The Theory of the Propagation of Gamma Rays through Matter

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The angular and energy distribution of gamma radiation is found as a function of depth of penetration into matter, for an initial energy of the order of several mev.

THE investigation of the energy spectrum and angular distribution of gamma rays as a function of depth of penetration in matter is of interest both for theory and for many experiments and applications, among which we may mention various questions concerning dosimetry, design of gamma-ray filters, etc.

In the present paper the following question is considered. A parallel beam of monochromatic γ - radiation is incident perpendicular on the plane surface of a layer of matter. How will the distribution of the γ - radiation in energy and angle change, as a function of depth, initial energy, and the properties of the material?

The energy spectrum of the scattered radiation has been found for several cases, without any simplifications, by a numerical method[1].

In contrast to this work, an analytic method is developed in the present paper for determining not only the energy spectrum, but also the angular distribution of the scattered γ - radiation. The calculations developed in the work are based on an equation of radiative transfer of the Boltzmann type. The angles of scattering are assumed to be small, as is the case for energies of the order of several mev. For the Klein-Nishina-Tamm cross section, we use an approximate formula which gives a good fit to the true cross section. No other omissions or simplifications are contained in the work.

In the special case where the absorption coefficient of the γ - rays can be taken as constant, this problem has been solved previously[2,3] by another method, more complicated than that applied here.

1. INTRODUCTION

During the passage of γ - rays through matter, absorption and scattering occur as a result of the following three fundamental processes: a) photo-effect, b) Compton effect, c) production of electron-positron pairs.

The photoeffect predominates at low energies. With increasing energy, the Compton scattering becomes important. For still higher energies, the probability of photo-and Compton effects decreases, while the probability of pair formation increases, so that pair formation begins to play the major role.

The energy interval within which the Compton effect is dominant is quite wide for light elements (0.03-25 mev for carbon, 0.05-15 mev for aluminum) and becomes narrower with increasing atomic number (0.2 -12 mev for copper, 0.6-5 mev for lead).

The intensity of the unscattered monochromatic radiation decreases exponentially during its passage through matter[4]

$$J = J_0 e^{-\tau x}, \quad (1)$$

where τ is the total absorption coefficient. Figure 1 (which is taken from reference 4) shows the dependence of the total absorption coefficient on energy for several elements. We note that for light elements, in the high energy region, (for example, for aluminum with $h\nu \gtrsim 10\ m_0c^2$), τ can be taken as approximately constant.

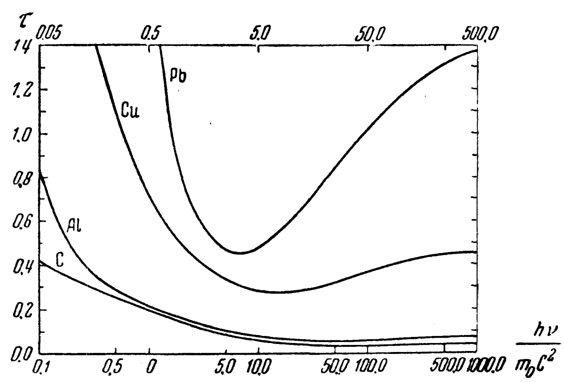

FIG. 1. Energy dependence of γ - ray absorption coefficient

[1] L. Spencer and U. Fano, Phys. Rev. 81, 464(1951); J. Res. Nat. Bur. Stand. 46, 446 (1951)

[2] L. L. Foldy, Phys. Rev. 81, 395,400 (1951)

[3] L. L. Foldy, Phys. Rev. 82, 927 (1951)

[4] W. Heitler, *Quantum Theory of Radiation*

In the energy range in which the Compton scattering is most important, we can neglect the radiation from photoelectrons, Compton recoils and pair particles.

We introduce certain formulas concerning the theory of Compton scattering. It is well known that the Compton effect is the incoherent scattering of γ - radiation by free electrons. Applying the laws of conservation of energy and momentum leads to the following expression for the wavelength of a photon scattered through an angle θ:

$$\lambda' = \lambda + 1 - \cos\theta, \qquad (2)$$

where λ' and λ are, respectively, the wavelengths of the scattered and initial photon, expressed in units of the Compton wavelength ($\lambda = m_0c^2/h\nu$).

The Klein-Nishina-Tamm formula, giving the probability that a photon of wavelength λ, in passing through a one centimeter layer of material, collides with an electron and is deflected through an angle θ while its wavelength becomes λ', is conveniently written in the form

$$d\,W_K\,(\lambda, \lambda', \theta) = n\pi r_0^2\left(\frac{\lambda}{\lambda'}\right)^2\left[\frac{\lambda}{\lambda'} + \frac{\lambda'}{\lambda} - \sin^2\theta\right] \qquad (3)$$

$$\times\ \frac{1}{2\pi}\,\delta\,(1 - \cos\theta - \lambda' + \lambda)\,d\lambda' d\Omega,$$

where n is the number of electrons per cc, r_0 is the classical electron radius, $d\Omega$ is the element of solid angle, and $\delta(\alpha)$ is the delta function.

In this form, $dW_K(\lambda, \lambda', \theta)$ automatically becomes zero if condition (2) is not satisfied.

We should emphasize that for gamma rays with an energy of the order of several mev, scattering through large angles, which is accompanied by large energy loss, is improbable compared to scattering through small angles. We may therefore neglect the $\sin^2\theta$ term in (3).

For our later work we also replace the factor $\frac{1}{2}(\lambda/\lambda')^2[(\lambda/\lambda') + (\lambda'/\lambda)]$ in (3) by $(\lambda/\lambda')^k$. If we determine the exponent k by the method of least squares, k turns out to be 1.69. Since our main concern is with values of λ' close to λ, we have chosen $k = 1.8$. In reference 2, k was taken equal to unity. In a note by the same author[3], the possibility of choosing k different from unity is pointed out. From. Fig. 2 it is clear that our choice gives a good fit to the true cross section. The expression for dW_K becomes:

$$d\,W_K\,(\lambda, \lambda', \theta) \qquad (3')$$

$$= a\,(\lambda/\lambda')^k\,(1/2\pi)\,\delta\,(1 - \cos\theta - \lambda' + \lambda)\,d\Omega d\lambda',$$

where $k = 1.8$ and $a = 2\pi n r_0^2$.

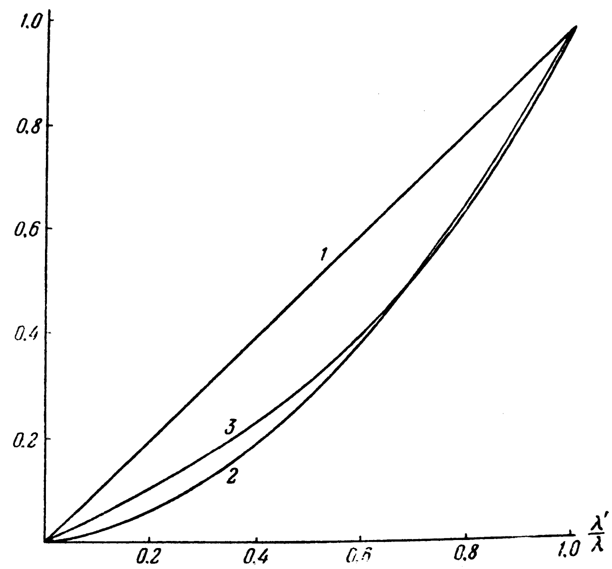

FIG. 2. Approximate replacements for the Klein-Nishina-Tamm formula. *1* - replacement of the factor $\frac{1}{2}(\lambda'/\lambda)^2[(\lambda'/\lambda) + (\lambda/\lambda')]$ by (λ'/λ); *2*- replacement of the same factor by $(\lambda'/\lambda)^{1.8}$, which is used in this paper; *3*- exact value.

The values of the constant a for various elements are given in the following table.

Values of the constant a in cm^{-1}

	C	Al	Fe	Cu	Pb
a	0.3435	0.3885	1.095	1.215	1.3425

The distribution of once-scattered radiation was found in reference 5.

[5] L. Cave, J. Corner and R. Liston, Proc. Roy. Soc. (London) **A204**, 223 (1950)

It is obvious that for a monochromatic unidirectional source, the distribution of unscattered radiation can be given in the form

$$\Gamma^0 (x, \lambda, \theta) = (1/2\pi)\,\delta(1 - \cos\theta)\,\delta(\lambda - \lambda_0)\, e^{-\tau(\lambda_0)x}, \tag{4}$$

where λ_0 is the wavelength of the primary radiation.

One then finds[5], for the distribution in angle and energy of the once-scattered radiation (in our notation), the following expression:

$$\Gamma^1 (x, \lambda, \theta)\, d\Omega d\lambda = d W_K (\lambda_0, \lambda, \theta)\, e^{-\tau(\lambda_0)x} \times \frac{1 - \exp\left\{-\dfrac{\tau(\lambda) - \tau(\lambda_0)\cos\theta}{\cos\theta} x\right\}}{\tau(\lambda) - \tau(\lambda_0)\cos\theta}. \tag{5}$$

Calculation by the method of reference 5 of higher orders of scattering becomes, even for the second, extremely involved, and is practically impossible. We therefore turn to the general equation of radiative transfer.

2. THE EQUATION OF RADIATIVE TRANSFER

We denote by $\Gamma(x, \lambda, \theta)$ the photon distribution function in wavelenght λ, angle θ with respect to the normal to the surface of the material , and depth x in cm. Then if the primary radiation has wavelength λ_0 and is incident perpendicular to the surface of the material, the change in $\Gamma(x, \lambda, \theta)$ is given by the equation

$$\cos\theta \frac{\partial \Gamma(x, \lambda, \theta)}{\partial x} = -\tau(\lambda)\Gamma(x, \lambda, \theta) + \int_{\lambda_0}^{\lambda}\int_{4\pi} \Gamma(x, \lambda', \theta')\, d W_K (\lambda', \lambda, \theta_1) + \frac{1}{2\pi}\delta(1 - \cos\theta)\,\delta(\lambda - \lambda_0)\,\delta(x), \tag{6}$$

where

$$\cos\theta_1 = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi').$$

Equation (6) can be described in words as follows:

The change in the distribution function $\Gamma(x,\lambda,\theta)$ (the left side of the equation) is caused by absorption (first term on the right), scattering (second term), and by newly arriving photons from the source (last term on the right).

We note that we could have omitted the source term in the equation, and taken account of it by a boundary condition at $x = 0$. Clearly, in our case the distribution function does not depend on azimuth.

We expand $\Gamma(x,\lambda,\theta)$ in a series of Legendre polynomials

$$\Gamma(x, \lambda, \theta) = \frac{1}{2\pi}\sum_{l=0}\frac{2l+1}{2}\Gamma_l(x, \lambda) P_l(\cos\theta). \tag{7}$$

Using Eqs. (6) and (3′), we obtain the following infinite system of equations for $\Gamma_l(x,\lambda)$:

$$\frac{l+1}{2l+1}\frac{\partial \Gamma_{l+1}(x, \lambda)}{\partial x} + \frac{l}{2l+1}\frac{\partial \Gamma_{l-1}(x, \lambda)}{\partial x} + \tau(\lambda)\Gamma_l(x, \lambda) = a\int_{\lambda_0}^{\lambda}\left(\frac{\lambda'}{\lambda}\right)^{1,8} P_l(1 - \lambda + \lambda')\,\Gamma_l(x, \lambda')\, d\lambda' + \delta(x)\,\delta(\lambda - \lambda_0). \tag{8}$$

In deriving this system of equations, we have expanded the delta function which appears in the transfer equation (6) in a series of Legendre polynomials and have used the addition theorem for Legendre polynomials.

The solution of this system of equations in the general case is very difficult. But, as already mentioned, the scattering through small angles is much more probable than scattering through large angles, especially for γ - rays of high energy (3-4 mev and greater). We may therefore assume that the angle θ is small. Since, for small angles, the terms in (7) with large l are most important, the expansion (7) in Legendre polynomials agrees approximately with the expansion in Bessel functions of zero order (Hankel transformation). We mention that this fact was first used in studying the scattering of electrons, by Kompaneets[6]:

$$\Gamma(x, \lambda, \theta) = (1/2\pi)\int_0^{\infty} J_0(l\theta)\,\Gamma_l(x, \lambda)\, l dl. \tag{9}$$

where we have used the relation

$$\lim_{l\to\infty} P_l\left(\cos\frac{\theta}{l}\right) = J_0(\theta). \tag{10}$$

For large l the system of equations (8) reduces approximately to the equation

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[6] A.S. Kompaneets, J. Exper. Theoret. Phys. USSR 15, 235 (1945)

$$\frac{\partial \Gamma_l(x,\lambda)}{\partial x} + \tau(\lambda)\Gamma_l(x,\lambda) = a\int_{\lambda_0}^{\lambda} J_0\left(l\sqrt{2(\lambda-\lambda')}\right)\left(\frac{\lambda'}{\lambda}\right)^{1,8}\Gamma_l(x,\lambda')\,d\lambda' + \delta(x)\,\delta(\lambda-\lambda_0). \quad (11)$$

Equation (11) is the definitive equation which we wanted to get for the distribution function of scattered γ - radiation in angle and energy.

3. SOLUTION OF THE EQUATION OF RADIATIVE TRANSFER

We expand the absorption coefficient $\tau(\lambda)$ in powers of $(\lambda - \lambda_0)$:

$$\tau(\lambda) = \tau_0 + \tau_1(\lambda-\lambda_0) + \tau_2(\lambda-\lambda_0)^2 + \ldots, \quad (12)$$

multiply Eq. (11) by $(\lambda/\lambda_0)^{1.8}\exp[-(\lambda-\lambda_0)s]$ and integrate with respect to λ from λ_0 to ∞. We then get

$$\frac{\partial F_l(x,s)}{\partial x} + \tau_0 F_l - \tau_1\frac{\partial F_l}{\partial s} + \tau_2\frac{\partial^2 F_l}{\partial s^2} - \ldots = \left(\frac{a}{s}\right)\exp[-l^2/2s]\,F_l(x,s) + \delta(x), \quad (13)$$

where $F_l(x,s)$ is related to $\Gamma_l(x,\lambda)$ by a Laplace transformation:

$$F_l(x,s) = \int_{\lambda_0}^{\infty}(\lambda/\lambda_0)^{1,8}\,\Gamma_l(x,\lambda)\exp(-(\lambda-\lambda_0)s)\,d\lambda. \quad (14)$$

Clearly, the unknown distribution function $\Gamma(x,\lambda,\theta)$ is expressed as:

$$\Gamma(x,\lambda,\theta) = \frac{1}{2\pi i}\left(\frac{\lambda_0}{\lambda}\right)^{1,8}\int_{\delta-i\infty}^{\delta+i\infty}\exp[(\lambda-\lambda_0)s]\,ds\,\frac{1}{2\pi}\int_0^{\infty}F_l(x,s)\,J_0(l\theta)\,l\,dl. \quad (15)$$

To solve Eq. (13), we look for $\Gamma_l(x,s)$ in the form

$$F_l(x,s) = e^{-\tau_0 x}\sum_{n=0}^{\infty} m_n(s,l)\,x^n. \quad (16)$$

We obtain the recursion relation

$$(n+1)\,m_{n+1}(s,l) = \frac{a}{s}\exp\left(-\frac{l^2}{2s}\right)m_n(s,l) + \tau_1\frac{\partial m_n(s,l)}{\partial s} - \tau_2\frac{\partial^2 m_n(s,l)}{\partial s^2} + \ldots \quad (17)$$

for $m_n(s,l)$.

Clearly, $m_0(s,l) = 1$. The expressions for $m_n(s,l)$ are sums of terms of the type

$$\frac{l^{2k}}{s^p}\exp\left(-r\,\frac{l^2}{2s}\right) \quad (18)$$

so that we can easily construct Laplace and Hankel transforms.

If we are interested only in the energy spectrum

$$\Gamma_0(x,\lambda) = 2\pi\int_0^{\infty}\Gamma(x,\lambda,\theta)\,\theta\,d\theta, \quad (19)$$

then the quantities $m_n^{(0)}(\lambda)$ in the expansion

$$\Gamma_0(x,\lambda) = \left(\frac{\lambda_0}{\lambda}\right)^{1,8} e^{-\tau_0 x}\sum_{n=0}^{\infty} m_n^{(0)}(\lambda)\,x^n \quad (20)$$

will be connected by the recursion relations:

$$(n+1)\,m_{n+1}^{(0)}(\lambda) = a\int_{\lambda_0}^{\lambda} m_n^{(0)}(\lambda')\,d\lambda' - [\tau(\lambda)-\tau_0]\,m_n^{(0)}(\lambda), \quad (21)$$

where $m_0^{(0)}(\lambda) = \delta(\lambda - \lambda_0)$. One can similarly obtain expansions for the angular moments.

We note that, by the method described above, we can find the angular distribution and energy spectrum not only by expanding the absorption coefficient $\tau(\lambda)$ in powers of $(\lambda - \lambda_0)$, but also if we replace it in any manner whatsoever, e.g. by a sum of exponentials

$$\tau(\lambda) = a\exp(b\lambda) + c\exp(d\lambda).$$

The expansions obtained for the distribution of scattered γ - radiation in energy and angle (16), and for the energy spectrum (20), converge rapidly, especially when the scattered radiation is softer than the unscattered (i.e., when the absorption coefficient does not decrease with decreasing energy). If the absorption coefficient increases with decreasing energy, we should replace

$\exp(-\tau_0 x)$ in the expansions (16) and (20) by $\exp(-\tau_m x)$, where τ_m is the absorption coefficient for the hardest component of radiation. This is easily done. It increases the rapidity of convergence for relatively large depths of penetration.

Clearly, in practical cases, in order that the calculation should not become too complicated, it is convenient to approximate the absorption coefficient by a simple formula of the type $\tau(\lambda) = \sum_{i=0}^{n} \tau_i (\lambda - \lambda_0)^i$, with small n.

Let us consider some simple cases.

4. THE CASE OF CONSTANT ABSORPTION COEFFICIENT

The calculations are most easily done when the absorption coefficient can be regarded as independent of energy, which is the case, as already mentioned above, in light elements at high energies: in carbon, for $h\nu \gtrsim 8\, m_0 c^2$; in aluminum, for $h\nu \gtrsim 10\, m_0 c^2$, etc.

In this case we can find all the terms in the expansion (16) and, taking Laplace and Hankel transforms, get the following expression for the distribution function of the scattered radiation in angle and energy:

$$\Gamma(x, \lambda, \theta) = \frac{1}{2\pi}\left(\frac{\lambda_0}{\lambda}\right)^{1,8} e^{-\tau_0 x}\left\{\delta(\lambda - \lambda_0)\,\delta\left(\frac{\theta^2}{2}\right) + \frac{\rho^2}{4(\lambda - \lambda_0)}\,\delta\left(\lambda - \lambda_0 - \frac{\theta^2}{2}\right) + \frac{1}{(\lambda - \lambda_0)^2}\sum_{n=2}^{\infty}\frac{n-1}{(n!)^2}\left(\frac{\rho}{2}\right)^{2n} \times \left(1 - \frac{\theta^2}{2n(\lambda - \lambda_0)}\right)^{n-2} u\left(\lambda - \lambda_0 - \frac{\theta^2}{2}\right)\right\}, \tag{22}$$

where

$$\rho = 2\sqrt{ax(\lambda - \lambda_0)}, \tag{23}$$

and $u(x)$ is the unit step function

$$u(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{24}$$

The solution has the form of a power series in the depth of penetration of the γ - radiation in the matter, multiplied by the exponential $\exp(-\tau_0 x)$ which describes the absorption of the unscattered radiation. Each term of the series has a simple physical meaning. The first term describes the radiation which has not undergone scattering. The second term represents the once-scattered radiation. It is clear from the relation between angle and wavelength that the succeeding terms represent the fractions of γ - radiation which have been Compton-scattered 2,3,... times.

As was to be expected, the maximum angle of deviation for twice-scattered radiation is $\theta^{(2)}_{max} = \sqrt{4(\lambda - \lambda_0)}$, for thrice-scattered radiation $\theta^{(3)}_{max} = \sqrt{6(\lambda - \lambda_0)}$, for radiation scattered n times – $\theta^{(n)}_{max} = \sqrt{2n(\lambda - \lambda_0)}$.

The intensity of the unscattered radiation is given by the product of two delta functions, that of the once-scattered radiation by a single delta function; the intensity of the twice-scattered radiation has a discontinuity at $\theta = \theta^{(2)}_{max}$, the intensity of the thrice-scattered radiation is continuous but has a discontinuous first derivative, the intensity of the radiation which has been scattered n times has a discontinuity in its $(n-2)$ *nd* derivative. The curve describing the intensity of the radiation which has been scattered n times naturally become smoother with increasing n. The angular distribution of γ-radiation which is scattered two or more times has a characteristic step function form.

These considerations about the continuity of the intensity of multiply scattered radiation are valid for any absorption coefficient.

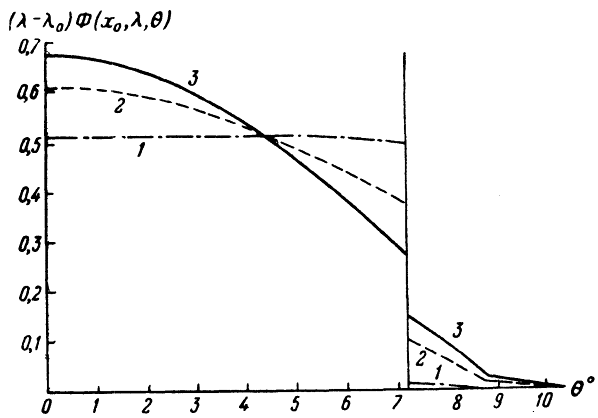

FIG. 3. Evolution of the angular distribution of scattered γ - radiation in the case of constant absorption coefficient. Normalized angular distributions are shown for radiation scattered more than once, at depths $\rho = 1.4$ and 6; $\lambda_0 = 1/34$, $\lambda = 1/30$. The discontinuities at $\theta = 7.1°$ and the kink at $\theta = 8.8°$ are caused, respectively, by the twice-and thrice-scattered radiation. With increasing depth of penetration, the distribution becomes smoother. Curve *1* - $\rho = 1$; *2* - $\rho = 4$; *3* - $\rho = 6$.

With increasing depth of penetration, the higher orders of scattering begin to be more and more important, the angular distribution becomes smoother and, as will be shown later, approaches a Gaussian $\exp[-\theta^2/2(\lambda-\lambda_0)]$.

The evolution of the angular distribution is shown in Fig 3.

To get the energy spectrum we must integrate Eq. (22) over all angles:

$$\Gamma_0(x,\lambda)=\delta(\lambda-\lambda_0)e^{-\tau_0 x} + \left(\frac{\lambda_0}{\lambda}\right)^{1,8}\frac{e^{-\tau_0 x}}{(\lambda-\lambda_0)}\left(\frac{\rho^2}{4}+\frac{1}{2}\left(\frac{\rho}{2}\right)^4 + \dots + \frac{1}{n!(n-1)!}\left(\frac{\rho}{2}\right)^{2n}+\dots\right) \tag{25}$$

$$= e^{-\tau_0 x}\left\{\delta(\lambda-\lambda_0)+\left(\frac{\lambda_0}{\lambda}\right)^{1,8}\frac{\rho}{2(\lambda-\lambda_0)}I_1(\rho)\right\},$$

where $I_1(\rho)$ is the Bessel function of first order and imaginary argument.

The physical meaning of the expansion (25) for the energy spectrum is the following: the n-th term refers to the n-times scattered radiation. This enables us to evaluate the contributions to the energy spectrum of multiply scattered radiations at different depths.

Thus, for $\rho = 1/2$, the intensity of once-scattered radiation makes up 96.95% of the total intensity of scattered radiation.

For $\rho = 1$, the once-scattered radiation contributes 88.5%, twice-scattered radiation gives 11.1%.

For $\rho = 2$, the intensity of once-scattered radiation drops to 62.9%, the intensities of twice- and thrice-scattered radiations are 31.4% and 5.2%.

For $\rho = 4$, the intensities of radiations scattered one, two, three and four times make up, respectively, 20.5, 41, 27.3 and 9.1% of the total intensity of scattered radiation.

For $\rho = 1$, it is already necessary to take into account the twice-scattered radiation. For initial energy $h\nu_0 = 12m_0c^2$ and final energy $h\nu = 10m_0c^2$, in carbon $\rho = 1$ corresponds to 44 cm, $\rho = 2$ to ~ 1.76 m.

As indications of the rapidity of convergence of the expansion (22), we may mention the following data: for $\rho = 2$ (an extremely great depth!), the third term is 4.5 times as small as the second, the fourth is 10.7 times as small as the third, etc.

In reference 2, an expression identical with Eq. (22) was obtained by another, more complicated, method. The physical meaning of the energy spectrum was not made clear in that reference.

5. VARIABLE ABSORPTION COEFFICIENT

Let us consider the case where the absorption coefficient is a quadratic function of the wavelength:

$$\tau(\lambda)=\tau_0+\tau_1(\lambda-\lambda_0)+\tau_2(\lambda-\lambda_0)^2. \tag{26}$$

We can approximate the absorption coefficient by an expression of the type of Eq. (26) both for light and heavy elements in various energy regions, including the neighborhood of the minimum absorption coefficient (cf. Fig. 1).

For the distribution of γ - radiation in energy and angle, we get

$$\Gamma(x,\lambda,\theta) = \Gamma^0(x,\lambda,\theta)+\Gamma^1(x,\lambda,\theta)+\frac{e^{-\tau_0 x}}{2\pi}\left(\frac{\lambda_0}{\lambda}\right)^{1,8} \times\left\{\frac{a^2}{2!2}u\left(1-\frac{\alpha}{2}\right)x^2+\left[\frac{a^3}{3!3\cdot 1!}\left(1-\frac{\alpha}{3}\right)u\left(1-\frac{\alpha}{3}\right) - \frac{a^2\tau_1}{8}u\left(1-\frac{\alpha}{2}\right)-\frac{a^2\tau_2}{96}(10+2\alpha-\alpha^2) \times(\lambda-\lambda_0)u\left(1-\frac{\alpha}{2}\right)\right](\lambda-\lambda_0)x^3+\dots\right\}, \tag{27}$$

where $\alpha = \theta^2/2(\lambda-\lambda_0)$, $\Gamma^0(x,\lambda,\theta)$ is the angular distribution (4) for unscattered radiation, $\Gamma^1(x,\lambda,\theta)$ is the angular distribution (5) for once-scattered radiation. The energy spectrum is given by

$$\Gamma_0(x,\lambda)=\Gamma_0^{\,0}(x,\lambda)+\Gamma_0^{\,1}(x,\lambda)+\left(\frac{\lambda_0}{\lambda}\right)^{1.8}e^{-\tau_0 x} \times\left\{\frac{a^2}{2}(\lambda-\lambda_0)x^2+\frac{1}{6}\left[\frac{a^3}{2}-\frac{3}{2}a^2\tau_1 - \frac{4}{3}a^2\tau_2(\lambda-\lambda_0)\right](\lambda-\lambda_0)^2x^3+\dots\right\}, \tag{28}$$

where $\Gamma_0^0(x,\lambda)$ and $\Gamma_0^1(x,\lambda)$ are the energy spectra of unscattered and once-scattered radiation, respectively, and are easily gotten from Eqs. (4) and (5).

Actually two more terms were calculated in the expansions (27) and (28), but these have not been given because of their complexity.

The distribution in energy and angle for the case of constant absorption coefficient (22) is obtained from (27) for $\tau_1 = \tau_2 = 0$.

In light elements, over a wide energy interval, the absorption coefficient may be assumed to be linearly dependent on wavelength: $\tau(\lambda) = \tau_0 + \tau_1 \times (\lambda - \lambda_0)$. Correspondingly, one should set $\tau_2 = 0$ in Eqs. (27) and (28).

In this case the expression for the energy spectrum is obtained in closed form:

$$\Gamma_0(x, \lambda) = e^{-\tau_0 x}\left\{\delta(\lambda - \lambda_0) + ax\left(\frac{\lambda_0}{\lambda}\right)^{1.8} \times {}_1F_1\left(1 - \frac{a}{\tau_1}; 2; -\tau_1(\lambda - \lambda_0)x\right)\right\}, \quad (29)$$

where ${}_1F_1(\alpha; \beta; x)$ is the confluent hypergeometric function. The energy spectrum for the case of a linear absorption coefficient was found in reference 7; our spectrum (29) coincides with the spectrum given there if we take into account the difference in the approximations used for the differential scattering cross section.

We now turn to a discussion of the general expression (27) for the distribution of scattered γ - radiation in angle and energy, and the energy spectrum (28).

First of all, it is easy to establish the physical meaning of these expansions: terms containing a to the nth power correspond to n-fold scattered radiation.

We have already dealt with the continuity properties of the intensity of the n-fold scattered radiation in Section 5: the intensity of the n-fold scattered radiation has a discontinuity in its $(n - 2)$nd derivative; with increasing order of scattering, the angular distribution becomes smoother.

For energies less than that at which the absorption coefficient reaches its minimum value (left of the minimum in Fig. 1), the primary radiation is more penetrating than the scattered radiation.

Photons deflected through large angles will have less energy, and will therefore suffer stronger absorption than photons scattered through small angles. With increasing depth of penetration, the angular spread of the γ - radiation will become narrower than in the case of constant absorption coefficient. The expansions for the angular distribution and energy spectrum will converge rapidly even for relatively large depths of penetration.

If, on the other hand, the initial energy is above the energy for minimum absorption, the primary radiation will be softer than the scattered radiation. With increasing depth of penetration the angular distribution will become diffuse.

The expansions (27) and (28) will converge well only for not too large depths of penetration. The rapidity of convergence can be somewhat increased by introducing $\exp(-\tau_m x)$ in place of $\exp(-\tau_0 x)$ and making corresponding changes in all the expansions, which is not hard to do. Figure 4 shows a sketch of the normalized angular distribution of the radiation, having energy 10.2 mev ($\lambda = 1/20$), which has been scattered two or more times in copper, at a depth of 30 cm, if the initial energy was 12.75mev ($\lambda_0 = 1/25$). Here $\tau_0 = 0.30\ \mathrm{cm}^{-1}$, $\tau_1 = -4\ \mathrm{cm}^{-1}$; $\tau_2 = 200\ \mathrm{cm}^{-1}$; $\tau_m = 0.28\ \mathrm{cm}^{-1}$.

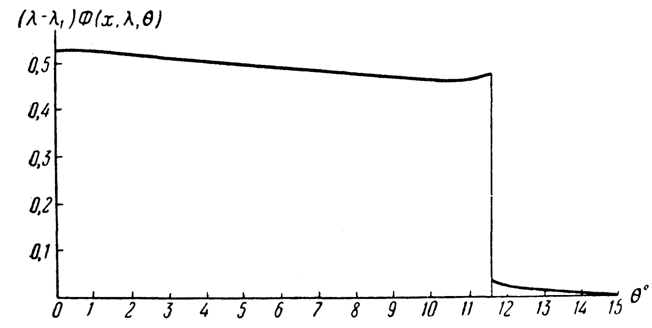

FIG. 4. Normalized angular distribution of γ -radiation of energy 10.2 mev, scattered two or more times in copper. The depth of penetration is 30 cm, the primary energy 12.75 mev.

To characterize the convergence in this case, we give the following: the second term in the sum in (27) is 2.5 times smaller than the first, the

[7] S. Z. Belen'kii, *Shower Processes in Cosmic Rays*, Gov't. Publ. House, Moscow-Leningrad, 1948

third term is 4.4 times smaller than the second, etc. At greater depths, the convergence is much poorer.

From Figure 4 we see that at $\theta = 11.5°$, (just before the twice scattered radiation drops out), the curve for the angular distribution actually rises a little, which can be explained by the methods given above.

The investigation of the angular distribution and energy spectrum of scattered γ - radiation at great depths of penetration in matter will be carried out in another paper.

In conclusion, I must express my profound gratitude to Prof. S. Z. Belen'kii for valuable suggestions, and to Acad. I. E. Tamm and Prof. E. L. Feinberg for supervising the work.

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Angular Distribution of Gamma Rays at Great Depths of Penetration in Matter

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The angular and energy distribution of γ-rays at great depths of penetration in matter is found for the cases of constant and linear dependence of the absorption coefficient on wavelength. The passage of γ-rays through an inhomogeneous medium is examined.

1. THE qualitative nature of the angular distribution at great depths of penetration depends strongly on the behavior of the γ-ray absorption coefficient. If the initial energy of the γ-ray is less than that at which the absorption coefficient is a minimum, then on the average, γ-rays scattered through small angles will be more penetrating than those scattered through large angles. With increase in the depth of penetration, the angular distribution will become narrower, or at any rate no wider. The small angle approximation applicable to scattering at energies of the order of several mev remains valid for great depths of penetration.

In the other case, where the absorption coefficient increases with energy, photons scattered through large angles will be more penetrating than those scattered through small angles. As the depth of penetration increases, the angular distribution will be smeared out, and the small angle approximation for each Compton scattering becomes incorrect.

In this respect, the results of reference 1, where the γ-ray energy spectrum at great depths of penetration is calculated using the small angle approximation, arouse some doubt.

In the present article, using the polynomial expansion of Spencer and Fano[2], the energy and angular distribution of γ-rays at great depths of penetration are found for the case of constant absorption coefficient (γ-rays of high energy in light elements, see reference 3), and for an absorption coefficient which increases linearly with wavelength. In these cases the small angle approximation is applicable, as can be seen from the final result.

In the case of constant absorption coefficient, the angular distribution tends to a gaussian one, although the approach to a gaussian distribution takes place significantly slower than indicated in reference 4. At the end of the article these results and those of reference 3 are generalized for an inhomogeneous medium. Below we shall use equations of radiation transport and the notation of the preceeding article[3].

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