Vertex Parts at Very High Energies in Quantum Electrodynamics

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A method is developed for calculating Feynman integrals with logarithmic accuracy, working to any order of perturbation theory. The method is applied to calculate the vertex part in quantum electrodynamics for a certain range of values of the momenta. The result is displayed as the sum of a perturbation series.

THE technique of Feynman[1] for calculating matrix elements in quantum electrodynamics is only suitable for the lowest-order approximations, since the algebraic complexities increase extremely rapidly when we consider contributions to the matrix element from higher-order perturbations. When perturbation theory is not applicable and it is necessary to consider the sum of the entire perturbation series*, another technique must be developed. For example, one elegent method[3] of calculating integrals with logarithmic accuracy depends on changing k_0 into ik_0. This method is, however, not applicable to all cases. In particular, it is inapplicable to the calculation of the vertex part $\Gamma_\sigma(p, q; l)$ in the case when the absolute value of the square of one of the vectors p, q, l is much larger than the absolute squares of the other two vectors. This case is especially important for concrete physical applications. There appear in this case terms with the structure $e^2L_1L_2$, a product of two big logarithms entering with each power of e^2 (we call these doubly-logarithmic terms). But the earlier method[3] can give only terms with the structure e^2L (singly-logarithmic terms), in which one large logarithm enters with each power of e^2.

1. To explain the method+ of obtaining the doubly-logarithmic terms, we shall consider as an example the integral*

$$I = \lim_{\varepsilon \to +0} \frac{d^4k}{[(p-k)^2 - m^2 + i\varepsilon]\,[(q-k)^2 - m^2 + i\varepsilon]\,(k^2 + i\varepsilon)}. \tag{1}$$

In what follows we shall everywhere omit the limiting process, simply choosing ϵ to be a positive number so small that it does not make any contribution in the final result. We shall evaluate (1) supposing that

$$|l^2| \gg |p^2|,\ |q^2|, \tag{2}$$

where $l = p - q$. For simplicity we assume

$$|p^2|,\ |q^2| \gg m^2, \tag{3}$$

which allows us to omit m^2 in the first two factors of the denominator in (1).

From (2) it follows that to a close approximation $l^2 = -2pq$, which allows us to rewrite (2) in the form

$$|pq| \gg |p^2|,\ |q^2|. \tag{2a}$$

Hence it is clear that the squares of the vectors p, q are very small compared with the squares of their components; the squares of the vectors p, q are almost null.

* We do not need to worry about the divergence of the perturbation series,[2] which occurs at much higher energies than those which we consider.

[1] R. P. Feynman, Phys. Rev. 76, 769 (1949)

[2] F. J. Dyson, Phys. Rev. 85, 631 (1952).

[3] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk. SSSR 95, 497, 773, 1177 and 96, 261 (1954).

+ In this paper the Feynman notations are used:

$$pq = p_0q_0 - PQ = p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3,$$

$$d^4k = (2\pi)^{-2}\,dk_0\,dk_1\,dk_2\,dk_3.$$

* The integral (1) is singular. To define it precisely we have to specify the Feynman rules for integrating round the poles. This is done by adding infinitesimal imaginary terms to the factors in the denominator.

It is very useful, and appropriate to the nature of the problem, to introduce variables u, v, x, in terms of which the vector k takes the form

$$k = k_{\parallel} + k_{\perp} = up + vq + k_{\perp}; \quad k_{\perp}^2 = -x. \tag{4}$$

The vector $k_{\perp}$ is perpendicular to p and q, while $k_{\parallel}$ lies in the plane of p and q. When the inequality (2) holds, the time-axis can be chosen to lie in the plane of p and q. Therefore the vector $k_{\perp}$ is space-like, and the variable x takes only positive values $x > 0$.

A rigorous argument (see Appendix 1) shows that the region in which doubly-logarithmic terms appear is limited by the inequalities

$$|q^2/l^2| \ll |u| \ll 1, \quad |p^2/l^2| \ll |v| \ll 1, \tag{5}$$

$$x \ll \min[|l^2 u|, |l^2 v|].$$

In this region the integrand, written in terms of the new variables, simplifies considerably, and the integral (1) takes the form

$$I = -\frac{1}{8\pi |l^2|} \int \frac{du}{u} \frac{dv}{v} \frac{dx}{x + l^2 uv - i\varepsilon}. \tag{6}$$

In (6) we carry out the integration with respect to x, $\min[|l^2 u|, |l^2 v|]$

$$\int_0 \frac{dx}{x + l^2 uv - i\varepsilon} = \ln \min\left[\frac{1}{|v|}, \frac{1}{|u|}\right] + \frac{i\pi}{2}[1 - \operatorname{sign}(l^2 uv)].$$

The first term vanishes after integration with respect to u or v, and the second gives

$$I = (i/4l^2) \ln|l^2/p^2| \ln|l^2/q^2|. \tag{7}$$

It is to be noted that only the imaginary part of the result of the x-integration is significant, and this imaginary part is different from zero and equal to $i\pi$ only when

$$l^2 uv < 0. \tag{8}$$

In the more complicated problem considered below, we carry out the integrations in the same order; after the x- integration we obtain an imaginary result, with the range of the subsequent u and v integrations limited by the same inequality (8).

We may define the region determined by the inequalities (5) as the region in which the integrand is logarithmically maximal As we see from (6), the integrand is in fact triply-logarithmic in this region. But various parts of the range of integration almost compensate each other. One of the logarithmic integrations gives an imaginary result of the order of unity instead of a large logarithm.

2. We calculate the vertex part $\Gamma_{\sigma}(p, q; l)$ in the region of values of the momenta defined by the inequalities (2), (3). In second-order perturbation theory it is given by the integral

$$\Gamma_{\sigma}^{(2)}(p, q; l) = \frac{e^2}{\pi i} \times \int \frac{\gamma_{\mu}(\hat{p} - \hat{k})\gamma_{\sigma}(\hat{q} - \hat{k})\gamma_{\mu} d^4k}{[(p-k)^2 + i\varepsilon][(q-k)^2 + i\varepsilon][k^2 + i\varepsilon]}. \tag{9}$$

The numerator may be expressed in terms of the new variables in the form

$$\gamma_{\mu}(\hat{p} - \hat{k})\gamma_{\sigma}(\hat{q} - \hat{k})\gamma_{\mu} = \gamma_{\mu}[\hat{p}(1-u) - v\hat{q} - \hat{k}_{\perp}] \times \gamma_{\sigma}[\hat{q}(1-v) - u\hat{p} - \hat{k}_{\perp}]\gamma_{\mu}. \tag{10}$$

We average over the direction of the vector $k_{\perp}$ Terms linear in $k_{\perp}$ then vanish, while the quadratic term becomes

$$\hat{k}_{\perp}\gamma_{\sigma}\hat{k}_{\perp} = {}^1/_2 k_{\perp}^2 \delta_{\perp\lambda\rho}\gamma_{\lambda}\gamma_{\sigma}\gamma_{\rho} = -{}^1/_2 x\delta_{\perp\lambda\rho}\gamma_{\lambda}\gamma_{\sigma}\gamma_{\rho}. \tag{11}$$

Here $\delta_{\perp\lambda\rho}$ is a symmetric matrix defined by

$$\delta_{\perp\lambda\rho} p_{\rho} = \delta_{\perp\lambda\rho} q_{\rho} = 0, \quad \delta_{\perp\lambda\rho} k_{\perp\rho} = k_{\perp\lambda}. \tag{12}$$

We are not interested in obtaining an explicit representation of $\delta_{\perp\lambda\rho}$ It will be enough for our purposes to know that the eigenvalues of the matrix $\delta_{\perp\lambda\rho}$ are 0 and 1 according to (12), so that the matrix is of the order of magnitude unity. Substituting (11) into (10) we obtain

$$\gamma_{\mu}[\hat{p}(1-u) - v\hat{q}]\gamma_{\sigma}[\hat{q}(1-v) - u\hat{p}]\gamma_{\mu} - {}^1/_2 x\delta_{\perp\lambda\rho}\gamma_{\mu}\gamma_{\lambda}\gamma_{\sigma}\gamma_{\rho}\gamma_{\mu}. \tag{13}$$

In the region of momenta defined by the inequalities (5), all terms other than the term $\gamma_\mu p \gamma_\sigma \bar{q} \gamma_\mu$ are negligible, since the presence of the small factors u, v, (x/l^2) converts one of the integrations from logarithmic to non-logarithmic. Consequently all such terms are singly-logarithmic and represent small corrections to the main term. As is well-known, the term with k^2 in the numerator contains a logarithmic divergence which is removed by renormalization, the remainder after renormalization being singly-logarithmic. Since we are neglecting such terms, the process of renormalization can for our purposes be reduced to the following simple rules: (1) all singly-logarithmic terms are dropped, whether they are convergent or divergent; (2) the symbol e denotes the renormalized experimental charge.

We consider the only surviving term in (13), namely

$$\gamma_\mu \hat{p} \gamma_\sigma \hat{q} \gamma_\mu. \tag{14}$$

The operator $\Gamma_\sigma(p, q; l)$ always appears in diagrams with an electron Green's function on either side,

$$G(p)\,\Gamma_\sigma(p, q; l)\,G(q). \tag{15}$$

Now for $p^2 >> m^2$, $G(p)$ takes the form $(\hat{p}/p^2)$. The operator (14), with the numerators of the Green's functions on each side, becomes

$$\hat{p} \gamma_\mu \hat{p} \gamma_\sigma \hat{q} \gamma_\mu \hat{q}. \tag{16}$$

When the operators are commuted by the usual rules, (16) may be written in the form

$$-p^2 \gamma_\mu \gamma_\sigma \hat{q} \gamma_\mu \hat{q} - 2q^2 \hat{p} \gamma_\sigma \hat{p} + 4pq \hat{p} \gamma_\sigma \hat{q}. \tag{17}$$

We mentioned earlier that p and q are vectors with large components but with small lengths. Therefore the terms containing p^2 and q^2 explicitly are negligible, and (17) can be replaced to a good approximation by its last term

$$\hat{p} \gamma_\mu \hat{p} \gamma_\sigma \hat{q} \gamma_\mu \hat{q} \to 4pq \hat{p} \gamma_\sigma \hat{q} = -2l^2 \hat{p} \gamma_\sigma \hat{q}. \tag{18}$$

The original operator (14) thus becomes

$$\gamma_\mu \hat{p} \gamma_\sigma \hat{q} \gamma_\mu = 4pq \gamma_\sigma = -2l^2 \gamma_\sigma. \tag{19}$$

Therefore

$$\Gamma_\sigma^{(2)}(p, q; l) = -\frac{2e^2 l^2}{\pi i} \gamma_\sigma \times \int \frac{d^4k}{[(p-k)^2 + i\varepsilon]\,[(q-k)^2 + i\varepsilon]\,[k^2 + i\varepsilon]}, \tag{20}$$

and so Eq. (7) gives

$$\Gamma_\sigma^{(2)}(p, q; l) = -(e^2/2\pi) \times \ln|l^2/p^2| \ln|l^2/q^2| \gamma_\sigma. \tag{21}$$

We now carry out the calculation of $\Gamma_\sigma(p, q; l)$ in the (2n)'th order of perturbation theory. Using the method described in Appendix 2, one can verify that the terms with the greatest number of large logarithmic factors, i. e. terms of the type $e^2 L_1 L_2$, are precisely those in which all the virtual photon lines overlap the point of emission of the external photon l. The general term of this type in the (2n)'th order contribution to $\Gamma_\sigma(p, q; l)$ is

$$\left(\frac{e^2}{\pi i}\right)^n \int \frac{\gamma_{\mu_1}(\hat{p} - \hat{k}_1)\gamma_{\mu_2}(\hat{p} - \hat{k}_1 - \hat{k}_2) \cdots \gamma_{\mu_n}\left(\hat{p} - \sum_1^n \hat{k}_i\right)\gamma_\sigma}{[(p-k_1)^2 + i\varepsilon]\,[(p-k_1-k_2)^2 + i\varepsilon] \cdots \left[\left(p - \sum_1^n k_i\right)^2 + i\varepsilon\right]} \times \frac{\left(\hat{q} - \sum_1^n \hat{k}_i\right)\gamma_{\mu_{i_n}}\left(\hat{q} - \sum_1^n \hat{k}_i + \hat{k}_{i_n}\right)\gamma_{\mu_{i_{n-1}}}}{\left[\left(q - \sum_1^n k_i\right)^2 + i\varepsilon\right]\left[\left(q - \sum_1^n k_i + k_{i_n}\right)^2 + i\varepsilon\right]} \times \frac{\left(\hat{q} - \sum_1^n \hat{k}_i + \hat{k}_{i_n} + \hat{k}_{i_{n-1}}\right)\gamma_{\mu_{i_{n-2}}} \cdots (\hat{q} - \hat{k}_{i_1})\gamma_{\mu_{i_1}}}{\left[\left(q - \sum_1^n k_i + k_{i_n} + k_{i_{n-1}}\right)^2 + i\varepsilon\right] \cdots [(q - k_{i_1})^2 + i\varepsilon]} \times \frac{d^4k_1}{k_1^2 + i\varepsilon} \frac{d^4k_2}{k_2^2 + i\varepsilon} \cdots \frac{d^4k_n}{k_n^2 + i\varepsilon}. \tag{22}$$

Here $i_1, i_2, \ldots, i_n$ is a permutation of the integers 1, 2, ..., n. When $i_1, i_2, \ldots, i_n = 1, 2, \ldots, n$, we have the so-called "ladder diagram" in which the virtual photon lines do not intersect. For all other permutations $i_1, \ldots, i_n$, some of the photon lines will intersect.

Just as in the calculation of the second-order contribution, all the $\hat{k}_i$ may be dropped from the numerator of the integrand. Any term containing even a single $\hat{k}_1$ gives rise to a lower power of the large logarithm and is therefore a small correction to the main term. After dropping the $\hat{k}_i$, the operator remaining in the numerator is

$$\gamma_{\mu_1}\hat{p}\gamma_{\mu_2}\hat{p}\ldots\gamma_{\mu_n}\hat{p}\gamma_\sigma\hat{q}\gamma_{\mu_{i_n}}\hat{q}\gamma_{\mu_{i_{n-1}}}\hat{q}\ldots\hat{q}\gamma_{\mu_{i_1}}. \tag{23}$$

As in the step from (14) to (16), we multiply (23) on each side by operators arising from electron Green's functions, and we commute the operators $\hat{p}$ with γ_{μ_i} and $\hat{q}$ with $\gamma_{\mu_{ik}}$, dropping the small terms in which p^2 or q^2 appears explicitely. After removing the $\hat{p}$ and $\hat{q}$ which belong to the electron Green's functions, we obtain the approximate equation

$$\gamma_{\mu_1}\hat{p}\gamma_{\mu_2}\hat{p}\ldots\gamma_{\mu_n}\hat{p}\gamma_\sigma\hat{q}\gamma_{\mu_{i_n}}\hat{q}\gamma_{\mu_{i_{n-1}}}\hat{q}\ldots\hat{q}\gamma_{\mu_{i_1}} \tag{24}$$
$$= (4pq)^n\gamma_\sigma = (-2l^2)^n\gamma_\sigma.$$

Equation (22) then takes the form

$$\gamma_\sigma\left(-\frac{2e^2l^2}{\pi i}\right)^n \tag{25}$$
$$\times\int[(p-k_1)^2+i\varepsilon]^{-1}[(p-k_1-k_2)^2+i\varepsilon]^{-1}$$
$$\ldots\left[\left(p-\sum_1^n k_i\right)^2+i\varepsilon\right]^{-1}\times\left[\left(q-\sum_1^n k_i\right)^2+i\varepsilon\right]^{-1}$$
$$\times\left[\left(q-\sum_1^n k_1+k_{i_n}\right)^2+i\varepsilon\right]^{-1}\ldots[(q-k_{i_1})^2+i\varepsilon]^{-1}$$
$$\times\frac{d^4k_1}{k_1^2+i\varepsilon}\frac{d^4k_2}{k_2^2+i\varepsilon}\cdots\frac{d^4k_n}{k_n^2+i\varepsilon}.$$

To evaluate (25), we introduce the variables u_i, v_i, x_i, which we used in the derivation of Eq. (7). As we remarked in connection with the integral (1), the region which gives doubly-logarithmic terms is limited by the inequalities (5) for each triad of variables u_i, v_i, x_i. The important region is in fact even smaller, and its true limits will be determined below.

By analogy with the derivation of (7), we shall suppose that $k_\perp i$ may be neglected in every factor of the denominator except k_i^2. This can be justified by an exact argument. In fact all our estimates reduce to the determination of the region in which the integral is logarithmically maximal.

We first integrate with respect to x_i, keeping only the imaginary part of the result. This gives

$$\int\frac{dx_i}{k_i^2+i\varepsilon}\to\begin{cases}-\pi i & l^2u_iv_i<0\\ 0 & l^2u_iv_i>0.\end{cases} \tag{26}$$

It remains to integrate with respect to u_i and v_i the integral

$$\gamma_\sigma\left[\frac{e^2}{4\pi}\operatorname{sign}(l^2)\right]^n\int\frac{du_1}{u_1}\frac{du_2}{u_1+u_2}\cdots\frac{du_n}{u_1+u_2+\cdots+u_n} \tag{27}$$
$$\times\frac{dv_{i_n}}{v_{i_n}+v_{i_{n-1}}+\cdots+v_{i_1}}$$
$$\times\frac{dv_{i_{n-1}}}{v_{i_{n-1}}+\cdots+v_{i_1}}\cdots\frac{dv_{i_1}}{v_{i_1}}.$$

All the integrations give a logarithm only in the ranges

$$|u_1|\ll|u_2|\ll\cdots\ll|u_n|, \tag{28}$$
$$|v_{i_1}|\ll|v_{i_2}|\ll\cdots\ll|v_{i_n}|.$$

Taking (28) together with (5) for each variable, we obtain

$$|q^2/l^2|\ll|u_1|\ll|u_2|\ll\ldots\ll|u_n|\ll 1, \tag{29}$$
$$|p^2/l^2\ll|v_{i_1}|\ll|v_{i_2}|\ll\ldots\ll|v_{i_n}|\ll 1.$$

Using (29) with (26), we can write down immediately the value of (22),

$$\gamma_\sigma\frac{1}{(n!)^2}\left(-\frac{e^2}{2\pi}\ln\left|\frac{l^2}{p^2}\right|\ln\left|\frac{l^2}{q^2}\right|\right)^n. \tag{30}$$

The result is identical for all diagrams of the type under consideration. The existence of diagrams

with intersecting virtual photon lines makes invalid the replacement of the summation of the perturbation series by an integral equation[3]. Therefore in our case the integral equation derived by Landau et al.[3] is incorrect.

The number of relevent diagrams in the (2n)'th order is equal to (n!) (the number of permutations $i_1, i_2, \ldots, i_n$). We can now sum the contributions from these diagrams over all values of n, and obtain

$$\Gamma_\sigma(p, q; l) = \gamma_\sigma \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{e^2}{2\pi}\ln\left|\frac{l^2}{p^2}\right|\ln\left|\frac{l^2}{q^2}\right|\right)^n = \gamma_\sigma \exp\left\{-\frac{e^2}{2\pi}\ln\left|\frac{l^2}{p^2}\right|\ln\left|\frac{l^2}{q^2}\right|\right\}. \tag{31}$$

This expression for $\Gamma_\sigma(p, q; l)$ may be regarded as the scattering matrix element of a bound electron by an external field or by a virtual photon. Equation (31) shows that the probability of such a scattering tends to zero as $|l^2|$ tends to infinity. Processes in which a large number of real photons are simultaneously emitted[4] will occur with much greater probability. The method developed in the present paper is in no way limited to the problem here considered.

In conclusion I express my thanks to Academician L. D. Landau and Professor I. Ia. Pomeranchuk for suggesting the problem and for their constant attention, and to A. A. Abrikosov for many valuable comments.

APPENDIX

1. For the purpose of calculating integrals with logarithmic accuracy, it is advantageous to introduce instead of the vectors p, q the pair p', q' given by

$$p' = p - \alpha_2 q, \quad q' = q - \alpha_1 p, \tag{A1}$$

with α_1, α_2 chosen so that the squares of p' and q' are rigorously zero. Since the squares of p and q are "almost" zero, α_1 and α_2 are small quantities. With high accuracy we may write

$$\alpha_1 = q^2/2pq = -q^2/l^2, \tag{A2}$$

$$\alpha_2 = p^2/2pq = -p^2/l^2.$$

Since α_1, α_2 are small, the transformation (A1) can be inverted to give

$$p = p' + \alpha_2 q', \quad q = q' + \alpha_1 p'. \tag{A3}$$

We resolve the vector $k_{||}$ along p' and q' and obtain

$$k_{||} = u'p' + v'q'. \tag{A4}$$

In terms of the new variables, the integral (1) becomes

$$I = \frac{|l^2|}{8\pi}\int \frac{1}{[l^2(1-u')(v'-\alpha_2) - x + i\varepsilon]} \times \frac{du'dv'dx'}{[l^2(1-v')(u'-\alpha_1) - x + i\varepsilon][-l^2u'v' - x + i\varepsilon]}. \tag{A5}$$

The advantage of the variables u', v' lies in the fact that according to (A5) the factors of the denominator are linear in each of u' and v'. This considerably simplifies the investigation of the region in which u', v' take very small or large values. It is especially easy to find the region in which the integrand is logarithmically maximal in terms of the variables u', v'; the upper bounds of the inequalities (5) are thus obtained immediately.

We calculate the integral (A5), integrating with respect to u' from $-\infty$ to $+\infty$, then with respect to x, and finally with respect to v'. The u' integration can be converted into a closed contour in the complex u' plane by the addition of a semi-circle of infinite radius either above or below the real axis; the integral then reduces to a sum of residues at the poles of the integrand. If the coefficients of u' in the three factors of the denominator have the same sign, then the closed contour encloses either all three poles or none at all. In either case the integral vanishes. To prove this we need only consider the second case. The sum of the three residues is proportional to the coefficient of u'^{-1} in the expansion of the integrand in negative powers of u', and this expansion obviously begins with a term in u'^{-3}.

Therefore the limits of the integration with respect to u' are given by

$$\min[0, \alpha_2] < v < 1. \tag{A6}$$

We can easily ascertain that the region $|v'| \sim |\alpha_2|$ does not give a doubly-logarithmic contribution. In the region $|\alpha_2| \ll v' \leq 1$, the expression obtained after the u' integration takes the form

[4] A. A. Abrikosov, J. Exper. Theoret. Phys. USSR **30**, 96 (1956); Soviet Phys. JETP **3**, 71 (1956)

$$I = \frac{i}{4}\int\limits_{v' \gg |\alpha_2|}^{1} \frac{(1-v')\,dv'dx}{[l^2v'(1-v')-x][x+\alpha_1 l^2 v'(1-v')]}. \tag{A7}$$

The value of the elementary integral (A7) coincides with Eq, (7), as was to be expected. The important region in the integral (A7) is limited by the inequalities

$$|\alpha_1 l^2 v'| \ll x \ll |l^2 v'|, \quad |\alpha_2| \ll v' \ll 1. \tag{A8}$$

We can now simplify the expression (A5) by using the condition $|v'| \ll 1$ at an earlier stage. We may also limit u' by the condition $|u'| \ll 1$. Consequently we can now carry through the whole calculation with the simplified integral

$$\frac{|l^2|}{8\pi}\int [l^2(v'-\alpha_2) - x + i\varepsilon]^{-1} \times [l^2(u'-\alpha_1) - x + i\varepsilon]^{-1} \frac{du'\,dv'\,dx}{[-l^2u'v' - x + i\varepsilon]}. \tag{A9}$$

We retained the x in all terms of (A9); if the u'-integration is now performed first, the condition $x \ll |v'|$ follows automatically, and also the condition $v' > 0$. Because of the symmetry of the integral (A9), we could equally well begin with the v'-integration, thus obtaining another upper bound to the important range for x ($x \ll |u'|$), and fixing the sign of u' ($u' > 0$). Therefore, so far as doubly-logarithmic terms are concerned, the integral (A9) supplemented by the conditions

$$|u'|, \ |v'| \ll 1, \tag{A10}$$

is equivalent to the integral (A7). The most symmetrical way of evaluating (A9) is to integrate first with respect to x between the limits $0 < x \ll |l^2 u'|, |l^2 v'|$, which allow us to retain x only in the last factor of the denominator. The neglect of the other poles is equivalent to dropping a part of the integrand which vanishes after further integration.

The calculation has been carried out with the same accuracy as the derivation of Eq. (7), with this difference, that we have not excluded from the beginning the region $|u'| \sim |\alpha_1|$, $|v'| \sim |\alpha_2|$, but have verified that this region is unimportant. In this region the variables u, v, can differ greatly from u', v'; so even if the contribution from the region can be easily estimated in terms of the variables u', v', the same is by no means true of the variables u, v. For practical calculations, however, it is more convenient to use the variables u, v, limiting the range of the variables beforehand to the region (15).

2. The same method can be applied to calculate terms of order e^2L, when they are the largest non-vanishing terms*. This situation arises in the calculation of $\Gamma_\sigma(p, q; l)$ by second-order perturbation theory, when $|p^2| \gg |q^2|$ and $|l^2| \gg m^2$.

As the simplest example which shows how to reformulate the method, we shall calculate an integral which differs from (A5) by an extra factor v'^m in the numerator, m being a positive integer. The integrations with respect to u' and x are completely unchanged; the integral with respect to v' ceases to be logarithmic and is to be taken between the precise limits $0 < v' < 1$. The integral which we shall calculate is

$$I[k_\mu, k_\mu k_\nu] = \int \frac{[k_\mu, k_\mu k_\nu]\,d^4k}{[(p-k)^2 + i\varepsilon][(q-k)^2 + i\varepsilon][k^2 + \varepsilon]} \tag{A11}$$

assuming the inequalities (2), (3) to hold. In the region $|k^2| \ll |l^2|$, the integral (A11) is equal to

$$\frac{i}{4l^2}\int\limits_{|q^2|/|l^2|}^{1} \frac{du}{u}\int\limits_{|p^2|/|l^2|}^{1} \frac{dv}{v}\,[k_{\|\,\mu}, k_{\|\,\mu}k_{\|\,\nu}], \tag{A12}$$

where it is understood that the vector $k_\|$ is written in the form $up + vq$, and that after integration only the highest (first) power of the logarithm is to be retained. For the integral $I[k_\mu]$ this is the only important region. For $I[k_\mu k_\nu]$ the region $|k^2| \gg |l^2|$ is also important; the integral diverges logarithmically for large k and is made finite only by being cut off for $|k^2| > \lambda^2$. In the region $|l^2| \ll |k^2| \ll \lambda^2$ the integral (A11) takes the form

$$\int \frac{[0, k_\mu k_\nu]\,d^4k}{[k^2 + i\varepsilon]^3},$$

or, after averaging over the direction of k,

$$\int \frac{[0, \tfrac{1}{4}\delta_{\mu\nu}]\,d^4k}{[k^2 + i\varepsilon]^2}.$$

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* The error in the evaluation of terms of order $e^2L_1L_2$, due to the uncertainty in the limits of the logarithmic integrals, is of the same order of magnitude as the terms of order e^2L.

The last integral is evaluated[3] by changing k_0 into ik_0, and has the value

$$(i/16)\,[0,\ \delta_{\mu\nu}]\ln(\lambda^2/|l^2|). \tag{A13}$$

The results we have obtained can be immediately applied to the evaluation of the second-order contribution to $\Gamma_\sigma(p, q; l)$ in the case $|p^2| \gg |q^2|$, $|l^2| \gg m^2$. For this we need only introduce variables u, v, defined by

$$k = q(1-u) + lv + k_\perp. \tag{A14}$$

It is clearly advantageous to decompose $k_\parallel$ into vectors whose squares are small. The variables u, v are defined in such a way that the main part of the logarithmic integral comes from small u and v. After some elementary algebra, we obtain the result

$$\Gamma_\sigma(p, q; l) = \frac{e^2}{4\pi p}\left[\hat{l}\gamma_\sigma\hat{l}\ln\left|\frac{p^2}{l^2}\right| - \hat{q}\gamma_\sigma\hat{q}\ln\left|\frac{p^2}{q}\right| - \hat{q}\gamma_\sigma\hat{l}\ln\left|\frac{p^2}{q^2}\right|\right] + \frac{e^2}{4\pi}\gamma_\sigma\ln\frac{\lambda^2}{p^2}. \tag{A15}$$

In this case no doubly-logarithmic terms appear.

Translated by F. J. Dyson
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The Infrared Catastrophe in Quantum Electrodynamics

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The infrared catastrophe is investigated by summing over diagrams. Expressions are obtained for $G(p)$ when $e^2/\pi \ln m^2/(p^2-m^2) \gtrsim 1$ and for $\Gamma_\mu(p, q; l)$ when $p \cdot q \gg p^2-m^2$, q^2-m^2. The problem of radiation of additional quanta during the scattering of an electron of arbitrary energy by an external field is considered.

IT is well known that the calculation of matrix elements of processes in quantum electrodynamics leads to infinities having various origins. Some of the infinities, which appear as a result of the divergence of integrals for large energies of virtual quanta and pairs, originate, as has already been pointed out[1], from an incorrect description of the interaction by means of the δ-function. In addition to these infinities (which are considered in references 1-4) there are also others, which result from integration over virtual quanta with k^2 close to zero, whenever the diagram under consideration includes a free electron with $p^2 = m^2$. This situation always occurs for matrix elements of real processes and has been named the infrared catastrophe. It is related to the fact that the very concept of a free line is a convention.

Actually, as shown by many authors[5], every process is accompanied by the radiation of a large number of low-energy quanta. For this reason, a properly formulated problem must take account of the possibility of such radiation with frequencies up to some maximum ω_{max}, corresponding to the fact that a real experimental apparatus always has a limited sensitivity to small changes in the energy of particles taking part in the process.

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