The Theory of the Electron Field Mass in the Presence of an External Medium

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Examination is made of the field correction to electron mass caused by an external medium. Second-order perturbation theory is used in the calculations.

IT was shown in many works[1,2] that the interaction between a charged particle and a quantum electromagnetic field produces in the particle an additional field mass, due to the radiation field (real-photon field), with which this charged particle is in equilibrium. This change in mass, depending on the temperature of the equilibrium radiation, causes a shift in the levels of the atomic systems. Under normal conditions this shift is small relative to the Bethe shift which, in particular, is related to the peculiar process of virtual emission and absorption of photons.

We shall consider in this article field corrections to the mass of an electron (or in general of any charged particle having a spin 1/2), produced by a medium characterized by a dielectric constant $\epsilon(\omega)$ and a magnetic permeability $\mu(\epsilon)$. The electromagnetic self-energy of a Fermion is known to diverge logarithmically, but the difference between its value in an external medium and in a vacuum is convergent[3], as will be shown below. These field corrections to the electromagnetic mass of the Fermion depend naturally on the parameters that characterize the medium.

A macroscopic description of the medium in terms of the parameters ϵ and μ is made possible by so renormalizing the self-energy (subtracting the self-energy of the particle in vacuum) that the higher frequencies, i.e., the shorter wavelengths, are cut off. A substantial share of the field correction is due here only to frequencies below the resonant frequencies that cause dispersion. This corresponds to wavelengths considerably in excess of the interatomic distances.

2. The electromagnetic field in a medium obeys the known differential equations:

$$\nabla^2 A'_\nu - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 A'_\nu}{\partial t^2} = 0, \quad \nu = 1, 2, 3, 4. \tag{1}$$

here: $A'_n = A_n/\mu\ (n = 1, 2, 3);\ A'_4 = \varepsilon A_4;$

A_n is the vector potential and $A_0 = -iA_4$ the scalar potential of the field. What matters now from here on is that ϵ and μ are frequency dependent as in the case of Cerenkov radiation. These quantities must therefore be considered as operators, the manipulation of which is defined, for example, by Eq. (27.5) of reference 4. One cannot therefore divide the first three equations of (1) by $1/\mu$ or the last of these equations by ϵ.

The quantization of the free transverse electromagnetic field in a medium has already been discussed[5-7] in connection with the quantum theory of the Cerenkov effect.

Let us consider the general quantization of the electromagnetic field in a medium with the aid of a Lagrangian that leads directly to (1) (see reference 8):

$$L = \frac{1}{8\pi}\varepsilon E^2 - \frac{1}{8\pi\mu}H^2 - \frac{1}{8\pi}\left(\operatorname{div}\mathbf{A} + \frac{\varepsilon\mu}{c}\frac{\partial A_0}{\partial t}\right)^2. \tag{2}$$

Assume the field to be contained in a large cube of side L, and expand the potentials in corresponding Fourier series, using standard methods; this results in an expression for the quantized potentials:

$$A_\nu = L^{-3/2}\sum_{\mathbf{q}}\sqrt{\frac{2\pi c'_\nu\hbar}{q}}\,[a_\nu(\mathbf{q}) \times \exp\{-ic'qt + i\mathbf{q}\mathbf{r}\} + a^+_\nu(\mathbf{q})\exp\{ic'qt - i\mathbf{q}\cdot\mathbf{r}\}], \tag{3}$$

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where $q = |\mathbf{q}|, \quad c'_n = c'\mu, \quad n = 1, 2, 3,$

$$c'_4 = c'/\varepsilon, \quad c' = c/\sqrt{\varepsilon\mu}, \quad a_4 = ia_0, \quad a_4^+ = ia_0^+,$$

and the amplitudes a_ν satisfy the commutation rules

$$[a_\nu(\mathbf{q}),\ a_{\nu'}^+(\mathbf{q}')] = a_\nu(\mathbf{q})\,a_{\nu'}^+(\mathbf{q}') - a_{\nu'}^+(\mathbf{q}')\,a_\nu(\mathbf{q}) = \delta_{\nu\nu'}\delta_{\mathbf{q}\mathbf{q}'}. \tag{4}$$

From this, it is also possible to find the commutation rules for the global fields:

$$[A_\nu(\mathbf{r}, t),\ A_\nu(\mathbf{r}', t')] = \frac{4\pi c'_\nu\hbar}{i}\frac{1}{8\pi^3}\int \exp\{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')\}\, d^3q\, \frac{\sin c'q\,(t-t')}{q}. \tag{5}$$

Comparing (5) with the commutation relations in vacuum we see that, owing to the frequency dependence of the velocity c' on the frequency, the function in the right half of Eq. (5) is spread out over the light cone. If $\epsilon=\mu=1$, the functions (5) become the ordinary commutation relations in vacuum (Δ-functions).

The quantized wave functions of the free Dirac field are of the known form:

$$\psi_\nu(\mathbf{r}, t) = \sum_n c_n\psi_n(\mathbf{r}, t) = L^{-3/2}\sum_n c_n b_{n\nu} e^{-icKt+i\mathbf{k}\mathbf{r}}, \tag{6}$$

where

$$K = \lambda\sqrt{k^2+k_0^2};\quad k_0 = mc/\hbar;\quad \lambda = \pm 1,\quad b_{n\nu}$$

are known matrices characterizing the spin states and

$$c_{n'}^+ c_n + c_n c_{n'}^+ = \delta_{nn'}. \tag{7}$$

Furthermore, the operators $\overline{\psi}$ and $\overline{A}_\nu$ of the interacting fields must satisfy the following equations:

$$\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - H\right)\overline{\psi} = -e\alpha_\nu\overline{A}_\nu\overline{\psi}, \tag{8}$$

$$\nabla^2\overline{A}_\nu - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\overline{A}_\nu}{\partial t^2} = 4\pi e\overline{\psi}^+\alpha_\nu\overline{\psi}, \tag{9}$$

where

$$\nu = 1, 2, 3, 4;\quad \overline{A}_n = \overline{A}_n/\mu,\quad n = 1, 2, 3,$$

$$\overline{A}_4 = \varepsilon\overline{A}_4;\quad e_- = -e$$

is the electron charge and $H=(c\hbar/i)\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2$ is the Hamiltonian of the Dirac equation. Equations (8) and (9) are general equations for the semi-phenomenological quantum electrodynamics in media (in the Heisenberg representation).

The operators $\overline{\psi}$ and A_ν can be expressed in terms of the free-field operators ψ and A_ν using the known operator S;

$$\overline{\psi} = S\psi S^{-1},\quad \overline{A}_\nu = SA_\nu S^{-1}, \tag{10}$$

where $S = e^v = 1 + v + v'v + v''v'v + \ldots,\ S^{-1}$

$$= e^{-v} = 1 - v + vv' + vv'v'' - \ldots,$$

$$vv' = \int_{-\infty}^{t} u(t')\,dt' \int_{-\infty}^{t'} u(t'')\,dt''$$

etc., and

$$v = \int_{-\infty}^{t} u(t')\,dt' = \frac{ie}{\hbar}\sum_{nn'} C_{n'n}v_{n'n}, \tag{11}$$

$$u(t) = \frac{ie}{\hbar}\int \psi^+(\mathbf{r}, t)\,\alpha_\nu A_\nu(\mathbf{r}, t)\,\psi(\mathbf{r}, t)\,d^3x = \frac{ie}{\hbar}\sum_{n'n} C_{nn'}u_{n'n},$$

$$v_{n'n} = \int_{-\infty}^{t} u_{n'n}\,dt, \tag{12}$$

$$u_{n'n} = \int d^3x\,\psi_{n'}^+(\mathbf{r}, t)\,\alpha_\nu A_\nu(\mathbf{r}, t)\,\psi_n(\mathbf{r}, t), \tag{13}$$

whereby, since the theory is symmetrical with respect to the sign of the charge, we have

$$C_{n'n} = {}^1/_2(C_{n'}^+C_n - C_nC_{n'}^+). \tag{14}$$

The validity of (10) is confirmed most simply by direct substitution into Eqs. (8) and (9), using the commutation relations (4), (5), and (7).

The fields $\overline{\psi}$ and $\overline{A}_\nu$ can be expanded in terms to the interaction constants as follows:

$$\overline{\psi} = \psi + [v\psi] + [v'[v\psi]] + \ldots \tag{15}$$

If second-order perturbation theory is used, the energy process of interest to us, that of emission and absorption of a virtual photon in the presence of a medium, is given by the third term of the right half of Eq. (15). To obtain the amplitude of the transition it is necessary to substitute Eqs. (11)-(13) and (3) into Eq. (15), and use the commutation relations corresponding to the absence of photons in the initial and final instant of time:

$$a_{\nu'}^+(\mathbf{q}')\,a_\nu(\mathbf{q}) = 0,\quad a_\nu(\mathbf{q})\,a_{\nu'}^+(\mathbf{q}') = \delta_{\nu\nu'}\delta_{\mathbf{q}\mathbf{q}'}, \tag{16}$$

and to the presence of one electron in state $n=0$ at the initial instant of time:

$$C_{n_+}^+ C_{n_+} = \delta_{n_+0}, \quad C_{n_+} C_{n_+}^+ = 1 - \delta_{n_+0}, \quad C_{n_-}^+ C_{n_-} = 1, \quad C_{n_-} C_{n_-}^+ = 0, \tag{17}$$

where n_+ denotes states with positive energy and n_- those with negative energy. From this it is possible to find the average value of the perturbation energy, which in the approximation employed will have the following form:

$$V = \frac{e^2}{2i\hbar} \sum_{n_+ n_-} (u_{0n_+} v_{n_+0} + v_{n_-0} u_{0n_-} - v_{0n_+} u_{n_+0} - u_{n_-0} v_{0n_-}). \tag{18}$$

3. We shall restrict ourselves from now on to a discussion of transparent dielectrics, for which $\mu=1$, $n=\sqrt{\varepsilon}$. In this case, substituting expressions (12), (13) and (3) into (18) and using (16) we obtain:

$$V = \frac{e^2}{4\pi^2} \int \frac{d^3q}{qn} \left\langle \psi_0^+ \alpha_n \left[K_0 - H_{k-q} - \frac{q}{n} \frac{H_{k-q}}{|H_{k-q}|} \right]^{-1} \alpha_n \psi_0 \right\rangle - \frac{e^2}{4\pi^2} \int \frac{d^3q}{qn^3} \left\langle \psi_0^+ \left[K_0 - H_{k-q} - \frac{q}{n} \frac{H_{k-q}}{|H_{k-q}|} \right]^{-1} \psi_0 \right\rangle. \tag{19}$$

Here ψ_0 and K_0 refer to the initial state of the electron, $H_{k-q} = \alpha(k-q) + \beta k_0$. Further, squaring the denominator, averaging the entire expression over all angles of vector q, and assuming the momentum of the electron $k=0$, we obtain:

$$V = \langle \psi_0^+ U \psi_0 \rangle = \int \psi_0^+ U \psi_0 d^3x,$$

where

$$U = \frac{e^2}{4\pi^2} k_0 \int \frac{d^3q}{q} \frac{q(3+n^{-2}) + 2K' n^{-1}}{n^2 K' q (q + qn^{-2} + 2K' n^{-1})}, \tag{20}$$

and $K' = \sqrt{q^2 + k_0^2}$. Assuming $n=1$ in Eq. (20) we obtain a well-known result, namely the logarithmically divergent field energy

$$U_0 = \frac{\alpha}{\pi} \left(\frac{3}{2} \int_0^\infty \frac{dq}{K'} - \frac{1}{4} \right) mc^2, \tag{21}$$

(see, for example, reference 9, page 399) where $\alpha = e^2/hc$. The energy (21) must be subtracted from the energy (20) if we want to compute the effect of the dielectric on the field mass.

To obtain the dependence of the index of refraction on the field mass, we employ the following equation

$$n^2 = 1 + \frac{4\pi N e_1^2}{m_1} \sum_k \mathrm{Re} \frac{f_k}{\omega_k^2 - \omega^2 - i\gamma\omega}, \tag{22}$$

where N is the density of the medium, e_1 and m_1 the charge in the mass of the particles forming the medium, ω_K the resonant oscillation frequencies, and γ the damping coefficient.* Then for $n-1 \ll 1$ the additional energy $U' = U - U_0$ will be

$$U' = \sum_k f_k U_k, \tag{23}$$

where

$$U_k = -mc^2 \frac{\alpha}{\pi} \frac{2\pi N e_1^2}{m_1 \omega_k^2} \left\{ \mathrm{Re} \int_0^\infty \frac{x\,dx}{(1 - x^2 - i\eta_k x)(x + \sqrt{x^2 + \zeta_k^2}) \sqrt{x^2 + \zeta_k^2}} + 2\,\mathrm{Re} \int_0^\infty \frac{dx}{\sqrt{x^2 + \zeta_k^2}\,(1 - x^2 - i\eta_k x)} \right\}, \tag{24}$$

and the parameter $\zeta_K = mc^2/\hbar w_K$ characterizes the ratio of the rest energy mc^2 of the particle under investigation to the binding energy of the electron (or other particles) forming the medium.

The integrals (24) can be evaluated in closed form. In this case we obtain:

$$U_k = -mc^2 \frac{\alpha}{4\pi \sqrt{\zeta_k^2 + 1}} \frac{2\pi N e_1^2}{m_1 \omega_k^2} \left\{ 5 \ln \frac{\sqrt{\zeta_k^2 + 1} + 1}{\sqrt{\zeta_k^2 + 1} - 1} - \frac{(\sqrt{\zeta_k^2 + 1} - 1)^2}{\zeta_k^2} \ln \frac{2}{\sqrt{\zeta_k^2 + 1} - 1} + \frac{(\sqrt{\zeta_k^2 + 1} + 1)^2}{\zeta_k^2} \ln \frac{2}{\sqrt{\zeta_k^2 + 1} + 1} \right\}. \tag{25}$$

4. Let us consider the limiting case, when the binding energy of the electrons in the medium is considerably less than the rest energy of the charged Fermion, $\hbar w_K \ll mc^2$, i.e., $\zeta \gg 1$. Then

* We introduced the damping term in (22) to establish the correct integration path around the singular points; in the final results we put $\gamma=0$ (transparent dielectric).

[9] A. Sokolov and D. Ivanenko, Quantum Field Theory, Moscow-Leningrad, 1952.

$$U_k \cong -mc^2 \frac{\alpha}{\pi}\left(\frac{2\pi N e_1^2}{m_1 \omega_k^2}\right)\frac{1}{\zeta_k^2}\left(2-\ln\frac{\zeta_k}{2}\right), \tag{26}$$

$$U=\Delta mc^2, \Delta m = -\frac{\alpha}{\pi} m \sum_k f_k \frac{2\pi N e_1^2}{m_1\omega_k^2}\frac{\hbar^2\omega_k^2}{m^2c^2}\left(2-\ln\frac{mc^2}{2\hbar\omega_R}\right). \tag{27}$$

Using the fact that $\sum_k f_k=1$, we can write the last result in the following form:

$$\Delta m = -\frac{\alpha}{2\pi} m (\varepsilon(0)-1) \tag{28}$$

$$\times \left(\frac{\hbar\omega_0}{mc^2}\right)^2\left(2-\ln\frac{mc^2}{2\hbar\omega_f}\right),$$

where $\epsilon(0)$ is the value of the dielectric constant at zero frequency; $w_f = \prod_k w_k^{f_k}$, and w_0 is determined from the following equation

$$\omega_0^{-2} = \sum_k f_k \omega_k^{-2}. \tag{29}$$

If the dispersion of the dielectric constant is due to the bound electrons of the medium, then $\hbar w_0/mc^2$ is of the order of α^2, since $m_1=m$ and $e_1=e$. In this case Δm is 10^{-11} $[\epsilon(0)-1]$ times the mass of the electron.

For the hydrogen atom, for example, it is possible to obtain from (29) the following level shift, caused by the effect of the dielectric on the mass:

$$\Delta E = \frac{Rh}{n^2}\frac{\alpha}{2\pi}(\varepsilon(0)-1)\left(\frac{\hbar\omega_0}{mc^2}\right)^2 \tag{30}$$

$$\times\left(2-\ln\frac{mc^2}{2\hbar\omega_f}\right),$$

of the order of $(Rh/n^2)\alpha^5$. If $n=2$, the change in frequency is of the order

$$\Delta\nu \sim 0.01\,(\varepsilon(0)-1) \tag{31}$$

It must be noted that under normal conditions (under normal temperature and pressure) the incremental mass and the change in frequency caused by the presence of a medium are small compared with the corresponding Bethe quantities, but are nevertheless greater than the corrections required by the radiation field.[1]

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