Further Discussion on Fluctuations in Gravitating Systems

IA. P. TERLETSKII
Moscow State University

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IN connection with the objections[1] relative to the conclusions arrived at by me in references 2 and 3, I must necessarily write some explanation verifying the correctness of these conclusions.

In a note and at a meeting we have pointed out that in sufficiently large gravitating systems the magnitude of fluctuations increases with increasing dimensions of the system. This statement was proved with the aid of the simplified model of the isothermal ideal gas, occupying a volume V for a constant external pressure P and temperature T (for example, an ideal gas in a container, closed by a piston on which a constant force is applied). An analogous model is used for calculating the value of the fluctuations in a real gas at the critical point, as is done, for example, by Leontovich[4].

For the proof we used the relations

$$\overline{(\Delta V)^2} = -kT\frac{\partial V}{\partial P}, \tag{1}$$

$$P = \frac{NkT}{V} - \frac{\partial U}{\partial V}, \quad U = -\alpha\frac{\varkappa M^2}{V^{1/3}},$$

the first of these can be proved as a rigorous theorem of the Gibbs statistical mechanics and hence is used only for systems in thermodynamic equilibrium; the expression for the potential energy U has the same validity as the law of gravitation of Newton.

From Eq. (1) the relative volume fluctuations of gravitating systems was derived as

$$\frac{\overline{(\Delta V)^2}}{V^2} = \frac{1}{N}\left(1 - \frac{4\alpha}{9}\,\frac{\varkappa m^2 N}{kTV^{1/3}}\right)^{-1}, \tag{2}$$

where m is the mass of the gas molecules, N is the total number of molecules in the system, $\varkappa$ is the gravitational constant and α is the numerical coefficient of the order of unity*. Since the last formula is used only for systems in thermodynamic equilibrium, the condition of applicability is

$$\partial P / \partial V < 0, \tag{3}$$

precisely as for the analysis of the fluctuations at the critical point [see Eq. (32.1) in reference 4].

According to Eq. (3), formula (2) can be applied to an ideal gravitating gas, enclosed in a shell under a sufficiently large positive external pressure ($P > NkT/V$), but is inapplicable to a confined gaseous cloud, occupying a volume V and located in an infinite empty space, since in this case $P = 0$, from which, according to Eq. (1), $\partial P/\partial V > 0$ and, consequently, condition (3) is not satisfied. Formula (2) is also inapplicable to the ideal gravitating gas, filling all of infinite space, since such a system obeys Newton's law of gravitation and is not thermodynamically stable.

Thus, if for the nature of the rough model of the real gravitating system of galactic (or metagalactic) form, we assume a gravitating ideal gas, enclosed in a shell, subjected to an external pressure (Model I) then, starting from formula (2), the applicability of which in the given case cannot be disputed, we come to the conclusion of increasing magnitude of fluctuations with increasing number of particles as the system approaches a gravitating unstable state. If as the model of a real gravitating system we use one gravitating according to the Newtonian law for an ideal gas filling all of infinite space (model II), then, by virtue of the Newtonian law of gravitation, such a system can not be in thermodynamic equilibrium, and, hence, formula (2) is inapplicable. If, finally, as the model of a real gravitating system we choose a spatially bounded gaseous cloud in an infinite empty space (Model III) then the system cannot be stable because of the condition $P = 0$ at the boundaries of the cloud. Clearly, such a model is unstable for an ideal non-gravitating gas, and can-

[1] M. I. Shakharonov, J. Exper. Theoret. Phys. USSR 27, 646 (1954)

[2] Ia. P. Terletskii, J. Exper. Theoret. Phys. USSR 22, 507 (1952)

[3] Ia. P. Terletskii, Proceedings of the Second Conference on the Problems of Cosmogony, Acad. Sci. USSR, 1953, p. 507

[4] M. A. Leontovich, *Statistical Physics*, Govt. Technical Printing Office, 1944, p. 124

* We note that in reference 3 this formula is given with two misprints. The accurate expression was derived in reference 2.

not be considered as a model of a real universe.

It is entirely obvious that in our notes[2,3] we implied Model I, and not models II or III, because only in the case of Model I is it valid to use formula (2) to fulfill the condition: $dP/dV < 0$ or consequently, $P > [(NkT)/(4V)]$. At first sight, Model II seems more accurate because the external shell of Model I is clearly an artificial structure. However, Model II results in a gravitationally unstable universe, and, hence, the usual laws of thermodynamics and statistical mechanics of systems in equilibrium are inapplicable. With Model II the assertions of the correctness of the thermodynamic laws or the small value of the fluctuations in macroscopic systems has as little basis as for its opposite. This model prohibits any theoretical analysis of thermodynamic or statistical questions regarding the universe because there does not exist a statistical mechanics or thermodynamics for absolutely nonequilibrium systems*.

It is known that a gravitationally unstable model of the universe is always inadequate, and repeated attempts were made to set aside the instability by generalizations from Newton's law of gravitation. It is not difficult to see that the instability of Model II is caused by a very rapid decrease of the potential energy of any part of the system as its dimensions increase. Actually, for the uniformly dense gas $\sigma = mN/V$, the potential energy of a part of the gas, enclosed in a volume V, can be expressed as $U = -a\kappa\sigma V^{5/3}$, while its kinetic energy $(3/2)NkT = (3\sigma kT/2m)V$ and consequently, for sufficiently large parts of the system the potential energy may exceed the kinetic energy by an arbitrary amount. However, a system is known to be unstable if the potential energy taken with a minus sign is greater than one-and-a-half times the kinetic energy, and, because of Eqs. (3) and (1), the system of a gravitating ideal gas can be stable only by fulfilling the condition

$$\frac{NkT}{V^2} + \frac{\partial^2 U}{\partial V^2} > 0, \tag{4}$$

and by virtue of Eq. (1),

$${}^3/_2({}^3/_2\, NkT) + U > 0. \tag{5}$$

Obviously, the stability of an unbounded system of an ideal gas may be guaranteed if we assume a violation of the Newtonian law of gravitation for very large systems and imagine, for example, above a critical value V_{cr} the potential energy of an isolated volume of a homogeneous gas increases not as $V^{5/3}$, but simply proportional to V. In this case, condition (4) is always fulfilled for sufficiently large systems. Thus, it is possible to have another model (Model IV) of a large gravitating system, in which the gas fills up all of infinite space, as in Model II, but in which the universe is gravitationally stable, as in Model I, as a result of the breakdown of Newton's law of gravitation for sufficiently large systems.

It is easy to see that for Model IV the fluctuations of the volume of any isolated part of the gas may be calculated from Eq. (2) if its volume does not exceed the critical volume V_{cr}, because in this case the law of gravitation has the Newtonian form. For volumes larger than V_{cr}, it is necessary to use another expression for U, and consequently, the expression for the fluctuations is different from Eq. (2). Thus, in the same domain where Newton's law of gravitation is fulfilled, the fluctuations can be calculated with the aid of Model I.

It is not difficult to cite a concrete example of Model IV. It is well-known that the Poisson equation for the gravitational potential φ does not have a finite solution for a uniformly distributed density for all space. In Einstein's theory of gravitation there occurs an analogous difficulty. It is known that this difficulty is removed in the classical Newtonian theory as well as in the more

* In his critical note, M. I. Shakharonov arbitrarily assumed that we considered Model II or Model III, as this may otherwise be understood from the reference to the formula which Shakharonov refers to as the virial theorem. We notice that Eq. (2) does not apply in these cases (in the first case this is obvious, owing to the gravitational instability of an unbounded system obeying Newton's law; in the second case, it is a consequence of the condition $P = 0$ at the boundary of the cloud). Shakharonov denies the correctness of my conclusions for the macroscopic character of the fluctuations in sufficiently large gravitating systems and the possibility of significant spontaneous departure from a state of thermodynamic equilibrium in a system of galactic dimensions. Denying our conclusions, Shakharonov considers the widespread opinion " that the fluctuations may not lead to a breakdown of thermodynamic equilibrium on a macroscopic scale" to be correct. Shakharonov does not note, however, the internal inconsistency of his conclusion. Using Model II, Shakharonov acknowledges a gravitationally unstable universe. But in this case it is impossible to come to any conclusions about the magnitude of the fluctuations, inasmuch as thermodynamics and statistical mechanics are inapplicable. The use of Model III is inconsistent as we demonstrated above because of the inevitable condition $P = 0$ and not by virtue of the virial theorem, as was maintained by Shakharonov. The virial theorem is in general inapplicable in the case of Model III, since the system of an ideal gas without an external potential barrier is unbounded.

precise theory of Einstein by means of the introduction of an additional term (the so-called cosmological term [see, for example, reference 5)]. It is easy to show that with the introduction of an additional term in the gravitational equations it is possible to eliminate the gravitational instability of an infinite gravitational system. Since we are concerned with the principal aspects of the problem, we will consider only the simplest generalization of Newton's theory, suggested by Neiman, according to which, in place of Poissons's equation, we postulate the equation of the form

$$\nabla^2\varphi - \lambda^2\varphi = 4\pi\sigma, \tag{6}$$

where $\lambda^2\varphi$ is the cosmological term.

According to Eq. (6) the potential energy of matter distributed with a density $\sigma(\mathbf{r})$ in a volume V can be expressed as:

$$U = -\frac{\varkappa}{2}\int_V\int_V \frac{\sigma(\mathbf{r})\,\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, e^{-\lambda|\mathbf{r}-\mathbf{r}'|}\, d\mathbf{r}\, d\mathbf{r}', \tag{7}$$

where $d\mathbf{r}$ and $d\mathbf{r}'$ are elements of volume.

For small volumes, where $V^{1/3} \ll \lambda^{-1}$, the exponential factor inside the integral expression can be neglected and in this case, for a system of mass M, we obtain the result of the Newtonian theory, that is, $U \sim M^2 V^{-1/3}$. For larger volumes, when $V^{1/3} \gg \lambda^{-1}$, then, according to Eq. (7) we obtain $U \sim M^2 V^{-1}$. For example, for the case of a volume filled with matter of uniform density $\sigma = M/V$, we calculate

$$U = -\frac{\varkappa M^2 2\pi}{\lambda^2 V} \text{ при } V^{1/3} \gg \lambda^{-1}. \tag{8}$$

Consequently, for a universe that is filled uniformly with matter, the potential energy taken with a minus sign for a sufficiently large portion, increasesproportionally with the volume, as well as does the kinetic energy. Thus, an ideal gas, gravitating according to Eq. (6), and filled uniformly to an infinite extent, can be considered as an example of Model IV.

Model IV may be rather somewhat of an approximation to the real universe, if we consider not an ideal, but a real gas, obeying Van der Waals equation and gravitating according to Eq. (6). For such a gas, for $V^{1/3} \ll \lambda^{-1}$, according to Eq. (8), we have

$$P = \frac{N(kT)}{(V - Nb)} - \frac{aN^2}{V^2} - \frac{2\pi\varkappa m^2 N^2}{\lambda^2 V^2}, \tag{9}$$

i.e., written differently, the gas obeys Van der - Waals equation and has a constant a, equal to

[5] D. Ivanenko and A. Sokolov, *Classical Theory of Fields*, GITTL, 1951, p. 70

$$A = a + \frac{2\pi\varkappa m^2}{\lambda^2}. \tag{10}$$

It is obvious that the question of stability of the model chosen is the same as for a real gas, obeying Van der Waal's equation, and the expression for the relative volume fluctuations has the form

$$\frac{\overline{(\Delta V)^2}}{V^2} = \frac{1}{N}\left[\frac{1}{\left(1 - \frac{Nb}{V}\right)^2} - \frac{2a}{\Theta}\frac{N}{V}\right]^{-1}. \tag{11}$$

It is obvious that the latter formula stands on the same basis in this case as in the case of an ordinary real gas.

From the point of view of Model IV the Boltzmann fluctuation hypothesis is more likely to be valid. Let the average density of the universe be such that on the average it is found in the gaseous phase and, according to Eq. (11), for sufficiently large volumes macroscopic fluctuations are possible. If fluctuations of gigantic dimensions are realized with increasing density in a definite portion of the universe, then in this region a condensed phase is developed, i.e., the formation of stars, planets, etc., results. In the course of time, however, this fluctuation is resolved, i.e., the density begins to decrease because the mass leaves the center of the fluctuation. After the transition to a sufficiently rarefied state there remains a region of condensed material (for example, a planet) which is gradually transformed into the gas phase owing to the statistical instability of such gravitating systems. In this manner, this portion of the universe is finally returned to its initial condition. Parallel to the fluctuations changing the volume on a macroscopic scale are changes in entropy, and, consequently, the law of monotonic increase in entropy is violated in a large part of the universe. Clearly, what this picture describes is only a roughly simplified model of an endlessly complicated phenomenon in a bounded portion of an infinite universe. However, it is not excluded that in this model there are shown features of cosmological processes in the course of large intervals of time in systems on a metagalactic scale.

An analysis of the methodological significance of the Boltzmann fluctuation hypothesis is given in articles in references 6 and 7 and in a monograph[8]

[6] Ia. P. Terletskii, Dokl. Akad. Nauk SSSR **72**, 1041 (1950)

[7] Ia. P. Terletskii, *Problems of Philosophy*, No. 5, 1951, p. 180

[8] Ia. P. Terletskii, *Dynamic and Statistical Laws of Physics*, IZD, Moscow State University, 1950

Another article[9] is devoted to a related problem. As for the paper of M. I. Shakharonov in the Reports of the Moscow University[10], which he cites in reference 1 as if it contains a philosophic analysis of Boltzmann's hypothesis and in which he draws a conclusion about the supposedly idealistic character of Boltzmann's theory, it cannot be regarded as convincing, especially in view of an evaluation of Boltzmann, given by V. I. Lenin[11], as a materialist who struggled systematically against Machism. It is strange that Shakharonov arrives at the conclusions of Boltzmann's idealism from numerous quotations of Engels, which merely give witness to the direct bond between the fluctuation hypothesis of Boltzmann and the exposition of the qualitative indestructibility of motion as developed by Engels.

[9] Ia. P. Terletskii, J. Exper. Theoret. Phys. USSR **17**, 837 (1947)

[10] M. I. Shakharonov, Reports, Moscow State University **6**, 15 (1953)

[11] V. I. Lenin, *Collected Works*, 4th printing, vol. 14, pp. 274-276

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