The last integral is evaluated[3] by changing k_0 into ik_0, and has the value

$$(i/16)\,[0,\ \delta_{\mu\nu}]\ln(\lambda^2/|l^2|). \tag{A13}$$

The results we have obtained can be immediately applied to the evaluation of the second-order contribution to $\Gamma_\sigma(p, q; l)$ in the case $|p^2| \gg |q^2|$, $|l^2| \gg m^2$. For this we need only introduce variables u, v, defined by

$$k = q(1-u) + lv + k_\perp. \tag{A14}$$

It is clearly advantageous to decompose $k_\parallel$ into vectors whose squares are small. The variables u, v are defined in such a way that the main part of the logarithmic integral comes from small u and v. After some elementary algebra, we obtain the result

$$\Gamma_\sigma(p, q; l) = \frac{e^2}{4\pi p}\left[\hat{l}\gamma_\sigma\hat{l}\ln\left|\frac{p^2}{l^2}\right| - \hat{q}\gamma_\sigma\hat{q}\ln\left|\frac{p^2}{q}\right| - \hat{q}\gamma_\sigma\hat{l}\ln\left|\frac{p^2}{q^2}\right|\right] + \frac{e^2}{4\pi}\gamma_\sigma\ln\frac{\lambda^2}{p^2}. \tag{A15}$$

In this case no doubly-logarithmic terms appear.

Translated by F. J. Dyson
12

SOVIET PHYSICS JETP VOLUME 3, NUMBER 1 AUGUST, 1956

The Infrared Catastrophe in Quantum Electrodynamics

A. A. ABRIKOSOV
Institute for Physical Problems, Academy of Sciences, USSR
(Submitted to JETP editor, October 28, 1954)
J. Exper. Theoret. Phys. USSR 30, 96-108 (January, 1956)

The infrared catastrophe is investigated by summing over diagrams. Expressions are obtained for $G(p)$ when $e^2/\pi \ln m^2/(p^2-m^2) \gtrsim 1$ and for $\Gamma_\mu(p, q; l)$ when $p \cdot q \gg p^2-m^2$, q^2-m^2. The problem of radiation of additional quanta during the scattering of an electron of arbitrary energy by an external field is considered.

IT is well known that the calculation of matrix elements of processes in quantum electrodynamics leads to infinities having various origins. Some of the infinities, which appear as a result of the divergence of integrals for large energies of virtual quanta and pairs, originate, as has already been pointed out[1], from an incorrect description of the interaction by means of the δ-function. In addition to these infinities (which are considered in references 1-4) there are also others, which result from integration over virtual quanta with k^2 close to zero, whenever the diagram under consideration includes a free electron with $p^2 = m^2$. This situation always occurs for matrix elements of real processes and has been named the infrared catastrophe. It is related to the fact that the very concept of a free line is a convention.

Actually, as shown by many authors[5], every process is accompanied by the radiation of a large number of low-energy quanta. For this reason, a properly formulated problem must take account of the possibility of such radiation with frequencies up to some maximum ω_{max}, corresponding to the fact that a real experimental apparatus always has a limited sensitivity to small changes in the energy of particles taking part in the process.

[1] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR 95, 497 (1954).

[2] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR 95, 773 (1954).

[3] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR 95, 1177 (1954)

[4] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR 96, 261 (1954).

[5] A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* GTTI (1953).

In practical calculations one usually uses the following device. One introduces a quantity λ_{min}, which serves as a fictitious photon mass. Then when we integrate over the momentum of the virtual photon the divergent result is replaced by a value proportional to $\ln(m/\lambda_{min})$. Next, one considers the same process, but replaces the virtual quantum by a real quantum (a process of the next higher order, in the preceeding approximation). It is not difficult to see that the cross section for the new process will be of the same order as the change in the first process which resulted from the introduction of the virtual quantum. We integrate over the momentum of the real quantum up to frequency ω_{max}, so that we again introduce the fictitious λ_{min}. When we combine the new cross section with the correction to the old cross section, the terms in $\ln \lambda_{min}$ cancel.

Such a procedure is completely valid for low energy processes and for large limiting frequencies ω_{max}. As we shall see later, the necessary conditions for applicability of this procedure are:

$$(e^2/\pi)\ln(m/\omega_{\max}) \ll 1 \quad \text{for} \quad E \lesssim m,$$

$$(e^2/\pi)\ln^2(E/m),$$

$$(e^2/\pi)\ln(m/\omega_{\max})\ln(E/m) \ll 1 \quad \text{for} \quad E \gg m,$$

where E is the order of magnitude of the energy of the process. When these conditions are violated, the procedure is invalid since, on the one hand, it becomes possible actually to radiate a large number of real quanta, and on the other hand virtual processes of arbitrary order begin to play an important role. The purpose of the present work is to investigate this question in general. In doing this, we shall here limit ourselves to considering the scattering of an electron by an external field. However, in principle, the results obtained are also applicable to other processes. In addition to the infrared catastrophe and the question of the radiation of soft quanta, we shall also consider the radiation of hard quanta during high energy scattering.

1. GREEN'S FUNCTION OF AN ELECTRON FOR $p^2 \approx m^2$

In reference 2 a calculation was made of the Green's function $G(p)$ and the vertex part $\Gamma_\mu(p, q; l)$ in the appropriate region. However, it is not hard to see that this calculation of $G(p)$ and Γ_μ applied only for $p^2 \gg m^2$. For the case where p^2 is so close to m^2 that $e^2/\pi \ln m^2/(p^2 - m^2) \gtrsim 1$, the calculation given in reference 2 is invalid. Since this region is directly related to the problem in which we are interested, we shall calculate $G(p)$ and the corresponding Γ_μ for this case.

The equations for Γ_μ and $G(p)$ are given in reference 1, Eq. (1.4). In solving them we can no longer (as was done in references 1,2) carry out a transition in the integrals to a four-dimensional Euclidean space by means of the substitution $k_0 \to ik_0$, since the vector p is essentially time-like. But for just this reason we can achieve this by the substitution $k_m \to -ik_m (m = 1, 2, 3)$.

Let us consider the integral in the equation for $G(p)$. The region of integration where $k^2 \gg m^2$ gives a result already known from reference 2, and is of no special interest when $p^2 \approx m^2$. But now there is a new region which gives rise to a logarithmic integral. This is the region $p^2 - m^2 \ll (p-k)^2 - m^2 \ll m^2$.

For the case where $p^2 \approx m^2$, we shall try to find a $G(p)$ of the form

$$G(p) = \beta(m^2)\,\nu\left(\frac{m^2}{p^2 - m^2}\right)\frac{\hat{p} + m}{p^2 - m^2}, \qquad (1)$$

where β was defined in reference 2, and ν is an unknown slowly varying function.

As for $\Gamma_\alpha(p, p-l; l)$, we make use of the fact that the Γ_μ always appears, in the cases which we need, with the factor $\hat{p} + m$ preceeding and following it (in the region of interest to us, $l^2 \ll m^2$, so that $\hat{p} \approx \hat{q}$), so we shall determine it up to terms which give a small contribution when bracketed in this way.

We shall show that $\Gamma_\sigma(p, p-l; l)$ should be of the form

$$(\hat{p} + m)\,\Gamma_\sigma(p, p-l, l)\,(\hat{p} + m) = (\hat{p} + m)\,\gamma_\sigma\,(\hat{p} + m)\,\alpha(m^2)\,\mu\left(\frac{m^2}{(p-l)^2 - m^2}\right), \qquad (2)$$

where μ is a slowly varying function. In fact, looking at the integral in the equation for Γ_σ, we see that the region of integration $k^2 \gg m$ together with the term γ_σ in the equation gives us the known result $\gamma_\sigma \alpha(m^2)$. In this way all the terms in the equation can be absorbed into $\alpha(m^2)$. But a logarithmic integral is obtained not only from this region but also from the region $q^2 - m^2 \ll (p-k)^2 - m^2 \ll m^2$. Here we can neglect k compared to m and, since $k^2 \ll m^2$, we can replace d_l and d_t by d_l^0 and $d_t^0 = 1$.

From the spinor factors appearing in the numerator of the integral we easily obtain

$$(\hat{p}+m)\gamma_\mu(\hat{p}-\hat{k}+m) \tag{3}$$
$$\times\ \gamma_\sigma(\hat{p}-\hat{l}-\hat{k}+m)\gamma_\nu(\hat{p}+m)$$
$$\approx 4\,p_\mu p_\nu(\hat{p}+m)\gamma_\sigma(\hat{p}+m).$$

The form of the spinor agrees with that assumed in (2). Furthermore it is easy to see that, except for the the rapidly varying part $D_{\mu\nu}$, the integrand depends only on the projection of k along the direction of p ($(p-k)^2 - m^2 \approx -2k\cdot p$). Because of this we can integrate over the other components of k.

Going over to a Euclidean space we obtain, as in reference 2, the relation (in our case $d^4k = (2\pi)^{-2}\, dk_0\, dk_1\, dk_2\, dk_3$),

$$d^4k \to (i/\pi)\, dk_p \mathbf{K}^2 d|\mathbf{K}|, \tag{4}$$

where the vector $\mathbf{K}$ is perpendicular to k_p.

Resolving the k which appears in $D_{\mu\nu}$ along k_p and $\mathbf{K}$, integrating over $\mathbf{K}$ and making a change of variables, we obtain for $\mu(\eta)$ the equation

$$\mu(\eta) = 1 - \frac{e^2}{2\pi}(3d_t^0 - d_l^0)\int_0^\eta \mu^3(z)\nu^2(z)\,dz, \tag{5}$$

where $\eta = \ln(m^2/[(p-l)^2 - m^2])$. In integrating over $\mathbf{K}$ we considered only those terms which give a logarithmic integral with respect to k_p. In addition we assumed that changing the sign of $q^2 - m^2$ in μ has no essential effect on its magnitude (as will be evident from the result). Equation (5) confirms the correctness of the form assumed for Γ_σ in Eq. (2).

We now look at the integral in the equation for $G(p)$ [1,4] in the interesting region where $p^2 - m^2 \ll (p-k)^2 - m^2 \ll m^2$. It is not hard to show that the integral in this region is of order $\int k^{-3} d^4k$ so that, as in reference 2, we must include contributions to $G(p-k)$ and to Γ_μ in the integral of order $(p^2 - m^2)/[(p-k)^2 - m^2]$. The correction to $G(p)$ is obtained simply by expanding the denominator of formula (1) for $G(p-k)$. The correction to Γ_μ is gotten from the equation for Γ_μ, given in reference 1, Eq. (1). In the present case the important range in the integral in the equation for Γ_μ is $p^2 - m^2 \ll (p-k)^2 - m^2 \ll (p-l)^2 - m^2$. In this region the integral is of order $\int k^{-3}\, dk$, and therefore, as is easily seen, the main terms are those of order $(p^2 - m^2)/[(p-k)^2 - m^2]$ in the first $G(p-k)$ and the first $\Gamma_\mu(p, p-k; k)$. The corrections to the other terms in Γ and G are unimportant.

The correction to Γ_μ should be of the form

$$(\hat{p}+m)\,\Gamma\,(p, p-l; l)(\hat{p}+m) \tag{6}$$
$$= \alpha(m^2)(\hat{p}+m)\gamma_\mu(\hat{p}+m)$$
$$\times \frac{p^2 - m^2}{(p-l)^2 - m^2}\,\delta[p^2 - m^2,\ (p-l)^2 - m^2].$$

Substituting in the equation for Γ_μ, we get an equation for δ:

$$\delta(\xi, \eta) = -\frac{e^2}{2\pi}(3d_t^0 - d_l^0)\nu(\eta)\mu^2(\eta) \tag{7}$$
$$\times \int_\eta^\xi [-\mu(z) + \delta(\xi, z)]\nu(z)\,dz.$$

Now substituting the corrections to G and Γ_μ in the equation for G and using previous results [2,4], we obtain an equation for ν:

$$\frac{1}{\nu(\xi)} = 1 + \frac{e^2}{2\pi}(3d_t^0 - d_l^0) \tag{8}$$
$$\times \int_0^\xi [-\mu(z) + \delta(\xi, z)]\nu(z)\,dz.$$

Equations (5), (7), (8), enable us to determine μ and ν. However there is no need to solve them directly. Actually, according to Ward's theorem [6], the relation:

$$\mu(\eta)\nu(\eta) = 1 \tag{9}$$

must hold. But then Eq. (5) gives us the function $\mu(\eta)$ in the form:

$$\mu(\eta) = \exp[-(e^2/2\pi)(3d_t^0 - d_l^0)\eta]. \tag{10}$$

The solution of equation (7) also presents no difficulties, and leads to the following form for $\delta(\xi, \eta)$:

$$\delta(\xi, \eta)/\mu(\eta) \tag{11}$$
$$= 1 - \exp\{(e^2/2\pi)(3d_t^0 - d_l^0)(\eta - \xi)\}.$$

Substituting this in equation (8), which can be written in the form $1/\nu(\xi) = 1 - \delta(\xi, 0)$, we get

---

[6] J. Ward, Phys. Rev. 78, 182 (1950).

$$\nu(\xi) = \exp[(e^2/2\pi)(3d_t^0 - d_l^0)\xi]. \quad (12)$$

This formula satisfies (9), as required.*

2. THE VERTEX PART WHEN $p \cdot q \gg p^2 - m^2$, $q^2 - m^2$.

In the preceding section we have already calculated one of the vertex parts of the type of interest to us here, namely $\Gamma_\mu(p,q;l)$ for $p^2 \approx m^2$, $q^2 \approx m^2$, $l^2 \ll m^2$. The other type of vertex $-i\Gamma_\mu(p,q;l)$ for $l^2 \gg p^2$, q^2, $p^2 \gg m^2$, $q^2 \gg m^2$, was found in a paper by Sudakov.[7] Here we consider the general case, making no restrictions on the vectors other than $p \cdot q \gg p^2 - m^2$, $q^2 - m^2$. As before, we shall sum only over the principal terms, i.e., those containing the maximum logarithmic terms in e^2. In particular, we shall not include terms of order $e^2\ln(l^2/m^2)$. For convenience of presentation, we shall use the symbols $p^2 - m^2 = 2m\Delta m_1$ and $q^2 - m^2 = 2m\Delta m_2$.

In the integrals representing the successive approximations to the vertex part, there are parts which include an integration over momenta of virtual quanta greater than the norms of the real momenta (in our case l^2 and m^2) which appear in the vertex part. If we separate this part in each of the integrals, it is not hard to see that together with the free term, the part containing only integrals over this region gives $\gamma_\mu\alpha(l^2)$ (just as in reference 2). In each of the remaining integrals, it turns out that the integration over $k \gg l$ (or m) must, if we neglect small terms, be carried out along a line lying closer to the principal vertex than that for the integrals with $k \ll l$ (or m). All this leads to the result that in the successive approximations for Γ_μ, we can integrate only over $k \ll l$ (or m), and take account of the integrals over the other region by a factor $\alpha(l)^2$ common to all terms, which determines the dependence of the integral on the "cutoff" limit. The dependence on l^2 given by this factor is unimportant in our case since, as we shall see later, there are terms in Γ_μ of order $[e^2\ln^2(l^2/m^2)]^n$, which are much greater than the terms arising from $\alpha(l^2)$ which are of order $[e^2\ln(l^2/m^2)]^n$. Because of this, we can always take $\alpha(l^2) \approx \alpha(m^2)$. In calculating the integrals for $k \ll l$, we use a method similar to that developed in the paper of Sudakov[7]. The principal part of the integral is gotten by resolving k along p, q, and in the plane perpendicular to these vectors. In integrating over the last component we must take the residue for $k^2 = 0$. Thus the additional vertex parts from the virtual quanta will be of the type considered in the preceding section, and each factor $\alpha(m^2)\mu$ from such a vertex will cancel against the factor $\beta(m^2)\nu$ from the factor G which is placed next to this Γ on the side toward the fundamental vertex. So, in calculating the vertex parts which we need, we can use the zeroth order result from neighboring Γ's and G's.

To make clear what the region of integration is, we shall first consider Γ_μ in first approximation. We begin with the part containing $d_t k^{-2}\delta_{\mu\nu}$. It has the well-known form

$$\frac{e^2}{\pi i}\int \frac{\gamma_\mu(\hat p - \hat k + m)\gamma_\sigma(\hat q - \hat k + m)\gamma_\mu d_t(k^2)\,d^4k}{[(p-k)^2 - m^2][(q-k)^2 - m^2]\,k^2} \quad (13)$$

We introduce variables u, v, x:

$$k = p\left(\frac{a^2u - av}{a^2-1}\right) + q\left(\frac{a^2v - au}{a^2-1}\right) + k_\perp, \quad (14)$$
$$x = -k_\perp^2,\ a = (pq)/m^2.$$

In terms of these variables, the volume element in k-space is

$$d^4k = (m^2/4\pi)\,a^2(a^2-1)^{-1/2}\,dx\,du\,dv. \quad (15)$$

Expressed in terms of these variables, the important expressions which appear in the denominators are equal to (16)

$$k^2 = m^2a^2(a^2-1)^{-1}(2auv - u^2 - v^2) - x,$$
$$(p-k)^2 - m^2 = 2m\Delta m_1 - 2(pq)v + k^2,$$
$$(q-k)^2 - m^2 = 2m\Delta m_2 - 2(pq)u + k^2.$$

Since x is positive (integration region $0 - \infty$), u and v must be chosen so that the point $k^2 = 0$ corresponds to $x > 0$. This requires that $2auv - u^2 - v^2 > 0$. We shall take $a > 0$, $a^2 - 1 > 0$. The contrary case is treated similarly, or even more simply by analytic continuation. Thus the integration must extend over the region between the two lines (17)

$$u_1 = v(a + \sqrt{a^2-1}),$$
$$u_2 = v(a - \sqrt{a^2-1}) = v/(a + \sqrt{a^2-1}).$$

* The product $\mu(\eta)\nu(\eta)$ does not depend on $p^2 - m^2$, so that at first glance it seems that the infrared catastrophe does not occur. Actually, the scattering matrix element contains terms in the exponent of order $\frac{e^2}{\pi}\frac{l^2}{m^2} \times \ln\frac{m^2}{p^2 - m^2}$ whose determination goes beyond the accuracy of this section. (cf. later sections).

[7] V. V. Sudakov, J. Exper. Theoret. Phys. USSR 30, 87, (1956). Soviet Phys. JETP 3, 65 (1956).

In addition it is clear from Eq. (16) that u and v are limited by the conditions

$$\Delta m_1/a \ll |v| \ll 1, \qquad (18)$$

$$\Delta m_2/a \ll |u| \ll 1.$$

The limit from above corresponds to the condition $k \ll l$. It is not hard to see that integration over positive and negative regions of u and v which satisfy our conditions gives the same contribution, and can be replaced by twice the integral over $u > 0$, $v > 0$. In the numerator, because of the smallness of u, v, and x, we can drop the k. When we bracket Γ on the left and right with $(\hat{p} + m)$ and $(\hat{q} + m)$ we get the expression $4\, p\cdot q(\hat{p} + m)\gamma_\sigma \times (\hat{q} + m)$ in the numerator.

We introduce the variables $\lambda = -\ln v$, $\mu = -\ln u$. In terms of these variables, the region of integration will have a form like that shown in Fig. 1 (the cross-hatched part). Of course, other relations between $\ln a$, $\ln(m/\Delta m_1)$ and $\ln(m/\Delta m_2)$ are possible. The integral will be proportional to the area cross-hatched in Fig. 1.

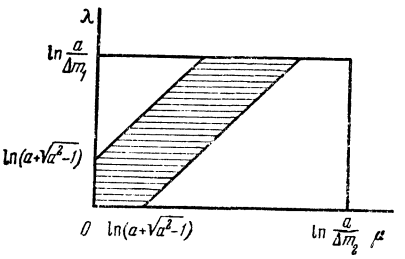

FIG. 1

We shall not yet give the corresponding expression, but will first consider the term in $(d_l - d_t)$ This term has the form

$$\frac{e^2}{\pi i}\int \frac{\hat{k}(\hat{p} - \hat{k} + m)\gamma_\sigma(\hat{q} - \hat{k} + m)\hat{k}[d_l(k^2) - d_t(k^2)]}{[(p-k)^2 - m^2][(q-k)^2 - m^2]\,k^4} d^4k\,. \qquad (19)$$

If we introduce in this integral the same variables as in (13), then the conditions (18) still hold. Then we can once more neglect $\hat{k}$ compared with $\hat{p}$ and $\hat{q}$ in the numerator, and upon multiplying on the left and right by $(\hat{p} + m)$ and $(\hat{q} + m)$ the numerator takes the form $4\, p\cdot kq\cdot k\,(\hat{p} + m)\gamma_\sigma(\hat{q} + m)$. Since the principal terms in square brackets of the expression in the denominator give the product $4\, p\cdot kq\cdot k$, upon cancelling these factors in numerator and denominator, we get in place of (19) the much simpler integral $\int k^{-4} d^4k$.

If we express this integral in terms of u, v, and x, it turns out that once more the important region in the integral is that in which $k^2 = 0$, but unlike what we had before, now both $k_{||}^2$ and x go to zero. As a result, the integral becomes proportional to the sum of the integrals $\int u^{-1} du$ (or, what is the same thing $\int v^{-1} dv$) taken along the two lines bounding the cross-hatched region of the type shown in Fig. 1. The particular form of the integral depends on the relation between Δm_1, Δm_2, and a.

Finally we obtain for the first approximation $\Gamma_\mu^{(1)}(p, q; l)$ the following formula:

$$(\hat{p} + m)\Gamma_\mu^{(1)}(p, q; l)(\hat{q} + m) = (\hat{p} + m)\gamma_\mu(\hat{q} + m)(-f),$$

where the function f has the form:

$$f = (e^2/2\pi)\{a(a^2 - 1)^{-1/2} \times ({}^3/_2\eta^2 + 2\zeta\xi_1) + (1 - d_l^0)\xi_1\} \qquad (20a)$$

for $\xi_1 > 0$, $\xi_2 > \xi_1 + \eta$;

$$f = (e^2/2\pi)\{a(a^2 - 1)^{-1/2}[\eta^2 + \zeta(\xi_1 + \xi_2) - {}^1/_2(\xi_1 - \xi_2)^2] + {}^1/_2(1 - d_l^0)(\xi_1 + \xi_2)\} \qquad (20b)$$

for $\xi_1 > 0$, $\xi_2 > 0$, $\xi_1 + \eta > \xi_2 > \xi_1 - \eta$;

$$f = (e^2/2\pi)({}^3/_2\eta^2 + 2\eta\xi_1 + {}^1/_2\xi_1^2) \qquad (20c)$$

for $-\eta < \xi_1 < 0$, $\xi_2 > \xi_1 + \eta$;

$$f = (e^2/2\pi)[\eta^2 + \eta(\xi_1 + \xi_2) + \xi_1\xi_2 - {}^1/_2\xi_2^2] \qquad (20d)$$

for $-\eta < \xi_1 < 0$, $0 < \xi_2 < \xi_1 + \eta$; (20e)

$$f = (e^2/2\pi)(\eta + \xi_1)(\eta + \xi_2)$$

for $-\eta < \xi_1 < 0$, $-\eta < \xi_2 < 0$,

where $\xi_1 = \ln(m/\Delta m_1)$, $\xi_2 = \ln(m/\Delta m_2)$, $\eta = \ln a$, $\zeta = \ln(a + \sqrt{a^2 - 1})$.

In all these terms, except those of the type $\zeta\xi$ in (20a) and (20b), we have taken $\zeta \approx \eta$, since taking account of their difference would go beyond our accuracy. For the same reason, the equations (20c), (20d) and (20e) should be taken for $a \gg 1$, and terms with one ξ or one η should be neglected.

We shall now assume that we are considering the n'th approximation for Γ_μ. In the corresponding diagram, n arbitrarily intersecting virtual lines can occur. We consider first the numerator of any diagram of this type. Upon multiplying on the left by $(\hat{p}+m)$ and on the right by $(\hat{q}+m)$ it will have the form

$$(\hat{p}+m)\gamma_\mu(\hat{p}+m)\gamma_\nu\ldots(\hat{p}+m)\gamma_\sigma(\hat{q}+m)\ldots\gamma_\xi(\hat{q}+m)\gamma_\eta(\hat{q}+m) \approx 2^{2n}p_\mu p_\nu\ldots q_\xi q_\eta\ldots(\hat{p}+m)\gamma_\sigma(\hat{q}+m). \tag{21}$$

Throughout all of this we have neglected $\hat{k}$ compared to $\hat{p}$ and $\hat{q}$, and the quantities p^2-m^2 and q^2-m^2 compared to $p_\mu q_\nu$. From formula (21) it is clear that the numerators of all diagrams of the same order are equal to one another. Differences can occur only in the denominators.

First we consider diagrams with a pair of lines. There can only be two such: one with parallel and one with intersecting lines. The integrand in both diagrams will be proportional to

$$\{(pk_1)[(pk_1)+(pk_2)][(qk_1)+(qk_2)](qk_1)\}^{-1} + \{(pk_1)[(pk_1)+(pk_2)][(qk_1)+(qk_2)](qk_2)\}^{-1}.$$

The integral with maximum degree of logarithmic divergence is obtained from the first term when $p\cdot k_2 \gg p\cdot k_1$, $q\cdot k_2 \gg q\cdot k_1$ and from the second if $p\cdot k_2 \gg p\cdot k_1$, $q\cdot k_1 \gg q\cdot k_2$. The expressions from both terms are the same, and so the integrations over $q\cdot k_1$ and $q\cdot k_2$ are independent. If we now make the change of variables $k_1 \rightleftarrows k_2$, and take half the sum of the expression thus obtained and the old expression, it turns out that $p\cdot k_1$ and $p\cdot k_2$ are also independent. Since the regions of integration are the same for both k's, we get the simple result

$$(\hat{p}+m)\Gamma_\mu^{(2)}(\hat{q}+m) = (\hat{p}+m)\gamma_\mu(\hat{q}+m)(-f)^2/2. \tag{22}$$

Generalizing this to a diagram of n'th order, we have

$$(\hat{p}+m)\Gamma_\mu^{(n)}(\hat{q}+m) = (\hat{p}+m)\gamma_\mu(\hat{q}+m)(-f)^n/n! \tag{23}$$

Summing all expressions of this type and noting, as pointed out at the beginning of this section, that the integrals over the momenta of virtual quanta in the region $k \gg l$ give a common factor $\alpha(m^2)$, we find the following result;

$$(\hat{p}+m)\Gamma_\mu(p,q;l)(\hat{q}+m) = \alpha(m^2)(\hat{p}+m)\gamma_\mu(\hat{q}+m)e^{-f} \tag{24}$$

This formula coincides with formula (10), if we take f from (20a) or (20b) and set $a \to 1$ ($\eta \to 0$). If, however, we take f in the form (20e) we arrive at the result obtained by Sudakov[7].

If we consider the process of scattering of an electron by an external field, then to get the matrix element we must multiply (24) by the expression corresponding to a free line. It is well known that to each free line there corresponds $\{G(p)\times(\hat{p}-m)\}^{1/2}$, which in our case is equal to

$$\sqrt{\beta(m^2)}\left(\frac{m}{\Delta m}\right)^{(e^2/4\pi)(3-d_l^0)}$$

Later we shall show that only the cases (20b), (20d) and (20e) have a direct physical interpretation. The results for these cases are:

$$M_0 = (\bar{u}_1\gamma_\sigma u_2)e^{-f_1}, \tag{25}$$

$$f_1 = (e^2/2\pi)\{a(a^2-1)^{-1/2}[\eta^2+\zeta(\xi_1+\xi_2)-{}^1/_2(\xi_1-\xi_2)^2]-(\xi_1+\xi_2)\} \tag{26a}$$

for $\xi_1>0$, $\xi_2>0$, $\xi_1+\eta>\xi_2>\xi_1-\eta$;

$$f_1 = (e^2/2\pi)[\eta^2+\eta\xi_1+\eta\xi_2+\xi_1\xi_2-{}^1/_2\xi_2^2] \tag{26b}$$

for $-\eta<\xi_1<0$, $0<\xi_2<\xi_1+\eta$;

$$f_1 = (e^2/2\pi)(\eta+\xi_1)(\eta+\xi_2) \tag{26c}$$

for $-\eta<\xi_1<0$, $-\eta<\xi_2<0$.

Generally speaking, when Γ_μ is an internal ver-

tex, the relation between Δm_1 and Δm_2 is arbitrary. However, in this case we must consider lines which encompass several vertices, as we shall do for the case of one additional quantum.

3. GENERALIZATION OF THE FEYNMAN DIAGRAMS FOR OBTAINING PROBABILITIES OF MULTIPLE PROCESSES.

Before we consider the question of the radiation of additional quanta during the scattering of an electron, we shall present a method which considerably simplifies the calculation of processes involving a large number of real particles. In this section we shall consider the emission of real pairs as well as real quanta. In the literature, the rules for calculating matrix elements with virtual quanta are presented very well (cf., for example, reference 5). However the situation with regard to the rules for finding transition probabilities is not so satisfactory. We shall give a simple scheme which enables one to find such probabilities without difficulty.

First we consider processes in which the only real particles are photons. Aside from the appearance of their momenta in the propagation factors for virtual electrons, the presence of these photons manifests itself as follows. For each initial photon, there is a factor $2\pi/|\mathrm{K}|$ in the probability, where K is the photon wave vector. For all except one of the final photons there is a factor $(2\pi/|\mathrm{K}|) \times (2\pi)^{-3} \mathrm{K}^2 d|\mathrm{K}|\, d\Omega$ in the probability. The momentum of the remaining photon is determined by the conservation laws so that it gives only a factor $2\pi/|\mathrm{K}|$. In addition, each photon gives rise, in the appropriate place in the matrix element, to the factor $\hat{\epsilon}$ where ϵ is the polarization vector. The sum over polarizations is carried out by replacing $\hat{\epsilon} \ldots \hat{\epsilon}$ in the expression for the probability by $-\gamma_\mu \ldots \gamma_\mu$ ($\mu = 0, 1, 2, 3$). Finally there enters into the expression for the probability the quantity $2\pi\delta(E_{\text{in.}} - E_{\text{fin.}})$ where $E_{\text{in.}}$ and $E_{\text{fin.}}$ are the total energies of the initial and final states. The δ-function is eliminated by integrating over one of the $d|\mathrm{K}|$'s.

It is easy to see that the quantity $1/|\mathrm{K}|$ can be written in the form $2\int_{\omega>0} d\omega\, \delta(k^2)$, where $k^2 = \omega^2 - \mathrm{K}^2$. If we integrate with respect to the momentum of the corresponding photon, this replacement results in a great similarity between real and virtual photons. In fact, in place of $\int \ldots \gamma_\mu \ldots \gamma_\mu \ldots \delta_+(k^2) \times d^4k$ ($\delta_+(x) = \delta(x) - 1/i\pi x$), which occurs for virtual photons, the expression

$$2\int_{\omega>0} \ldots \gamma_\mu \ldots \gamma_\mu \ldots \delta(k^2)\, d^4k$$

appears in the formula for the probability for real photons. The requirement $\omega > 0$ is relativistically invariant, and in particular signifies that the projection of the four-vector k on any time-like vector n with $n_0 > 0$ is positive. We note that in place of the factor $1/|\mathrm{K}|$ for the second photon, whose momentum is determined by the conservation laws and $\delta(E_{\text{in.}} - E_{\text{fin.}})$, we can simply put $2\delta_{\omega>0}(k^2)$. In doing this we still satisfy the energy conservation law and retain the correct factor, since $dE_{\text{fin}} = d\omega$.

We now go over to the case when the only real particles are electrons, and there are no photons or pairs. In this case each electron in the initial state contributes the expression $(1/2E)(\bar{u} \ldots u)$ to the probability, if we use the normalization $\bar{u}u = 2m$. Each of the electrons in the final state, except one whose momentum is determined by the conservation laws, gives the factor $(2E)^{-1}(\bar{u} \ldots u) \times (2\pi)^{-3} \mathbf{p}^2 d|\mathbf{p}|\, d\Omega$, where E is the energy of the electron. The last of the final state electrons gives simply $(2E)^{-1}(\bar{u} \ldots u)$. If we carry out the sum over electron spins, we replace $\bar{u}u \to \hat{p} + m$. In addition there is a factor $2\pi\delta(E_{\text{in.}} - E_{\text{fin.}})$ in the probability.

Just as in the case of photons, we can make the replacement

$$\frac{1}{E} = 2\int_{E>0} \delta(p^2 - m^2)\, dE.$$

If we are dealing with an electron for which we integrate over the momentum and sum over the spin, then we arrive at the result that each such electron gives a factor

$$\frac{\hat{p} + m}{2\pi} \int_{E>0} \delta(p^2 - m^2)\, d^4p.$$

As for the last of the final state electrons, after summing over its spin we can write for it $(\hat{p} + m)\, \delta_{E>0}(p^2 - m^2)$, which also includes $\delta(E_{\text{in.}} - E_{\text{fin.}})$.

The similarity to virtual processes still does not appear explicitly in this form. In order to see the similarity, let us consider, for example, electron-electron scattering in zeroth approximation. From the conservation laws, it is clear that $d^4p = d^4q$, where q is the momentum of the virtual quantum which the electrons exchange with one another. If we make this substitution, then the expression for the probability suggests the matrix element for the interaction of a pair of electrons with exchange of a pair of identical virtual photons. A similar situation will occur in the more general case.

Now we consider the case when pairs are produced in a process. Since the momentum of one of the components of the pair is determined by the

conservation laws if the momentum of the pair-producing photon is given, the integration need be taken over only one of the components, say the electron, and over the momentum of the photon producing the pair. In the integrand the factor $(\hat{p}+m)\,\delta_{E_p>0}(p^2-m^2)$ refers to the electron, and the factor $(\hat{p}-\hat{k}+m)\,\delta_{E_{p-k}>0}((p-k)^2-m^2)$ to the positron. The generalization to the case where real quanta, electrons, and pairs are present, presents no difficulty.

The rules we have given can be presented in the form of generalized Feynman diagrams. In fact we shall draw a diagram which shows a sort of doubled diagram for the matrix element of our process--a diagram together with its mirror image (the latter corresponds to the Hermitian adjoint matrix element). We now join those photon and electron lines for which there is a summation over spin and integration over momentum, provided the latter is not excluded by the conservation laws. We shall put a dash through such connected lines in order to distinguish them from the usual ones. After this the diagram will be completely analogous to the usual Feynman diagram for a matrix element. In particular, real pairs will be described by a loop of dashed electron lines. We note that if the matrix element is a sum of diagrams, then in the generalized diagrams each of the initial ones is joined in turn to the mirror images of all the others as well as to its own image.

In addition to virtual and intermediate lines, the fundamental elements of the generalized diagram will be:

1. A real photon line. The factor for it is

$$2\delta_{\omega>0}(k^2).$$

2. A real electron line. Its factor is

$$\frac{\hat{p}+m}{2\pi}\,\delta_{E>0}(p^2-m^2).$$

3. Two identical photon lines between a pair of electrons, corresponding to the Coulomb interaction. The factor appearing for these lines is

$$(4\pi/q^2)^2.$$

The integration in the generalized diagram is taken over all closed photon lines, dashed as well as undashed, and over electron loops made up of ordinary and dashed lines. In the case of a loop, one also takes the spur. The factor $1/\pi i$ for photons and $-1/\pi i$ for a loop occurs only for virtual processes.

All of the rules given here refer to the case where the detailed characteristics of the emitted particles are not of interest (or are determined by conservation laws), i.e., we have closed lines. If this is not the case, then the usual factors for real particles occur. As examples, generalized diagrams are shown in Fig. 2 for: a – Compton effect, b – electron-electron scattering, c – materialization of a pair in the field of an electron. We show only one of the graphs for each effect.

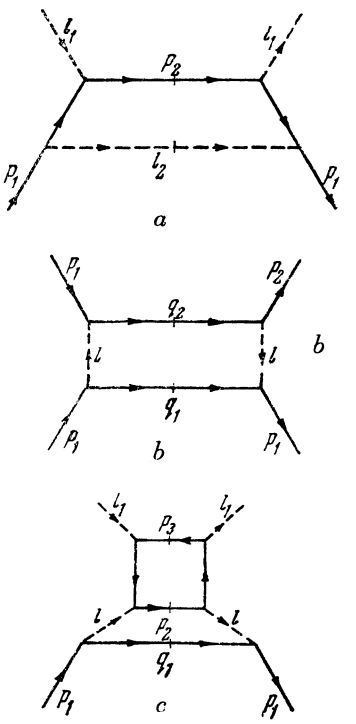

FIG. 2

4. SCATTERING ACCOMPANIED BY RADIATION OF ADDITIONAL QUANTA.

We now consider scattering with radiation of a single additional quantum $\varkappa$. We shall supose that the quantum is free ($\varkappa^2=0$) and that to the free line there corresponds certain Δm_1 and Δm_2, subject to the conditions $m\Delta m \ll q\varkappa \ll p\cdot q$. It is easy to see that the additional factor from the new vertex cancels against the factor from the internal electron line. There remain only lines which surround the fundamental vertex, or which circle both vertices. Since lines of these two types can be shifted arbitrarily, integrations over them are carried out independently, i.e., we must find the contribution from the lines surrounding both vertices, and then

multiply by the already known factor for the fundamental vertex. As for lines surrounding both vertices, we see that, because of the appearance of a new denominator, we must impose the condition $|p \cdot qu| \ll q \cdot \varkappa$ in order to separate out the logarithmic part of the integral over u. We easily see that the new lines again give an exponential factor, while the sum of the exponents of the two exponentials does not depend on $q \cdot \varkappa$, but is determined only by the values of a, Δm_1 and Δm_2. Now considering an arbitrary number of additional quanta, we arrive at the conclusion that this same situation always occurs, i.e., the net effect of the virtual lines is a factor common to all the diagrams of scattering with radiation.

We now go on to consider the case of an arbitrary number of real quanta. We shall consider the probabilities of processes involving radiation of 1,2,...n quanta, making the following assumptions:

a). The total energy carried off by the quanta does not exceed ω_1 in the rest system of the initial electron, and does not exceed ω_2 in the rest frame of the final electron.

b). We associate with free electron lines, in all diagrams considered, certain values Δm_1 and Δm_2, such that $\Delta m_1 \ll m$, $\Delta m_2 \ll m$ and $a\Delta m_1 > \Delta m_2 > \Delta m_1 / a$. In the present case the Δm are auxiliary quantities and can be chosen arbitrarily.

Let us consider the generalized diagrams corresponding to scattering with radiation. It is not hard to see that four groups of real lines occur (see Fig. 3): lines joining *a--the q* line on the left

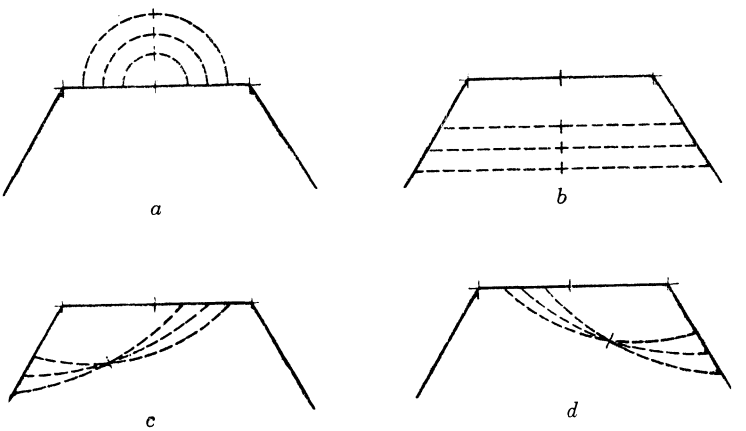

FIG. 3

to the one on the right, *b*--the p line on the left to the one on the right, *c*--the left q line to the right p line, *d*--the left p line to the right q line. If we consider the possible transpositions of lines we see that each group can be treated independently. For the same reason as before, we can use zeroth order values for the new vertices and electron lines. Consider a line of type *a*. There corresponds to it the factor

$$-2e^2 \int \frac{\gamma_\sigma (\hat{q} + \hat{k} + m) \gamma_\mu (\hat{q} + m) \gamma_\mu (\hat{q} + \hat{k} + m) \gamma_\tau \delta (k^2)\, d^4k}{[2m\Delta m_2 + 2(qk) + k^2]^2}. \tag{27}$$

If we choose the time axis of the vector k along the direction of q and keep only the logarithmic part of the integral, we see at once that it is equal to

$$-(e^2/\pi) \ln (\omega_2/\Delta m_2)\, \gamma_\sigma (\hat{q} + m) \gamma_\tau, \tag{28}$$

if $\omega_2 \ll m$. In the opposite case, ω_2 in (28) should be replaced by m.

If we consider successive lines of type *a*, it turns out that to get the maximum degree of logarithmic divergence in the integral, the lines must not intersect. Upon summing the integrals, we get the total result (for $\omega_2 \ll m$)

$$\gamma_\sigma (\hat{q} + m) \gamma_\tau (\omega_2/\Delta m_2)^{-e^2/\pi}. \tag{29}$$

Similar expressions also occur for type *b*.

Now we go to lines of type *c* and *d*. Both types of lines give the same contribution. If we consider one of the lines, for example of type *c*, then we find that the integral obtained differs from the integral for the case where the photon is virtual only by the fact that in case *c* we have $\hat{q}+\hat{k}$ in the denominator, in place of $\hat{q}-\hat{k}$ for a virtual photon, i.e., the difference is merely a sign change. Actually, just as for the case of a real photon, the main term for the virtual photon case comes from taking the δ-function in the photon propagation factor. The change in the domain of integration also produces no difference, since the double integral over positive frequencies corresponds to a double integral over $u > 0$ and the corresponding v. But we know that the region $u < 0$ gives the same contribution.

Thus we obtain for a single line of type *c* an expression proportional to the area cross-hatched in Fig.1, from which the lower part is cut off by the lines $\lambda = \ln(am/\omega_1)$, $\mu = \ln(am/\omega_2)$. This expression appears as the exponent in the sum over all lines. Multiplying the common exponential factor from all the real lines by the common exponential for all the virtual lines, we find that we get in the exponent an expression of the type of (26), with ξ_1 and ξ_2 replaced by $\ln(m/\omega_1)$ and $\ln(m/\omega_2)$. Now it is no longer difficult to explain the physical meaning of (20), and also to give the reasons why the other cases of (20) have no physical interpretation.

Upon comparing the formulas with the ω's with formula (26) it is evident the latter determine the matrix element for scattering with radiation of additional quanta with energy not exceeding Δm_1 in the coordinate system of the initial electron and no greater than Δm_2 in the frame of the final electron. If we assign the quantum frequency in one coordinate system, say ω_1, then upon transforming to the other system, we find that ω_2 lies within the limits

$$(a+\sqrt{a^2-1})\,\omega_1 > \omega_2 > \omega_1/(a+\sqrt{a^2-1}). \quad (30)$$

We note that the sloping lines bounding the integration region in Fig. 1 correspond to just this condition. For this reason, if we impose the condition $\omega \ll \omega_1$ in one coordinate system, then the maximum frequency in the other system cannot exceed $\omega_1(a+\sqrt{a^2-1})$, and conversely, i.e., it can actually be varied independently of ω_1 only within the interval given above. It is therefore clear that all the cases of (20) which are not contained in (26) have no physical meaning.

In a physical formulation of the problem, the frequency is limited in some one definite system. In general, such a bound would appear in Fig. 1. as some line bounding the cross-hatched region at its upper right corner. This line can be found by determining the maximum value of ω_2 for a given limiting frequency ω and a given ω_1. We shall not try to find the appropriate expressions for all cases which may arise, but merely point out that the resulting expression for the probability of a process with radiation of an arbitrary number of quanta with energy less than ω in the system of the initial electron, and of n additional quanta in the energy interval from ω to ω' is:

$$W = (W_0/n!)\, e^{-2f_1(\omega)}\,[f_1(\omega) - f_1(\omega)]^n, \quad (31)$$

where $f_1(\omega)$ is the f_1 of formula (26) with $\ln(m/\omega)$ substituted for ξ_1 and $\ln(m/\omega)+\eta$ in place of ξ_2.

Formula (31) is the familiar Poisson formula, and corresponds to the probability of n independent events. This is understandable, since we are dealing with quanta of such low energy that the radiation of these quanta essentially does not affect the electron's momentum, so that the radiation processes are actually independent. This result which we have obtained by a direct summation of diagrams was also demonstrated by Glauber[8] for the radiation of photons by an electron described by a given current.

We should point out that in order to get the correct result by such a method, we must impose restrictions on the domain of integration; these actually amount to a determination of the limits of energy and quantum frequency up to which the current can be regarded as classical. Strictly speaking, these limits can be obtained only through a quantum description of the current. This situation is not serious so long as one deals with quantum frequencies which are so small that $\ln(m/\omega) \gg \ln(E/m)$, but is very important for higher quantum energies.

In conclusion, I express my gratitude to Acad. L. D. Landau and V. V. Sudakov for many valuable comments.

---

[8] R. Glauber, Phys. Rev. **84**, 395 (1951).

Translated by M. Hamermesh
13