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Excitation of the Collective Levels of Heavy Nuclei by Neutrons*

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On the basis of the modal representation of A. Bohr and B. Mottelson[1,2], the inelastic scattering of neutrons by the collective levels of the target nucleus are calculated for the case in which there is no compound nucleus formation.

UNTIL recently, it has been assumed that reactions involving heavy nuclei (among which we include the inelastic scattering of neutrons) always pass through a compound nucleus phase. However, in the light of the recent researches of A. Bohr and B. Mottelson on a collective model of the nucleus[1,2], another mechanism for inelastic scattering can be considered: the neutron, in passing close to the target nucleus, produces a "tidal wave" in it, thus exciting surface vibrations (see reference 2, p. 158).

1. CALCULATION OF THE EFFECTIVE CROSS SECTION OF INELASTIC SCATTERING (GENERAL CASE)

We begin with the Hamiltonian of the total system: target nucleus and passing particle:

$$H_{\text{tot}}(\mathbf{r}, \alpha, \mathrm{x}) = -\frac{\hbar^2}{2M}\nabla_r^2 + \hat{H}_{\mathrm{n}}(\alpha, \mathrm{x}) + U(\mathbf{r}, \alpha). \quad (1)$$

Here $\mathbf{r}$ represents the coordinates of the passing particle relative to the center of the target nucleus; α and $\mathbf{x}$ are, respectively, the set of collective coordinates and the coordinates of the individual particles of the target nucleus. The operator H_{n} defines the internal motion of the nucleus, $U(\mathbf{r}, \alpha)$ is the energy of interaction between the scattered particle and the nucleus. Here we assume that the scattered particle interacts only with the collective degrees of freedom of the target nucleus. We can assume, approximately, that the incident neutron moves in an average field, changes in which are connected to the collective motions in the nucleus. In view of the small compressibility of nuclear matter, this collective motion can be associated only with the motion of the nuclear surface; for example, we can take as collective coordinates the expansion coefficients of the equation of the surface in spherical harmonics:

$$R(\vartheta, \varphi) = R_0\Big[1 + \sum_{\lambda,\mu}\alpha_{\lambda\mu} Y_{\lambda\mu}(\vartheta, \varphi)\Big], \quad (2)$$

where R_0 is the mean radius of the nucleus.

For a spherical nucleus, the mean field which acts on the nucleon is spherically symmetric:

$$U(\mathbf{r}, \alpha) = V(r) \equiv f(r/R_0); \quad (3)$$

the equipotential surfaces, which include the surface of the nucleus, are concentric spheres. In adiabatic deformations of the nuclear surface, all equipotential surfaces will be deformed similarly; the potential in this case will no longer be spherically symmetric:

$$U(\mathbf{r}, \alpha) = f(r/R(\vartheta,\varphi)) \equiv f(\xi), \quad (4)$$

where $R(\vartheta,\varphi)$ is the expression for the deformed surface (2); $\xi_0 = r/R$.

We expand the function (4) about the point $\xi_0 = r/R_0$:

$$f(\xi) = f(\xi_0) + \sum_{n=1}^{\infty}\frac{1}{n!}\frac{\partial^n f(\xi)}{\partial\xi^n}\Big|_{\xi=\xi_0}(\xi-\xi_0)^n.$$

From Eq. (3) we obtain

$$\partial^n f(\xi)/\partial\xi^n|_{\xi=\xi_0} = R_0^n\,\partial^n V(r)/\partial r^n.$$

Taking Eq. (2) into account, we have

$$U(\mathbf{r}, \alpha) = V(r) + \sum_{n=1}^{\infty}\frac{(-)^n r^n}{n!}\frac{\partial^n V(r)}{\partial r^n}\left[\frac{\sum_{\lambda,\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}(\vartheta,\varphi)}{1+\sum_{\lambda,\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}(\vartheta,\varphi)}\right]^n. \quad (5)$$

In the study of processes involving excitation of the collective levels of the nuclei, we limit ourselves to the first two terms of the expansion (5), thus assuming the deformations to be small. We have for the total Hamiltonian in this case:

$$\hat{H}_{\text{tot}}(\mathbf{r}, \alpha, \mathrm{x}) = -(\hbar^2/2M)\nabla_r^2 + V(r) + \hat{H}_{\mathrm{n}}(\alpha, \mathrm{x}) + W(\mathbf{r}, \alpha), \quad (6)$$

where the perturbation term is

* This work was completed in November, 1954.

[1] A. Bohr, Dan Mat. Fys. Medd. 26, 14 (1952).

[2] A. Bohr and B. R. Mottelson, Dan. Mat. Fys. Medd. 27, 16 (1953).

$$W(\mathbf{r}, \alpha) = -r\frac{\partial V(r)}{\partial r}\sum_{\lambda,\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\vartheta, \varphi). \qquad (7)$$

In order to estimate the magnitude of the effectivecross section of inelastic scattering, we make use of the general formula for the probability of transition between states of the continuous spectrum,under the action of a perturbation, which does not depend upon time:

$$dW_{\nu\nu'} = (2\pi/\hbar)\,|W_{\nu'\nu}|^2\,\delta(E_{\nu'} - E_\nu)\,d\nu', \qquad (8)$$

where ν represents the set of quantum numbers k and n (k is the propagation vector of the particle at infinity, n is the quantum number which characterizes the discrete levels of the target nucleus), $E_\nu = E_n + E_k$, where E_n is the energy level of the target nucleus and E_k is the kinetic energy of the particle at infinity.

The matrix element $W_{\nu'\nu} \equiv W^{n'k'}_{nk}$ is determined by means of the zeroth approximation wave functions:

$$\psi_{n,\mathbf{k}} = F_{\mathbf{k}}(\mathbf{r})\,\varphi_n(\alpha, \mathbf{x}), \qquad (9)$$

where the $\varphi_n(\alpha, x)$ are determined from the equation

$$H_{\mathbf{n}}(\alpha, \mathbf{x})\,\varphi_n(\alpha, \mathbf{x}) = E_n\varphi_n(\alpha, \mathbf{x}), \qquad (10)$$

and the functions $F_k(r)$ satisfy the equation

$$[-(\hbar^2/2M)\nabla_r^2 + V(r)]\,F_{\mathbf{k}}(\mathbf{r}) = E_k F_{\mathbf{k}}(\mathbf{r}). \qquad (11)$$

We write the matrix element $W^{n'k'}_{nk}$ in the form:

$$W^{n'k'}_{nk} = \int F^*_{k'}(\mathbf{r}')\,\varphi^*_{n'}(\alpha, \mathbf{x})\,W(\mathbf{r}', \alpha)\,\varphi_n(\alpha, \mathbf{x})\,F_{\mathbf{k}}(\mathbf{r}')\,(d\mathbf{r}')\,(d\alpha)\,(d\mathbf{x}) \qquad (12)$$
$$= \int F^*_{k'}(\mathbf{r}')\,W_{n'n}(\mathbf{r}')\,F_{\mathbf{k}}(\mathbf{r}')\,(d\mathbf{r}'),$$

where the matrix element $W_{n'n}$ is determined by means of the wave functions of the stationary state of the target nucleus: (13)

$$W_{n'n}(\mathbf{r}') = \int \varphi^*_{n'}(\alpha, \mathbf{x})\,W(\mathbf{r}', \alpha)\,\varphi_n(\alpha, \mathbf{x})\,(d\alpha)\,(d\mathbf{x})$$
$$= -r'\frac{\partial V(\bar{r}')}{\partial r'}\sum_{\lambda,\mu}\langle\alpha_{\lambda\mu}\rangle Y_{\lambda\mu}(\vartheta', \varphi').$$

For the differential $d\nu'$ in Eq. (8) we use the element of volume in wave space: $dk'_x dk'_y dk'_z$. In spherical coordinates,

$$d\nu' = k'^2\,dk'\,d\Omega = {}^1/_2\,k'd(k'^2)\,d\Omega.$$

As wave functions for the incident and scattered neutrons we have the "distorted" waves $F_k(r)$ and $F_{k'}(r)$, where F_k is normalized to unit current density and $F_{k'}$ to the δ-function in the space of the propagation vector:

$$F_{\mathbf{k}} = (4\pi M/\hbar k)^{1/2} \qquad (14)$$
$$\times\sum_{l=0}^{\infty} i^l\sqrt{2l+1}\,R_{kl}(r')\,Y_{l0}(\vartheta', \varphi'),$$

$$F^*_{\mathbf{k}'} = (2\pi)^{-3/2}\,4\pi \qquad (15)$$
$$\times\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'}(-i)^{l'}R_{k'l'}(r')\,Y^*_{l'm'}(\vartheta', \varphi')\,Y_{l'm'}(\vartheta, \varphi),$$

where ϑ, φ are the angles of scattering, R_{kl} are the radial wave functions.

For such normalized functions F_k and $F_{k'}$, the expression $dW_{\nu\nu'}$ is the differential inelastic scattering cross section

$$d\sigma_{\nu\nu'} = 4\pi M\hbar^{3}\,k'\,|W_{\nu'\nu}|^2 d\Omega. \qquad (16)$$

We have already carried out integration over $\mathbf{k}'$ in Eq. (16), which yields the value for k':

$$k' = [2M\hbar^{-2}(E_k - \Delta E_{n'n})]^{1/2},$$

$$\Delta E_{n'n} = E_{n'} - E_n.$$

The effective cross section of inelastic scattering of an unpolarized beam of neutrons is equal to

$$d\sigma_{AB} = (2I_A + 1)^{-1}\sum_{M_A}\sum_{M_B} d\sigma_{\nu\nu'}, \qquad (17)$$

where M_A and M_B are the projections of the spin of the target nucleus in the ground and excited states. Carrying out a series of calculations, we get

$$d\sigma_{AB} = d\Omega\sum_{J=0}^{\infty} B_J(k', k)\,P_J(\cos\vartheta), \qquad (18)$$

where $P_J(\cos\vartheta)$ are the Legendre polynomials, and the coefficients $B_J(k', k)$ are equal to

$$B_J(k', k) = \frac{1}{9(2I_A + 1)} \qquad (19)$$
$$\times\sum_{\lambda=|I_A - I_B|}^{I_A + I_B}(2\lambda + 1)(R_0^\lambda)^{-2}\,|Q^{(\lambda)}_{BA}|^2 A_{J\lambda}(k', k),$$

$$A_{J\lambda}(k', k) = \frac{k'}{k} \sum_{L=0}^{\infty} \sum_{L'=|L-J|}^{L+J} \sum_{l=|L-\lambda|}^{L+\lambda} \sum_{l'=|L'-\lambda|}^{L'+\lambda} (-)^{J-\lambda}(2L+1)(2L'+1) \tag{20}$$

$$\times (2l+1)(2l'+1)(LL'00|LL'J0)(ll'00|ll'J0)\, W(LL'll'; J\lambda)$$

$$\times (Ll00|Ll\lambda 0)(L'l'\lambda 0|L'l'\lambda 0)\ \mathrm{Re}\ [(-i)^{L+l'}\, i^{L'+l}\, P_{Ll}(k', k)\, P^{*}_{L'l'}(k', k)];$$

$$P_{Ll}(k', k) = 2M\hbar^{-2}\int_0^{\infty} R^{*}_{k'L}(r)\,\frac{\partial V(r)}{\partial r}\,R_{kl}(r)\,r^3 dr, \tag{21}$$

$W(LL ll'; J\lambda)$ are the Racah coefficients[3,4], $(ab00|abc0)$ are the Klebsch-Gordon coefficients.

In Eq. (13) we have replaced the matrix element $W_{n'n}$ by the matrix element $\langle\alpha\lambda\mu\rangle$ of the multipole moment of transition according to the equation (see Appendix)

$$\langle\alpha_{\lambda\mu}\rangle = (4\pi/3R_0^{\lambda})\left(\frac{2\lambda+1}{4\pi}\right)^{1/2} \tag{22}$$

$$\times(-)^{M_B}(I_B I_A - M_B M_A | I_B I_A \lambda\mu)\, Q_{BA}^{(\lambda)}.$$

It can be seen from Eq. (18) that the total cross section of inelastic scattering is equal to

$$\sigma_{AB} = 4\pi B_0(k', k), \tag{23}$$

with B_0 from Eq. (19), Eq. (20) reduces to

$$A_{0\lambda}(k', k) = \frac{k'}{k}\sum_{L=0}^{\infty}\sum_{l=|L-\lambda|}^{L+\lambda}(2L+1) \tag{24}$$

$$\times(2l+1)(Ll00|Ll\lambda 0)^2 |P_{Ll}(k', k)|^2.$$

Knowing the quantities Q_{BA}, which characterize the collective levels of the target nucleus, we can calculate the differential and total cross sections of the inelastic scattering of neutrons by Eqs. (18)-(24).

2. EXCITATION OF THE ROTATIONAL LEVELS OF EVEN-EVEN NUCLEI

We now investigate the important case of the excitation of the rotational levels in heavy even-even nuclei. It is known from experiments that the lowest energy levels of even-even nuclei, far from closed shells, satisfy the relation:

$$E_I = (\hbar^2/2J)\,I(I+1),\quad I = 0, 2, 4\ldots, \tag{25}$$

where $J=3B\beta$ is the effective moment of inertia, β characterizes the total deformation of the nucleus, B is a parameter associated with the mass.

In the energy region close to the ground state of the nucleus, the most important type of collective motion is the quadrupole vibration of the surface.

In the adiabatic approximation of the Bohr-Mottelson theory, the wave function describing even-even nucleus in the lower energy states (25) has the form

$$\psi_n = \sqrt{\frac{2I+1}{8\pi^2}}\,\varphi_{n_\beta n_\gamma}(\beta, \gamma)\,\chi_0\, D_{M0}(\theta_i), \tag{26}$$

where the functions $D^I_{M0}(\vartheta, \varphi\, \psi)$ describe the nuclear rotation; $\vartheta, \varphi\, \psi$ are the Euler angles which fix the principal axes of the deformed nucleus relative to the fixed system of coordinates; I and M are the spin of the nucleus and its projection on the fixed axis z.

We consider transitions between states which are characterized by the quantum numbers $I_A=0$ and $I_N=I$. We seek the quantity $Q^{(\lambda)}_{BA} \equiv Q^{(I)}_{I0}$. It can be shown (see Appendix) that

$$Q^{(I)}_{I0} = \prod_{k=1}^{I/2}\left[\frac{10(4k-1)(4k-3)}{3\cdot 2k(2k-1)(4k+1)}\right]^{1/2} Q^{(2)}_{2k,2k-2} \tag{27}$$

We have for the wave functions (26)[2]

$$Q^{(2)}_{2k,2k-2} = \left[\frac{3\cdot 2k(2k-1)}{10(4k-1)}\right]^{1/2}\frac{Q_0}{2Z}, \tag{28}$$

whence

$$Q^{(I)}_{I0} = (2I+1)^{-1/2}(Q_0/2Z)^{I/2}, \tag{29}$$

where Q_0 is the internal quadrupole moment of the target nucleus[2]. The magnitude of Q_0 characterizes the degree of deformation of the nuclear material.

Substituting Eq. (29) into Eq. (23), we obtain the total effective cross section of inelastic scattering of neutrons on the rotational level of the heavy even-even nucleus:

$$\sigma_{0I} = (4\pi/9)(Q_0/2ZR_0^2)^I A_{0I}(k', k), \tag{30}$$

$$I = 2, 4, 6, \ldots$$

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[3] L. C. Biedenharn, J. M. Blast and M. E. Rose, Rev. Mod. Phys. 24, 249 (1952).

[4] K. Alder, Helv. Phys. Acta 25, 235 (1952).

$$A_{0I}(k', k) = \frac{k'}{k} \sum_{L=0}^{\infty} \sum_{l=|L-I|}^{L+I} (2L+1)(2l+1)(L\,l00\,|\,LlI0)^2\,|\,P_{Ll}(k', k)\,|^2, \quad (31)$$

$$k'^2 = 2M\hbar^{-2}(E_k - \Delta E_{I0}), \quad \Delta E_{I0} = E_I - E_0,$$

where E_I is the energy of the rotational level (25).

If we assume $V(r)$ to be a potential well with sufficiently steep walls, we get from Eq. (21) (with a high degree of accuracy)

$$P_{Ll}(k', k) = 2M\hbar^{-2} R^*_{k'L}(R_0)\, V_0\, R_{kl}(R_0)\, R_0^3, \quad (32)$$

where V_0 is the effective depth of the potential well, R_0 is the radius of the nucleus. (For a square well of depth V_0, and radius R_0, the derivative $\partial V(r)/\partial r = V_0\delta(r-R_0)$; therefore Eq. (32) is satfied exactly).

The radial functions $R_{kl}(r)$ can be written in the form[5]

$$R_{kl}(r) = e^{i\delta_l}(kr)^{-1} G_{kl}(r).$$

The functions $G_{kl}(r)$ have the asymptotic form:

$$G_{kl}(r) \sim \sin(kr - \tfrac{1}{2} l\pi + \delta_l),$$

where $\delta_l = \delta_l(V(r), k)$ is the phase shift. We can now write

$$|\,P_{Ll}(k', k)\,|^2 \quad (33)$$

$$= (x')^{-2}\,|\,G_{k'L}(R_0)\,|^2\, x^{-2}\,|\,G_{kl}(R_0)\,|^2\, x_0^4\, R_0^2,$$

where $x = kR_0$, $x' = k'R_0$ and $x_0^2 = 2M\hbar^2 V_0 R_0^2$.

Let us consider the square well. In this case the functions $G(R_0)$ are represented by Bessel functions of half integer order. Chief interest lies in the case of such values of L and x_0 for which the relation

$$J_{L-1/2}(x_0) = 0, \quad (34)$$

is satisfied. This equation is the condition for the existences, in the case of the square well, of a one particle level with energy $E_{lev} = 0$ and angular momentum L. In this case, for $x' \ll 1$,

$$G_{k'L}(R_0) = (-)^L (2/\pi x')^{1/2} \{J^2_{-(L-1/2)}(x') + J^2_{L-1/2}(x')\}^{-1/2}. \quad (35)$$

It is seen from Eqs. (30), (31), (33) and (35) that for $x' \to 0$ (at the threshold of inelastic scattering) $\sigma_{0I} \sim 1/x' \to \infty$ as a result of the terms in (31) with $L=0$ and $L=1$.

For $L=0$ or $L=1$, we get [from Eq. (34)] $\cos x_0 = 0$ and $\sin x_0 = 0$, i.e.,

$$x_0 = \tfrac{1}{2} n\pi \quad (n = 1, 2, 3, \ldots). \quad (36)$$

If we take as the depth of the well $V_0 = 28$ mev, then, for example for $n=8$, we get from Eq. (36) $R_0 \approx 8.2 \times 10^{-13}$ cm. Thus for nuclei with mass number $A \approx 180$ ($R_0 \approx 8.2 \times 10^{-13}$ cm), the effect of excitation of the rotational levels can be very pronounced (elements Ta, W, etc.).

It can be assumed that, within the framework of the assumptions we have made, the general character of the behavior of the cross section of inelastic scattering ought not to depend essentially on the form of the potential well (for given effective depth and radius).

Calculations of the cross section of inelastic scattering on the first rotational level of W^{186}_{74} $E_{lev} = 123$ kev, $|Q_0| \approx 18 \times 10^{-24}\,\text{cm}^2$ *) give $\sigma_{02} \approx 10^{-24}$ to $10^{-23}\,\text{cm}^{-2}$, in dependence on the depth V_0 of the square well and the energy E of the incident neutron:

[5] Mott and Massay, *Theory of Atomic Collisions*

* Such a value for Q_0 is obtained from the energy of the rotational levels of W^{186}. To use the value of Q_0 obtained from electromagnetic data (spectroscopic measurements, half life measurements, Coulomb excitation, etc) would be incorrect since the deformation of the nuclear material can be distinguished from deformation of the charge. Measurements of the cross section of inelastic scattering of neutrons with excitation of rotational levels can appear to be new (i.e., besides the energy of the rotational levels) precise data on the degree of deformation of the nuclear material.

$V_0 = 20$ mev

E (mev)	0.200	0.300	1.000
σ_{02} (barns)	0.39	0.59	0.68

$V_0 = 28$ mev

E (mev)	0.150	0.200	0.300	0.500
σ_{02} (barns)	2.97	11.42	11.02	8.78

It is evident from the table that in the case of one particle resonance the cross section of inelastic scattering without the formation of an intermediate nucleus can reach a considerable size. For comparison, the cross section of inelastic scattering with excitation of the first level of W^{186}_{74} passing through an intermediate nucleus, has been calculated according to the theory of Hauser and Feshbach[6]. For energies of the incident neutron of $E=0.2$ mev, the cross section was equal to 0.6 barn. Thus in some cases the two mechanisms of excitation of the lowest level of the target nucleus can concur. The problem as to which of these mechanisms dominates can be solved only by experiment, on the basis of investigation of the path of the excitation curve of the first levels of the target nucleus in their dependence on the energy of the incident neutron. We note that, recently, Guernsey and Goodman[7,8] observed inelastic scattering of neutrons with $E=1.1$ mev on the first rotational level of Ta ($E_{\text{lev}} = 136.5$ kev) which occurs, in their opinion, without formation of the intermediate state.

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APPENDIX

1. CONNECTION BETWEEN DEFORMATIONS AND MULTIPOLE MOMENTS

1. We define the multipole moment of a system of nucleons as follows:

$$Q_{\lambda\mu} = \sum_{p=1}^{A} r_p^\lambda \, Y^*_{\lambda\mu}(\vartheta_p, \varphi_p), \tag{A.1}$$

If the number of particles is large, we can change to a continuous distribution of nucleons, i.e.,

$$Q_{\lambda\mu} = \int r^\lambda \, Y^*_{\lambda\mu}(\vartheta, \varphi)\, \rho\, dV, \tag{A.2}$$

where ρ is the density of nucleons in the nucleus. Taking ρ to be constant, we have $\rho = A/(4\pi/3)R_0^3$ and carrying out the integration over r in (A.2), we obtain

$$Q_{\lambda\mu} = \frac{A}{4\pi/3R_0^3(\lambda+3)} \int Y^*_{\lambda\mu}(\vartheta, \varphi)\, R^{\lambda+3}(\vartheta, \varphi)\, d\Omega.$$

For small deformations,

$$R^{\lambda+3} \approx R_0^{\lambda+3}\,[1 + (\lambda+3) \sum_{\lambda,\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\vartheta, \varphi)],$$

therefore

$$Q_{\lambda\mu} = (3A/4\pi)\, R_0^\lambda\, \alpha_{\lambda\mu}, \qquad \alpha_{\lambda\mu} = \frac{4\pi}{3AR_0^\lambda} \sum_{p=1}^{A} r_p^\lambda \, Y^*_{\lambda\mu}(\vartheta_p, \varphi_p). \tag{A.3}$$

2. We have for the matrix elements of the operator $\alpha_{\lambda\mu}$

$$\langle \alpha_{\lambda\mu} \rangle = \frac{4\pi}{3AR_0^\lambda} \sum_{p=1}^{A} \int \psi_B^* \, r^\lambda_p \, Y^*_{\lambda\mu}(\vartheta_p, \varphi_p)\, \psi_A \, d\tau. \tag{A.4}$$

We make use of the well known expression:[9]

$$(-)^{M_B}(I_B I_A - M_B M_A | I_B I_A \lambda\mu) \tag{A.5}$$

$$\times\, Q^{(\lambda)}_{BA}\, [(2\lambda+1)/4\pi]^{1/2}$$

$$= \int \psi_B^* \, r_p^\lambda \, Y^*_{\lambda\mu}(\vartheta_p, \varphi_p)\, \psi_A \, d\tau = \langle r_p^\lambda \; Y^*_{\lambda\mu}(\vartheta_p, \varphi_p)\rangle,$$

Whence we obtain Eq. (22)

[6] W. Hauser and H. Feshbach, Phys. Rev. 87, 366 (1952).

[7] J. B. Guernsey and C. Goodman, Bull. Am. Phys. Soc. 29, 48 (1954).

[8] J. B. Guernsey and C. Goodman, Phys. Rev. 95, 636 (1954).

[9] K. A. Ter-Martirosian, J. Exper. Theoret. Phys. USSR 22, 284 (1952).

2. RELATION BETWEEN MULTIPOLE MOMENTS OF TRANSITION OF VARIOUS ORDERS

1. For the spherical harmonics $Y_{\lambda\mu}(\vartheta, \varphi)$ it is easy to obtain the relation

$$(l'l00 \mid l'l\lambda 0)\left(\frac{4\pi}{2\lambda+1}\right)^{1/2} Y_{\lambda\mu}(\vartheta, \varphi) = \frac{4\pi}{[(2l'+1)(2l+1)]^{1/2}} \times \sum_{m,m'} (l'lm'm|l'l\lambda\mu)\, Y_{l'm'}(\vartheta, \varphi)\, Y_{lm}(\vartheta, \varphi).$$

From this we get

$$(l'l00 \mid l'l\lambda 0)\left(\frac{4\pi}{2\lambda+1}\right)^{1/2} (r^\lambda Y^*_{\lambda\mu}) = \frac{4\pi}{[(2l'+1)(2l+1)]^{1/2}} \times \sum_{m,m'} (l'lm'm|l'l\lambda\mu)\,(r^{l'} Y^*_{l'm'})(r^l Y^*_{lm}),$$

where $\lambda = l' + l$.

2. For the matrix elements that are defined by means of the wave functions (26), we have

$$(l'l00 \mid l'l\lambda 0)(4\pi/2\lambda+1)^{1/2} \langle r^\lambda Y^*_{\lambda\mu}\rangle = \sum_{m,m'} (l'lm'm|l'l\lambda\mu) \sum_I \sum_M \langle r^{l'} Y^*_{l'm'}\rangle \langle r^l Y^*_{lm}\rangle, \tag{A.6}$$

where the summation is carried out over the intermediate states. Making use of (A.5) we get from (A.6)

$$\begin{aligned}(l'l00 \mid l'l\lambda 0)&\left(\frac{4\pi}{2\lambda+1}\right)^{1/2} (-)^{M_B} (I_B I_A - M_B M_A \mid I_B I_A \lambda\mu)\, Q^{(\lambda)}_{I_B I_A} \sqrt{\frac{2\lambda+1}{4\pi}} \\ &= \frac{4\pi}{[(2l+1)(2l'+1)]^{1/2}} \sum_I \sum_{m,m',M} (-)^{M_B} \left(\frac{2l'+1}{4\pi}\right)^{1/2} (I_B I - M_B M \mid I_B I l' m') \\ &\times (l'lm'm|l'l\lambda\mu)(-)^M \sqrt{\frac{2l+1}{4\pi}} (I I_A - M M_A \mid I I_A\, lm)\, Q^{(l')}_{I_B I} Q^{(l)}_{I I_A}.\end{aligned} \tag{A.7}$$

In Eq. (A.7) we can carry out the summation over the magnetic quantum numbers:

$$\begin{aligned}\sum_m \sum_{m'} \sum_M &(-)^M (I_B I - M_B M | I_B I l' m')(l'lm'm|l'l\lambda\mu)(I I_A - M M_A | I I_A\, lm) \\ &= (-)^{I_A - l}[(2l+1)(2l'+1)]^{1/2} (I_B I_A - M_B M_A \mid I_B I_A \lambda\mu)\, W(I_B I \lambda l;\, l' I_A).\end{aligned} \tag{A.8}$$

Now

$$(l'l00 \mid l'l\lambda 0)\, Q^{(\lambda)}_{I_B I_A} = (-)^{I_A - l}[(2l+1)(2l'+1)]^{1/2} \sum_I Q^{(l')}_{I_B I} Q^{(l)}_{I I_A}\, W(I_B \lambda I l;\, I_A l'), \tag{A.9}$$

where $\lambda = l' + l$.

3. Making use of the relation

$$W(I_B \lambda I l;\, 0\, l') = (-)^{I_B + l - l'} (2I_B + 1)^{-1/2} (2l+1)^{-1/2} \delta_{I_B \lambda} \delta_{Il},$$

we get

$$(l'l'00\,|\,l'l\lambda 0)\,Q^{(\lambda)}_{\lambda 0} = (-)^{\lambda - l'} \sqrt{\frac{2l'+1}{2\lambda+1}}\,Q^{(l')}_{\lambda l'}\,Q^{(l)}_{l0}.$$

We set $l'=2$; then it follows that

$$Q^{(\lambda)}_{\lambda 0} = \left[\frac{10\,(2\lambda - 3)\,(2\lambda - 1)}{3\,(2\lambda+1)\,(\lambda - 1)\,\lambda}\right]^{1/2} Q^{(2)}_{\lambda\lambda-2}\,Q^{(\lambda-2)}_{\lambda-2,0}\,(\lambda = 2, 4, 6\ldots) \qquad \text{(A.10)}$$

Eq. (27) follows readily from Eq. (A.10).

Translated by R. T. Beyer
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