. \\ &\quad \left. + \Delta x - k_1 \frac{d^2y}{dt^2} \right\} + \mu^2 F(t), \\ \frac{d^2y}{dt^2} + y &= -\mu^2 \left(h \frac{dy}{dt} + k_2 \frac{d^2x}{dt^2} \right). \end{aligned} \quad (4.3)$$

Here as before, $t = \omega t_1$, $x = I/I_0$ and, in addition, $y = I_1/I_0$, where I_1 is the current in the "crystal".

We have a solution in the form

$$\begin{aligned} x &= R \cos(t - \varphi) + \mu \{ P \cos 3(t - \varphi) \\ &\quad + Q \sin 3(t - \varphi) + \dots \}, \end{aligned} \quad (4.4)$$

$$y = U \cos(t - \varphi) + V \sin(t - \varphi) + \mu \{ \dots \}.$$

There are no even harmonics in y because in the stabilized mode, in which we are interested, the "crystal" overtones appear only in order μ^3 ².

Substituting Eq. (4.4) in Eq. (4.3), setting to zero the coefficients of $\sin(t - \varphi)$, $\sin 3(t - \varphi)$ and $\cos(t - \varphi)$, $\cos 3(t - \varphi)$ and expressing R, P, Q, U, V and φ in a power series in μ , we obtain the successive approximation equations. We will not, however, undertake the solution of these in their general form since we are interested only in the steady-state conditions ($R'_0 = 0$), and the stabilized mode, in which $\varphi'_0 = 0$, i.e., there are no first-order frequency corrections². Under these conditions, the equations for the first approximation are as follows:

*In standard crystal oscillators there are also used simple circuits in which the crystal appears as the sole oscillatory element. When considering the properties of the equivalent circuit of the crystal in such oscillators one applies the theory of a system having one degree of freedom.

$$R_0 \left(1 - \frac{R_0^2}{4}\right) = k_1 V_0, \quad P_0 = 0, \\ -\Delta R_0 = k_1 U_0, \quad Q_0 = R_0^3/96, \quad (4.5)$$

and the equation for the second approximations have the form:

$$2R' - \left(1 - \frac{3R_0^2}{4}\right) R_1 + k_1 V_1 = F_{\perp}, \quad (4.6)$$

$$2R_0 \varphi_1' - \Delta R_1 - k_1 U_1 = \frac{R_0^5}{128} + F_{\parallel};$$

$$P_1 = -\frac{R_0^3}{256} \left(1 - \frac{R_0^2}{2}\right), \quad (4.7)$$

$$Q_1 - \frac{R_0^2}{32} R_1 = -\frac{\Delta}{768} R_0^3;$$

$$2U_1' - 2V_0 \varphi_1' = -hU_0, \quad (4.8)$$

$$2V_1' + 2U_0 \varphi_1' = -hV_0 + k_2 R_0.$$

We now separate the constant terms r_1, u_1, v_1, δ_1 in R_1, U_1, V_1, φ_1 :

$$R_1 = r_1 + \rho(\tau), \quad U_1 = u_1 + \xi(\tau), \quad (4.9)$$

$$V_1 = v_1 + \eta(\tau), \quad \varphi_1 = \delta_1 \tau + \chi(\tau).$$

Substituting these relations in Eqs. (4.6) and (4.8), we get equations for the constant terms

$$k_1 v_1 = \left(1 - \frac{3R_0^2}{4}\right) r_1, \quad 2V_0 \delta_1 = hU_0, \quad (4.10)$$

$$k_1 u_1 = 2R_0 \delta_1 - \Delta r_1 - \frac{R_0^5}{128},$$

$$2U_0 \delta_1 = -hV_0 + k_2 R_0.$$

From the last two equations, it follows, first of all, that

$$h(U_0^2 + V_0^2) = k_2 R_0 V_0; \quad (4.11)$$

this equation, in conjunction with Eq. (4.5), determines R_0, U_0 and V_0 . Secondly, using these same two equations we obtain the following expression for the second-order frequency correction δ_1 :

$$\delta_1 = hU_0 / 2V_0. \quad (4.12)$$

For r_1, u_1 and v_1 there remain only the first two equations of Eq. (4.10). The missing third equation is obtained only in the next approximation*.

The variable terms satisfy the equations

$$2\rho' - \left(1 - \frac{3R_0^2}{4}\right) \rho + k_1 \eta = F_{\perp}, \quad (4.13)$$

$$\xi' - V_0 \chi' = 0,$$

$$2R_0 \chi' - \Delta \rho - k_1 \xi = F_{\parallel}, \quad \eta' + U_0 \chi' = 0.$$

In considering ξ and η one may take initial conditions corresponding to the vanishing of the quantities of interest at $\tau = -\infty$. Then, as follows from the last two equations of (4.13),

$$U_0 \xi + V_0 \eta = 0.$$

We may note that $\mu(U_0 U_1 + V_0 V_1)$ is the first-order correction to the amplitude in the "crystal". Thus, in first order, the "crystal" amplitude is not subject to fluctuations.

If we assume that the value of χ is zero at $\tau = 0$, then

$$\xi = \xi(0) = V_0 \chi, \quad \eta - \eta(0) = -U_0 \chi, \quad (4.14)$$

in which $U_0 \xi(0) + V_0 \eta(0) = 0$. Substituting $\eta = -(U_0/V_0)\xi$ and $\chi' = (1/V_0)\xi'$ in the first two equations of (4.13), we obtain for ρ and ξ :

$$\rho' + p_1 \rho + p_2 \xi = F_{\perp} / 2, \quad (4.15)$$

$$\xi' + q_1 \rho + q_2 \xi = (V_0 / 2R_0) F_{\parallel},$$

where

$$p_1 = \frac{1}{2} - \frac{3R_0^2}{4} - 1, \quad p_2 = -\frac{k_1 U_0}{2V_0}, \quad (4.16)$$

$$q_1 = -\frac{\Delta V_0}{2R_0}, \quad q_2 = -\frac{k_1 V_0}{2R_0}.$$

Before solving these equations, we present the the solutions of the first-approximation equations and the pertinent data on the stability limits of the stabilized mode². For the stabilized branch we have, from Eqs. (4.5) and (4.11),

$$\frac{R_0^2}{4} = 1 - \sigma + \sqrt{\sigma^2 - \Delta^2}, \quad U_0^2 + V_0^2 = \frac{hU_0}{2V_0} \\ = \frac{8\sigma}{k_1^2} [(1 - 2\sigma)(\sigma - \sqrt{\sigma^2 - \Delta^2}) + \Delta^2], \quad \sigma_1 = \frac{hU_0}{2V_0} \\ = -\frac{h}{2\Delta} (\sigma + \sqrt{\sigma^2 - \Delta^2}), \quad (4.17)$$

where $\sigma = k_1 k_2 / 2h$.

In Fig. 3 is shown the region of stability in terms of the variables Δ^2 and σ . Formally, the stabilization should extend up to $\Delta = 0$, but in reality, at $\Delta \sim \mu^{1/2}$, the approximation on which our analysis is based is no longer valid (in particular, $\delta_1 \rightarrow \infty$).

* It is derived, like Eq. (4.11), from the requirement that there be no regular increase in the second-order correction to the amplitude in the crystal and has the form:

$$2hU_0 u_1 + h \frac{V_0^2 - U_0^2}{V_0} v_1 - k_2 V_0 r_1 = 0.$$

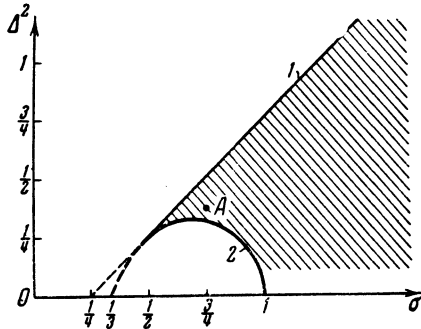


FIG. 3. 1 - $\Delta^2 = \sigma - 1/4$; 2 - $\Delta^2 = (1 - \sigma)(3\sigma - 1)$

At this point there occurs a region of unstable solutions which carry over to the non-stabilized mode² and therefore the lower edge of the stability region is not extended to the abscissa axis in Fig. 3. The non-stabilized mode, for which the crystal is not excited in the zeroth approximation, will not be considered, since it is very much like the case of the simple oscillator (I, Sec. 5).

5. FLUCTUATION IN A COUPLED SYSTEM

Equations (4.15) coincide with Eqs. (2.5) if we replace $F_{||}$ and χ of the latter by $V_0 F_{||}$ and ξ . Using these substitutions we can apply formulas (2.7) - (2.10) to find ρ and ξ , thus we have

$$\overline{\rho^2} = \frac{\mu C}{4(p_1 + q_2)} \left\{ 1 + \frac{q_2^2 + (V_0^2 p_2^2 / R_0^2)}{p_1 q_2 - p_2 q_1} \right\} \quad (5.1)$$

$$\overline{\xi(\tau) \xi(\tau')} = \frac{\mu C}{4(\lambda_1^2 - \lambda_2^2)} \quad (5.2)$$

$$\times \left\{ \left[q_1^2 + \frac{V_0^2}{R_0^2} (p_1^2 - \lambda_1^2) \right] \frac{1}{\lambda_1} e^{\lambda_1 |\tau' - \tau|} - \left[q_1^2 + \frac{V_0^2}{R_0^2} (p_1^2 - \lambda_2^2) \right] \frac{1}{\lambda_2} e^{\lambda_2 |\tau' - \tau|} \right\}$$

$$\overline{\xi^2} = \frac{\mu C}{4(p_1 + q_2)} \left\{ \frac{q_2^2 + (V_0^2 / R_0^2) p_1^2}{p_1 q_2 - p_2 q_1} + \frac{V_0^2}{R_0^2} \right\}$$

Using Eq. (4.14), we find the mean-square value of the phase fluctuations:

$$\overline{\chi^2(\tau)} = \frac{1}{V_0^2} \left[\overline{\xi^2(\tau)} - 2\overline{\xi(\tau) \xi(0)} + \overline{\xi^2(0)} \right] = \frac{2}{V_0^2} \left[\overline{\xi^2} - \overline{\xi(\tau) \xi(0)} \right] \quad (5.3)$$

and consequently formulas (5.2) can also be used to obtain $\overline{\chi^2}$. We now consider the amplitude fluctuations.

Substituting the values of the coefficients p_1, p_2, q_1, q_2 from Eq. (4.16) into Eq. (5.1), and using Eqs. (4.5), it is not difficult to reduce the expression for $\overline{\rho^2}$ to the form:

$$\overline{\rho^2} = \frac{\mu C \left[\left(\frac{R_0^2}{4} - 1 \right) \left(\frac{3R_0^2}{4} - 1 - 2\sigma \right) + \Delta^2 \right]}{4 \left(\frac{R_0^2}{2} - 1 \right) \left[\left(\frac{R_0^2}{4} - 1 \right) \left(\frac{3R_0^2}{4} - 1 \right) + \Delta^2 \right]}$$

In turn, substituting the expression for $R_0^2/4$ given in Eq. (4.17), the above expression is transformed into the following:

$$\overline{\rho^2} = \frac{\mu C}{4(1 - 2\sigma)} \left\{ \frac{1 - 4\sigma}{1 - 2\sigma + 2\sqrt{\sigma^2 - \Delta^2}} + \frac{\sigma}{1 - 2\sigma + \sqrt{\sigma^2 - \Delta^2}} \right\} \quad (5.4)$$

The denominator of the first term becomes zero for $\Delta^2 = \sigma - 1/4$, i.e., at the outer limit of the region of stability of the stabilized mode (Fig. 3). The denominator of the second term becomes zero for $\Delta^2 = (1 - \sigma)(3\sigma - 1)$, i.e., at the inner limit which obtains when $1/2 < \sigma < 1$. The appearance of stability limits at which the generating circle becomes completely "dissolved" means that the smallest value of $\overline{\rho^2}$ obtained outside the region of stability is higher than that of the simple oscillator.

If we assume a sufficiently strong coupling to the "crystal" ($\sigma > 1$, see Fig. 3), then the minimum $\overline{\rho^2}$ is obtained at a deviation Δ_{opt} determined by the equation

$$\sqrt{\sigma^2 - \Delta_{opt}^2} = \frac{4(1 - 2\sigma) + \sqrt{6(1 - 2\sigma)(2 - 3\sigma)}}{2}$$

This minimum is

$$\overline{\rho^2}_{min} = \frac{\mu C}{4(2\sigma - 1)} \left\{ \frac{4\sigma - 1}{3(2\sigma - 1) - \sqrt{6(1 - 2\sigma)(2 - 3\sigma)}} - \frac{2\sigma}{2(2\sigma - 1) - \sqrt{6(1 - 2\sigma)(2 - 3\sigma)}} \right\}$$

For $\sigma = 1$ we have $\Delta_{opt}^2 = 0.4$ and $\overline{\rho_{min}^2} = 10\mu C_4$, i.e., it is 10 times larger than the level of amplitude fluctuations in the non-stabilized oscillator [see Eq. (5.9) in I]. With an increase of σ , $\overline{\rho_{min}^2}$ is reduced, but even for $\sigma \rightarrow \infty$ (in which case $\Delta_{opt}^2 \rightarrow \sqrt{\sigma/2}$) we have $\overline{\rho_{min}^2} \rightarrow \mu C$, i.e., it is 4 times larger than in the simple oscillator. In the case of weak coupling to the "crystal", when the limits of the stability region become narrower, $\overline{\rho^2}$ becomes even larger inside the region. Thus, for example, at the point A in Fig. 3 ($\Delta^2 = 3/8$, $\sigma = 3/4$) we have $\overline{\rho^2} = 33.2 \mu C/4$.

Hence, stabilization leads to the reduction of "stability" of the generating circle and to an increase of the strength of the amplitude fluctuation in the oscillator circuit. The relative amplitude fluctuation $\mu\sqrt{\overline{\rho^2}}/R_0$ is still larger since the radius of the generating circle R_0 in the stabilization case is always smaller than two; this is apparent from Eq. (4.17).

We now consider the phase fluctuations. From Eq. (5.2) the correlation function of ξ tends towards zero with an increase of $|\tau' - \tau|$. Consequently, for large τ the second term in Eq. (5.3) vanishes and $\overline{\chi^2}$ attains a stationary value:

$$\overline{\chi^2} = \frac{2\overline{\xi^2}}{V_0^2} = \frac{\mu C}{4(p_1 + q_2)} \left\{ \frac{(q_1^2/V_0^2) + (p_1^2/R_0^2)}{p_1 q_2 - p_2 q_1} + \frac{1}{R_0^2} \right\}.$$

Using Eqs. (4.5) and (4.16), this expression can be transformed to the same as Eq. (3.3), that of the synchronization case:

$$\overline{\chi^2} = \frac{\mu C [(3Z - 1)(2Z - 1) + \Delta^2]}{8Z(2Z - 1)[(3Z - 1)(Z - 1) + \Delta^2]} \quad (5.5)$$

$$\left(Z = \frac{R_0^2}{4} \right).$$

It is not necessary for us to analyze this expression. The important feature is that in the first-order correction to the phase there is no diffusion; a stationary process is set up very much like that which takes place in the synchronization of the oscillator by an external synchronizing force. Hence, in the first approximation the effect of the "crystal" can be interpreted as that of a synchronizing force - not only as regards the dynamic behavior of the oscillator but also with regard to phase fluctuations. Using Eqs. (3.4) and (4.17), we find the amplitude of the equivalent synchronizing emf due to the "crystal";

we have

$$\begin{aligned} H^2 &= Z [(Z - 1)^2 + \Delta^2] \\ &= 2(1 - \sigma + \sqrt{\sigma^2 - \Delta^2})(\Delta^2 \\ &\quad + \sigma^2 - \sigma\sqrt{\sigma^2 - \Delta^2}). \end{aligned}$$

There are, however, certain differences from the synchronization case. Firstly, in the synchronization case the phase φ_0 is determined by the external force, while here the random spread takes place about the value φ_0 which depends on the initial conditions. Secondly, and this is more important, in the isolated system, phase diffusion always takes place, while the result we have obtained here only indicates that, in the case of stabilization, the diffusion coefficient is of a smaller order of magnitude. If the phase fluctuation is written in the form $\Delta\varphi = \mu\chi + \mu^2\chi_1 + \mu^3\chi_2 \dots$ then

$$\overline{\Delta\varphi^2} = \mu^2\overline{\chi^2} + 2\mu^3\overline{\chi\chi_1} + \mu^4(\overline{\chi_1^2} + 2\overline{\chi\chi_2}) + \dots$$

One is easily convinced that (for large τ) $\overline{\chi^2}$ does not contain terms proportional to τ . For calculation of the following terms we must turn to the equations for the higher approximations - the third for χ_1 and the fourth for χ_2 . The calculation of χ_1 shows that the term $\overline{\chi\chi_1}$ also does not yield a diffusion dependence on τ , but undoubtedly it will appear in $\overline{\chi_1^2}$. Because it is excessively complicated the calculation of the diffusion coefficient cannot be carried out successfully and one can only draw the conclusion that stabilization reduces the phase diffusion coefficient by a factor $\sim \mu^2$, i.e., it multiplies the time required to establish a regular phase distribution along the generating circle by a factor $\sim 1/\mu^2$.

6. THERMAL NOISE

Up to this point only the shot-noise of the tube has been considered in the physical examples of the random force $F(t)$. However, our method based on the use of symbolic equation and correlation theory can also easily be applied to the calculation of thermal fluctuations. For this purpose, in forming the equations of motion one merely introduces a random thermal emf localized in the physical resistances of the circuit.

We will consider the coupled system which has already been treated (Fig. 2). If the branch containing R and R_1 now contains the emf sources $\mathcal{E}(t_1)$ and $\mathcal{E}_1(t_1)$, then in place of equations (4.3), we have

$$\frac{d^2x}{dt^2} + x = \mu \left\{ \frac{dx}{dt} \left[1 - \frac{1}{3} \left(\frac{dx}{dt} \right)^2 \right] \right. \tag{6.1}$$

$$\left. + \Delta x - k_1 \frac{d^2y}{dt^2} \right\} + \mu^2 F(t) + \frac{1}{L\omega^2 I_0} \frac{d\mathcal{E}}{dt_1},$$

$$\frac{d^2y}{dt^2} + y = -\mu^2 \left(h \frac{dy}{dt} + k_2 \frac{d^2x}{dt^2} \right) + \frac{1}{L_1\omega^2 I_0} \frac{d\mathcal{E}_1}{dt_1}.$$

Here we have introduced the dimensionless parameters and the time $t = \omega t_1$ everywhere except in the last terms containing \mathcal{E} and \mathcal{E}_1 . In accordance with the Nyquist formula for the spectral intensity of a thermal emf, the correlation function of $\mathcal{E}_1(t)$ is

$$\overline{\mathcal{E}(t_1)\mathcal{E}(t'_1)} = 2RkT\delta(t_1 - t'_1). \tag{6.2}$$

Inasmuch as R_1 and R_2 are taken to be quantities of the same order of magnitude (the first) with respect to μ , the emf \mathcal{E} and the emf \mathcal{E}_1 are also of the same order. If the random force appears to second order in the first equation of (6.1), i.e.,

$$\frac{1}{L\omega^2 I_0} \frac{d\mathcal{E}}{dt_1} = \mu^2 G(t), \tag{6.3}$$

it will appear to order μ^3 in the second equation. Hence the thermal noise of the ‘‘crystal’’ need not be considered in calculations having an accuracy of the order μ^2 .

In order to calculate the correlation function of $G(t)$ one must know the correlation function

$$\frac{d\mathcal{E}(t_1)}{dt_1} \frac{d\mathcal{E}(t'_1)}{dt'_1}. \text{ The latter cannot be obtained}$$

from Eq. (6.2) in the usual manner because $\mathcal{E}(t_1)$ does not have a finite mean-square value. However, this difficulty can be circumvented if one notes that in the approximation which is of interest to $\frac{d\mathcal{E}}{dt_1} = \frac{\partial \mathcal{E}}{\partial t_1} = \dot{\mathcal{E}}$ so that the dependence of \mathcal{E} on the

fast time t_1 has the form

$$\mathcal{E}(t_1) = \mathcal{E}_{\parallel}(\tau_1) \cos \omega [t_1 - \varphi(\tau_1)] - \mathcal{E}_{\perp}(\tau_1) \sin \omega [t_1 - \varphi(\tau_1)],$$

where $\tau_1 = \mu t_1$. Consequently,

$$\begin{aligned} d\mathcal{E}(t_1)/dt_1 \approx & \\ & -\omega \mathcal{E}_{\parallel}(\tau_1) \sin \omega [t_1 - \varphi(\tau_1)] \\ & -\omega \mathcal{E}_{\perp}(\tau_1) \cos \omega [t_1 - \varphi(\tau_1)]. \end{aligned}$$

Now, since $\overline{\mathcal{E}_{\parallel}(\tau_1)\mathcal{E}_{\parallel}(\tau'_1)} = \overline{\mathcal{E}_{\perp}(\tau_1)\mathcal{E}_{\perp}(\tau'_1)}$ and \mathcal{E}_{\parallel} and \mathcal{E}_{\perp} are not cross-correlated, we have

$$\begin{aligned} \overline{\frac{d\mathcal{E}(t_1)}{dt_1} \frac{d\mathcal{E}(t'_1)}{dt'_1}} &= \omega^2 \overline{\mathcal{E}_{\parallel}(\tau_1)\mathcal{E}_{\parallel}(\tau'_1)} \\ &= \omega^2 \overline{\mathcal{E}(\tau_1)\mathcal{E}(\tau'_1)} = 2\omega^2 RkT\delta(t_1 - t'_1). \end{aligned} \tag{6.4}$$

Thus, $G(t)$ has a correlation function of the form

$$\overline{G(t)G(t')} = D\delta(t - t'), \tag{6.5}$$

in which, according to Eqs. (6.3) and (6.4),

$$\begin{aligned} \mu^4 \overline{G(t)G(t')} &= \frac{\mu^4 D}{\omega} \delta(t_1 - t'_1) \\ &= \frac{1}{L^2\omega^4 I_0^2} \frac{d\mathcal{E}(t_1)}{dt_1} \frac{d\mathcal{E}(t'_1)}{dt'_1} = \frac{2RkT}{L^2\omega^2 I_0^2} \delta(t_1 - t'_1), \end{aligned}$$

i.e.,

$$\mu^4 D = 2RkT / L^2\omega I_0^2. \tag{6.6}$$

In our approximation the thermal emf calculation amounts to adding a random force $G(t)$ to $F(t)$ in the first equation of Eq. (6.1); this force is also characterized by δ -correlation and is independent of $F(t)$. Thus, in the problem of the simple oscillator (see Figure in I), taking $k_1 = 0$ and $\Delta = 0$ (i.e., $\omega = \omega_0$) in the first equation of (6.1), we have

$$\begin{aligned} \frac{d^2x}{dt^2} + x &= \mu \frac{dx}{dt} \left[1 - \frac{1}{3} \left(\frac{dx}{dt} \right)^2 \right] \\ &+ \mu^2 F(t) + \mu^2 G(t). \end{aligned}$$

It is apparent that all the bilinear quantities characterizing the correlation and intensity of the fluctuations will, in the approximation being considered, be formed additively from terms calculated before which depend on $F(t)$, and these same terms as determined by $G(t)$. Thus, for example, in place of formulas (5.9) of I for the fluctuations of amplitude and phase in the simple oscillator, we now have

$$\overline{\rho^2} = \frac{\mu}{4} (C + D), \quad \overline{\chi^2} = \frac{\mu}{8} (C + D)\tau.$$

The ratio of shot-noise to thermal noise is given by C/D , i.e., from Eqs. (5.7), (5.8) of I and Eq. (6.6) it is

$$\frac{\text{shot}}{\text{thermal}} = \frac{\mu^4 C}{\mu^4 D} = \frac{e}{2kT} \frac{L^2 \omega_0^2 \bar{I}_a}{R}.$$

This expression can be written in a simple and descriptive form due to Gorelik. The second factor, since $\frac{L\omega_0}{R} \bar{I}_a = I_{\text{amp}}$, is equal to the voltage which appears across the condenser in the circuit:

$$\frac{L^2 \omega_0^2 \bar{I}_a}{R} = L\omega_0 I_{\text{amp}} = \frac{I_{\text{amp}}}{\omega_0 C} = U_C.$$

Consequently,

$$\frac{\text{shot}}{\text{thermal}} = \frac{eU_C}{2kT}$$

The ratio of shot-noise to thermal noise is equal to the ratio of the work done in carrying an electron across the condenser (at maximum voltage) to twice the energy of the thermal noise in the circuit. For $T = 300^\circ \text{K}$, we have

$$\frac{\text{shot}}{\text{thermal}} = 17 U_C,$$

Where U_C is given in volts.

Translated by H. Lashinsky
198