The Problem of the Radiation of Ultra High Velocity Electrons Moving in a Constant Magnetic Field

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As a further development of researches[1-4] on the quantum theory of the "luminous" electron, a formula has been obtained for the differential spectrum of radiation which is uniformly valid for any spectral range, for the comparitively small energies of the radiating electron, $E \ll E_{1/2} = mc^2(2Rmc/3h)^{1/2}$ as well as for energies $E \sim E_{1/2}$ and $E \gg E_{1/2}$. A closed formula for the total radiation energy (valid for arbitrary energies of the radiating electron) and a formula for the "critical" radiation frequency of the "luminous" electron have also been obtained.

1. INTRODUCTION

In works on the quantum theory of the "luminous" electron[1,3], a formula has been found for the total radiation energy, taking into account the first quantum correction. This formula is valid for energy of the radiating electron $E \ll E_{1/2}$. A similar result has been obtained more recently by Schwinger.[4] A formula was obtained in reference 3 (see also reference 4) for the differential spectrum at these energies. Klepikov[5] has obtained formulas which characterize the angular distribution of the radiation. These formulas are also correct for energies $E \gtrsim E_{1/2}$. Formulas for the differential spectrum were not obtained in this research. The differential spectrum was calculated for a series of energy values, with the aid of numerical integration.

In the present research a formula is found for the differential spectrum of radiation of the "luminous" electron which is uniformly valid for any spectral range for arbitrary energies of the radiating electron. A closed formula for the total radiation energy (valid for arbitrary energies of the radiating electron) and a formula for the "critical" radiation frequency of the "luminous" electron have also been obtained. These formulas permit us to obtain a number of new physical results.

2. ANGULAR DISTRIBUTION OF THE RADIATION

The formula which characterizes the spectral composition and angular distribution of the radiation (see, for example, reference 3) can be written in the following form:

$$W_{n,n';s,s'} = \frac{ce^2}{4\pi}\int d\Omega\, \varkappa^2 d\varkappa\, \Phi_{n,n';s,s'}\delta(K-K'-\varkappa), \tag{2.1}$$

where

$$\Phi_{n,n;s,s'} = |\bar{\alpha}_x|^2 + \cos^2\vartheta\,|\bar{\alpha}_y|^2 + \sin^2\vartheta\,|\bar{\alpha}_z|^2 - (\bar{\alpha}_y^+\bar{\alpha}_z + \bar{\alpha}_z^+\bar{\alpha}_y)\sin\vartheta\cos\vartheta,$$

$$|\bar{\alpha}_x|^2 = I_{s,s'}^2\left\{\frac{KK'-k_0^2}{2KK'}(I_{n,n'-1}^2 + I_{n-1,n'}^2) - \frac{4\gamma\sqrt{nn'}}{KK'}I_{n,n'-1}I_{n-1,n'}\right\},$$

$$|\bar{\alpha}_y|^2 = I_{s,s'}^2\left\{\frac{KK'-k_0^2}{2KK'}(I_{n,n'-1}^2 + I_{n-1,n'}^2) + \frac{4\gamma\sqrt{nn'}}{KK'}I_{n,n'-1}I_{n-1,n'}\right\},$$

[1] A. A. Sokolov, N. P. Klepikov and I. M. Ternov, J. Exper. Theoret. Phys. USSR 23, 632 (1952).

[2] A. A. Sokolov, N. P. Klepikov and I. M. Ternov, J. Exper. Theoret. Phys. USSR 24, 249 (1953); Dokl. Akad. Nauk SSSR 89, 665 (1953).

[3] A. A. Sokolov and I. M. Ternov, J. Exper. Theoret. Phys. USSR 25, 698 (1953); Dokl. Akad. Nauk SSSR 92, 537 (1953).

[4] J. Schwinger, Proc. Nat. Acad. Sci. 40, 133 (1954).

[5] N. P. Klepikov, J. Exper. Theoret. Phys. USSR 26, 19 (1954).

$$|\bar{\alpha}_z|^2 = I^2_{s,s'}\left\{\frac{KK'-k_0^2}{2KK'}(I^2_{n,n'}+I^2_{n-1,n'-1}) - \frac{4\gamma\sqrt{nn'}}{KK'}I_{n,n'}I_{n-1,n'-1}\right\},$$

$$\bar{\alpha}_y^+\bar{\alpha}_z + \bar{\alpha}_z^+\bar{\alpha}_y = -I^2_{s,s'}\frac{\sqrt{4\gamma}\,\varkappa\cos\vartheta}{KK'}\{I_{n,n'-1}I_{n-1,n'-1}+I_{n-1,n'}I_{n,n'}\},$$

$$K=\sqrt{k_0^2+4\gamma n},\quad K'=\sqrt{k_0^2+4\gamma n'+\varkappa^2\cos^2\vartheta},\quad x=\frac{\varkappa^2\sin^2\vartheta}{4\gamma}\quad \gamma=\frac{eH}{2ch},$$

$$I_{n,n'}(x) = (n!n'!)^{-1/2}e^{-x/2}x^{(n-n')/2}Q_{n'}^{(n-n')}(x),$$

n is the principal quantum number, s the radial quantum number. The function $Q_n^{(n-n')}(x)$ is the associated Laguerre polynomial.

The expression for $\Phi_{n,n';s,s'}$ can be greatly simplified by means of relations among the functions $I_{n,n'}$. Using recurrence relations among the Laguerre polynomials, we get the following relations among the $I_{n,n'}$:

$$\begin{aligned}\sqrt{x}I_{n,n'-1} &= \sqrt{n}I_{n-1,n'-1}-\sqrt{n'}I_{n,n'};\\ \sqrt{x}I_{n-1,n'} &= \sqrt{n}I_{n,n'}-\sqrt{n'}I_{n-1,n'-1}.\end{aligned} \tag{2.2}$$

By differentiation of the $I_{n,n'}(x)$ with respect to x and use of the relations just written down, we can show that

$$\begin{aligned}\sqrt{n}I_{n-1,n'}-\sqrt{n'}I_{n,n'-1} &= \left(\frac{d}{d\sqrt{x}}+\sqrt{x}\right)I_{n,n'};\\ \sqrt{n'}I_{n-1,n'}-\sqrt{n}I_{n,n'-1} &= \left(\frac{d}{d\sqrt{x}}-\sqrt{x}\right)I_{n-1,n'-1}.\end{aligned} \tag{2.3}$$

Further, taking into account the equality

$$\tfrac{1}{2}(KK'-k_0^2) = \gamma(n+n'-x), \tag{2.4}$$

which arises from the conservation of energy (K-$K'=\varkappa$), the expression for $\Phi_{n,n';s,s'}$ can be reduced to the following form:

$$\begin{aligned}\Phi_{n,n';s,s'} = I^2_{s,s'}(x)\frac{4\gamma}{KK'}\Big\{&\frac{K^2}{4\gamma}\operatorname{ctg}^2\vartheta\,(I^2_{n,n'}+I^2_{n-1,n'-1})\\ &+x\,(I'^2_{n,n'}+I'^2_{n-1,n'-1})\\ &+x\,(I_{n,n'}I'_{n,n'}-I_{n-1,n'-1}I'_{n-1,n'-1})\Big\}.\end{aligned} \tag{2.5}$$

Problems involving a change in trajectory, where it is important to take radial quantum transitions into account, were considered in reference 3. Since we are presently interested only in the problem of the intensity of the radiation, we can carry out the summation over s' with the aid of the relation $\sum_s I^2_{s,s'}=1$. If we further change to the dimensionless variable $\vec{\xi}=\vec{\varkappa}/2\sqrt{\gamma}$ we can write the expression for the total radiation energy as

$$W=\sum_{m=0}^{n}W_m; \tag{2.6}$$

$$W_m = \frac{\gamma}{\pi}ce^2A\int\frac{\Phi_m(\vec{\xi})}{A-\xi}\delta\,[\varphi_m(\vec{\xi})]\,d^3\xi; \tag{2.6a}$$

$$A=K/2\sqrt{\gamma},$$

$$\varphi_m(\vec{\xi}) = \xi - A + \sqrt{A^2-m^2+\xi^2\cos^2\vartheta},$$

$$\begin{aligned}\Phi_m(\vec{\xi}) = {}&\operatorname{ctg}^2\vartheta\,(I^2_{n,n-m}+I^2_{n-1,n-1-m})\\ &+\frac{\xi^2\sin^2\vartheta}{A^2}(I'^2_{n,n-m}+I'^2_{n-1,n-1-m})\\ &+\frac{\xi^2\sin^2\vartheta}{A^2}(I_{n,n-m}I'_{n,n-m}-I_{n-1,n-1-m}I'_{n-1,n-1-m}),\end{aligned}$$

where the argument of the function $I_{n,n'}$ is $\xi^2\sin^2\vartheta$, and the prime on the I indicates, the derivative with respect to this argument.

For further computation, in order to avoid the complicated argument which results from integration with δ-functions, it is appropriate to transform the integral in Eq. (2.6a) to a surface integral by means of the well-known formula

$$\int\ldots\delta(\varphi)\,d\tau = \int_{\varphi=0}\ldots d\sigma/|\nabla\varphi|, \tag{2.7}$$

where $d\sigma$ is an element of the surface φ=0. We then get, in place of Eq. (2.6a):

$$W_m = \frac{\gamma}{\pi}ce^2A\int_{S_m}\frac{\Phi_m(\vec{\xi})}{(\partial\varphi_m/\partial n)(A-\xi)}\,d\sigma, \tag{2.8}$$

Here S_m is the surface in the space of the dimensionless quantity $(\vec{\xi})$, determined by the equation

$\varphi_m(\vec{\xi})=0$; $d\sigma$ is an element of this surface; n is the outward drawn normal to this surface. There is axial symmetry relative to the z axis.

Each of the surfaces S_m gives a spatial picture of the distribution of the radiation frequencies for the quantum transition $n \to n'$ $(n-n'=m)$. The wave number of the radiation in a particular direction, in the dimensionless units employed here, is equal to the modulus of the radius vector drawn in this direction from the origin to the intersection with the surface S_m. It must be noted that, in the transition to very high frequencies, radiation will not be possible in all directions. The maximum possible angle between the direction of radiation of a given frequency and the plane of motion of the electrons (in the classical sense) decreases all the more strongly with increase in frequency. Such a decrease in the angle of possible radiation is a consequence of the law of conservation of energy--momentum.

In the entire region of frequencies of interest to us, the discrete spectrum is equivalent to the continuous, and we can therefore make the transition from summation over m to the integral

$$W=\int_0^n W_m\,dm. \tag{2.9}$$

in forming the total energy of radiation. In this transition, the family of surfaces S_m fills the entire space, the boundary of which is defined by the equation $\varphi_n=0$.

We take it into account that in the transition from one surface to another, the identity $\varphi_m=0$ exists. It follows from this identity that

$$dm=2(A-\xi)\frac{\partial\varphi_m}{\partial n}\,dn, \tag{2.10}$$

$$m=2A\xi-\xi^2\sin^2\vartheta,$$

where dn is the differential normal to the surface $\varphi_m=0$. Substituting (2.10) into (2.9), taking the value of W_m from (2.8) and considering that $d\sigma dn=d^3\vec{\xi}$ is an element of volume in the dimensionless space in which we are working, we finally obtain

$$W=\frac{2\gamma}{\pi}ce^2A\int_{V_n}\Phi_{2A\xi-\xi^2\sin^2\vartheta}(\vec{\xi})\,d^3\vec{\xi}, \tag{2.11}$$

where V_n is the volume bounded by the surface S_n.

For what follows, it is appropriate to change to other units. In the units employed here, the wave number of the fundamental classical vibration (the "fundamental" in the sense of the classical theory of the "luminous" electron) is equal to $\xi_0=1/2A$. As the new independent variable, we select the ratio of ξ to ξ_0, i.e., what in the classical theory of the "luminous" electron in the discrete spectrum is known as the number of the harmonic. Setting $\vec{\nu}=2A\vec{\xi}$ we obtain

$$W=\frac{ce^2\beta^2}{4\pi R^2}\int_{V_n}\Phi_{\nu'}\left(\frac{\vec{\nu}}{2A}\right)d^3\vec{\nu}, \tag{2.12}$$

where

$$\nu'=\nu\left(1-\frac{\nu}{4n}\beta^2\sin^2\vartheta\right),$$

$$\beta=\frac{\sqrt{n}}{A},\quad \sqrt{1-\beta^2}=\frac{mc^2}{E},$$

$R=\sqrt{n/\gamma}$ is the classical radius of the trajectory of the motion of the electron.

We further make use of approximations of the function $I_{n,n'}(x)$, correct over the entire spectrum[2,5]:

$$I_{n,n'}(x)=\frac{1}{\pi\sqrt{3}}\sqrt{1-\frac{x}{(\sqrt{n}-\sqrt{n'})^2}} \tag{2.13}$$

$$\times K_{1/3}\left\{\frac{2}{3}\sqrt[4]{nn'}(\sqrt{n}-\sqrt{n'})\left(1-\frac{x}{(\sqrt{n}-\sqrt{n'})^2}\right)^{3/2}\right\};$$

$$\dot{I}'_{n,n'}(x)=\frac{1}{n\sqrt{3}(\sqrt{n}-\sqrt{n'})}\sqrt[4]{nn'}\left(1-\frac{x}{(\sqrt{n}-\sqrt{n'})^2}\right) \tag{2.13a}$$

$$\times K_{2/3}\left\{\frac{2}{3}\sqrt[4]{nn'}(\sqrt{n}-\sqrt{n'})\left(1-\frac{x}{(\sqrt{n}-\sqrt{n'})^2}\right)^{3/2}\right\}$$

and neglect terms of order $\sqrt{1-\beta^2}$ in comparison with the fundamental terms, we get the following formula, which characterizes the angular distribution of the radiation:

$$W=\frac{ce^2}{R^2}\frac{1}{3\pi^2} \tag{2.14}$$

$$\times\int\frac{\nu^2d\nu}{1-\nu/2n}\sin\vartheta\,d\vartheta\left\{\varepsilon_\vartheta K^2_{1/3}\left(\frac{1}{3}\frac{\nu}{1-\nu/2n}\varepsilon_\vartheta^{3/2}\right)\cos^2\vartheta\right.$$

$$+\varepsilon_\vartheta^2K^2_{2/3}\left(\frac{1}{3}\frac{\nu}{1-\nu/2n}\varepsilon_\vartheta^{3/2}\right)$$

$$+\frac{1}{2}\left(\frac{\nu}{2n}\right)^2\left(1-\frac{\nu}{2n}\right)^{-1}\varepsilon_\vartheta^2\left[K^2_{1/3}\left(\frac{1}{3}\frac{\nu}{1-\nu/2n}\varepsilon_\vartheta^{3/2}\right)\right.$$

$$\left.\left.+K^2_{2/3}\left(\frac{1}{3}\frac{\nu}{1-\nu/2n}\varepsilon_\vartheta^{3/2}\right)\right]\right\},$$

where $\varepsilon_\vartheta=1-\beta^2\sin^2\vartheta$, $d^3\nu^2 d\nu\sin\vartheta d\vartheta d\varphi$ is the element of volume in spherical coordinates. Integration over φ can easily be carried out in Eq. (2.14), since the integrand does not depend on φ.

3. DIFFERENTIAL SPECTRUM

In order to get the differential spectrum, we must integrate Eq. (2.14) with respect to ϑ. In this integration, taking into account the exponential fall off of the integrand as one departs from the angle $\vartheta=\pi/2$, we can make the change of variable $\cos\vartheta=x$ and extend the limits of integration to infinity.

The integrals appearing in Eq. (2.14) are computed by means of the theory of the Mellin transformation. For example, for calculation of the integral

$$I_1=\int_0^\infty x^2\varepsilon_x K^2_{1/3}(p\varepsilon_x^{3/2})\,dx; \quad (\varepsilon_x=1-\beta^2+x^2) \tag{3.1}$$

we must take into consideration the Nicholson integral

$$K_\mu(z)K_\nu(z)=2\int_0^\infty K_{\mu-\nu}(2z\,\mathrm{ch}\,t)\,\mathrm{ch}\,(\mu+\nu)t\,dt, \tag{3.2}$$

and also the value of the integrals

$$\int_0^\infty K_\nu(x)\,x^{\mu-1}\,dx=2^{\mu-2}\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right), \tag{3.3}$$

$$\int_0^\infty\frac{x^{2p-1}dx}{(a+x^2)^{p+q}}=\frac{1}{2a^q}\,\mathrm{B}(p,\ q),\quad (a>0), \tag{3.4}$$

$$\int_0^\infty\frac{\mathrm{ch}\,2px}{\mathrm{ch}^{2q}x}\,dx=2^{2(q-1)}\mathrm{B}(q+p,\ q-p), \tag{3.5}$$

$$(\mathrm{Re}\,q>|\mathrm{Re}\,p|).$$

We express the function $K^2_{1/3}$ in the integrand of Eq. (3.1) with the help of Eq. (3.2), and expressing the function K_0 (which enters into the result of this representation) by means of the equality following from Eq. (3.3) [by Mellin's theorem]. This lowers the order of integration and is possible because of the absolute convergence of the integrals Finally, making use of Eqs. (3.4) and (3.5), we get

$$I_1=\frac{\varepsilon_0^{5/2}}{8\pi i}\int_{k-i\infty}^{k+i\infty}p_0^{-\mu}2^{\mu-2}\varphi_1(\mu)\,d\mu, \tag{3.6}$$

where

$$\varphi_1(\mu)=\Gamma^2\left(\frac{\mu}{2}\right)\mathrm{B}\left(\frac{3}{2},\ \frac{3}{2}\mu-\frac{5}{2}\right)\mathrm{B}\left(\frac{\mu}{2}-\frac{1}{3},\ \frac{\mu}{2}+\frac{1}{3}\right),$$

$$\varepsilon_0=1-\beta^2,\quad p_0=p\varepsilon_0^{3/2}.$$

If we transform the expression for $\varphi_1(\mu)$ by means of the product formula for the gamma function,

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right) \tag{3.7}$$

$$=(2\pi)^{1/2(n-1)}n^{1/2-nz}\Gamma(nz),$$

we get

$$I_1=\frac{\pi}{4\sqrt{3}}\frac{\varepsilon_0}{p}\left[\int_{2p_0}^\infty K_{5/3}(x)\,dx-K_{2/3}(2p_0)\right]. \tag{3.8}$$

The other integrals are calculated in a similar way. As a result, we get the following formula for the differential spectrum of the "luminous" electron:

$$W=\int_{\nu=0}^{2\pi}dW_\nu, \tag{3.9}$$

$$dW_\nu=\frac{1}{\pi\sqrt{3}}\frac{ce^2}{R^2}\varepsilon_0\nu\,d\nu\left\{\int_{\frac{2}{3}\frac{\nu}{1-\nu/2n}\varepsilon_0^{3/2}}^\infty K_{5/3}(x)\,dx\right.$$

$$\left.+\left(\frac{\nu}{2n}\right)^2\left(1-\frac{\nu}{2n}\right)^{-1}K_{2/3}\left(\frac{2}{3}\frac{\nu}{1-\nu/2n}\varepsilon_0^{3/2}\right)\right\}.$$

This formula for the differential spectrum is applicable for the entire spectrum, both for energies $E\ll E_{1/2}$ as well as for $E\sim E_{1/2}$ and $E\gg E_{1/2}$. A formula for the differential spectrum which is valid for $E\ll E_{1/2}$ was obtained in reference 3:

$$dW=\frac{3\sqrt{3}}{4\pi}\frac{ce^2}{R^2}\left(\frac{E}{mc^2}\right)^4 y\,dy\left\{\int_y^\infty K_{5/3}(x)\,dx\right. \tag{3.10}$$

$$\left.-\frac{3}{2}y^2\frac{h}{Rmc}\left(\frac{E}{mc^2}\right)^2K_{5/3}(y)\right\},$$

$$y=(2\omega/3\omega_0)(mc^2/E)^3,\quad \omega_0=c/R.$$

This formula can be obtained from Eq. (3.9) if we limit ourselves to accuracy of first order in ν/n. We have

$$\int\limits_{\frac{2}{3}\frac{\nu}{1-\nu/2n}\varepsilon_0^{3/2}}^{\infty} K_{5/3}(x)\,dx \approx \int\limits_{\frac{2}{3}\nu\varepsilon_0^{3/2}}^{\infty} K_{5/3}(x)\,dx - \frac{\nu}{2n}\frac{2}{3}\nu\varepsilon_0^{3/2}K_{5/3}\left(\frac{2}{3}\nu\varepsilon_0^{3/2}\right)+\dots$$

Since

$$\frac{2}{3}\nu\varepsilon_0^{3/2} = \frac{2}{3}\frac{\omega}{\omega_0}\left(\frac{mc^2}{E}\right)^3 = y \tag{3.11}$$

and

$$1/2n = (h/Rmc)(mc^2/E), \tag{3.12}$$

we can write Eq. (3.9) in the form of Eq. (3.10) with accuracy to first order in ν/n.

The functions which enter into Eq. (3.9) for the differential spectrum have been studied in the classical theory of the "luminous" electron. Therefore this expression makes it possible to study in full detail the entire spectrum of the "luminous" electron for arbitrary energies. In the study of the spectrum it is expedient to take the overall magnitude of the spectrum as unity. Then Eq. (3.9) takes the following form in the independent variable $\xi=\nu/2n=h\omega/E\,(0\leq\xi<1)$:

$$dW = \frac{ce^2}{\pi\sqrt{3}}\left(\frac{mc}{h}\right)^2\xi d\xi\left\{\int\limits_{\xi/(1-\xi)\zeta}^{\infty} K_{5/3}(x)\,dx + \frac{\xi^2}{1-\xi}K_{2/3}\left(\frac{\xi}{1-\xi}\frac{1}{\zeta}\right)\right\}, \tag{3.13}$$

where

$$\zeta = (3h/2Rmc)(E/mc^2)^2 = (E/E_{1/2})^2.$$

In the case of extreme ~~ultrarelativistic~~ energy ($\zeta\gg1$) we have, for almost the entire spectrum (with the exception of the immediate vicinity of the far limit of the spectrum ($\zeta\to1$) : $\zeta/(1-\xi)\zeta\ll1$, so that use can be made of the asymptotic formula

$$K_\nu(x)\approx 2^{\nu-1}\Gamma(\nu)\,x^{-\nu}, \quad x\ll1, \quad \nu>0. \tag{3.14}$$

The asymptotic formula for the differential spectrum has the following form:

$$dW \approx \frac{ce^2}{R^2}\frac{3^{1/6}\Gamma(2/3)}{\pi}\left(\frac{Rmc}{h}\right)^{4/3} \times\left(\frac{E}{mc^2}\right)^{4/3} d\xi\xi^{4/3}(1-\xi)^{2/3}\left(1+\frac{\xi^2}{2(1-\xi)}\right). \tag{3.15}$$

In what follows we verify the applicability of this formula for the description of almost the entire spectral range in the ultrarelativistic limit by the calculation of the total energy of radiation.

The form of the spectra for the values $E=E_{1/2}$ and $E=10E_{1/2}$ is shown in Figs. 1 and 2, where the

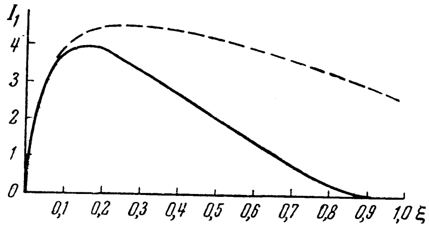

FIG. 1. Quantum and classical (dashed curve) spectra for $E=E_{1/2}$

$I_1=\frac{\pi\sqrt{\xi}}{ce^2}\frac{h}{mc}^2\frac{dW}{d\xi}$; $\xi=\frac{h\omega}{E}$

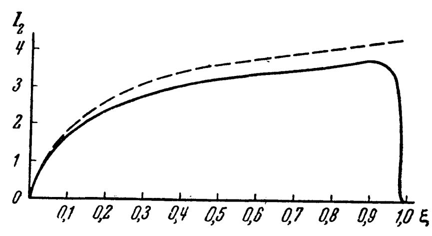

FIG. 2. Quantum and classical (dashed curve) spectra for $E=10E_{1/2}$

$I_2=\frac{\pi\sqrt{3}}{ce^2}\frac{h}{mc}^2\frac{dW}{d}$; $\xi=\frac{hW}{E}$

spectra given by classical theory for the same conditions have been plotted for comparison.

Analysis of the spectra for all energies shows that the criterion for the applicability of the classical theory of radiation of the "luminous" electron, which was established in the previous researches,

$$\zeta\ll1, \tag{3.16}$$

has meaning only in relation to the calculation of the total radiation energy in the classical theory. So far as the applicability of the classical theory to the analysis of differential spectrum, the criterion mentioned tells us nothing. Relative to the applicability of the classical theory to the analysis of the differential spectrum, we must note that first, the classical theory is in general not applicable for the study of the far portions of the spectrum, independent of whether the criterion (3.16) is satisfied or not; second, the relative region of applicability of the classical theory for the analysis of the differential spectrum, i.e., the region of applicability for fixed relative error, taken in relation to the total magnitude of the spectrum, does not depend on the criterion (3.16) and is approximately constant. In particular, for example, if we set ourselves the goal of comparing the results of the quantum and classical theory relative to the density of radiation in a certain frequency interval (say in the visible light spectrum), then the agreement between the results of the classical and quantum theory will improve considerably with increase in energy.

It follows from the analysis of the spectra that the dependence of the form of the spectrum on the energy is weakened with increase of the energy of the radiating electron. In the limiting ultrarelativistic case, the form of the spectrum is practically independent of the energy.

4. THE "CRITICAL" FREQUENCY

First of all, it is evident directly from Eq. (3.9) that the number of the "critical" harmonic ν_c (in the continuous spectrum, this is the ratio of the "critical" frequency to the frequency of the fundamental $\omega_0 = c/R$) at which the maximum energy radiation density is located, can be formally defined, just as in the classical theory, by the condition

$$\frac{2}{3}\frac{\nu_c}{1-\nu_c/2n}\varepsilon_0^{3/2} = 1. \qquad (4.1)$$

If we take Eq. (3.12) into account, we arrive at the following expression for the "critical" frequency:

$$\omega_c = (3c/2R)(E/mc^2)^3(1+\zeta)^{-1}. \qquad (4.2)$$

This coincides with what was obtained by Klepikov[5] without development, on the basis of a consideration only of the density of energy radiation in the direction of the maximum angle of radiation $\vartheta = \pi/2$.

However, Eq. (4.2) is not sufficiently rigorous. Actually, as is known, it is assumed in classical theory that the "critical" frequency ω_c^{cl} is defined by the following expression

$$\omega_c^{Cl} = (3c/2R)(E/mc^2)^3. \qquad (4.3)$$

As a direct construction of the classical spectrum of the radiation of the "luminous" electron shows, the critical frequency actually amounts to approximately one fourth this value. The quantum formula (4.2) reduces to the classical formula (4.3) at comparitively small energies of the radiating electron $\xi \ll 1$. The "critical" frequency is virtually the same as in the classical case, being about one fourth this value. Thus, in order to obtain the "critical" frequency, it is necessary to multiply the value obtained from Eq. (4.2) by the factor q, which is approximately $1/4$ in the case $E \ll E_{1/2}$. If the value of this factor did not depend on the energy and was equal to the same constant value $1/4$ even at energies $E \sim E_{1/2}$, and $E \gg E_{1/2}$, then Eq. (4.2) would have been rigorous, since the limitation on the constant factor is not essential, as is also the case in the classical theory. However, as direct construction of the spectrum shows, this factor is actually not a constant quantity, and increases somewhat with increase in energy, approaching unity in the extreme relativistic case $\zeta \gg 1$. Therefore, the more rigorous formula for the "critical" frequency has the following form:

$$\omega_c = q(\zeta)(3c/2R)(E/mc^2)^3(1+\zeta)^{-1}. \qquad (4.4)$$

The function $q(\zeta)$ appearing in this formula can be found by graphical methods. The results of such a series of calculations are shown in Fig. 3.

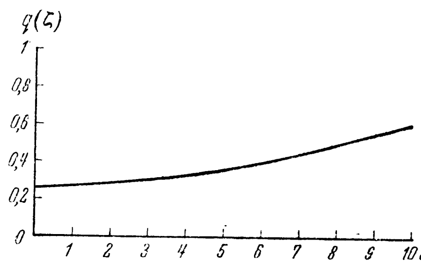

FIG. 3. Graph of the function $q(\zeta)$

5. TOTAL ENERGY OF RADIATION

To obtain the total energy of radiation, it is necessary to carry out integration in Eq. (3.9) over ν. Making the change of integration variable

$$x = \frac{2}{3}\frac{\nu}{1-\nu/2n}\varepsilon_0^{3/2}, \qquad (5.1)$$

we can represent the total radiation energy in the form

$$W = \frac{2}{3}\frac{ce^2}{R^2}\left(\frac{E}{mc^2}\right)^4\varphi(\zeta); \quad \varphi(\zeta) = \varphi_1(\zeta) + \zeta^2\varphi_2(\zeta), \qquad (5.2)$$

where

$$\varphi_1(\zeta) = \frac{9\sqrt{3}}{16\pi}\int_0^\infty \frac{x^2 K_{5/3}(x)\,dx}{(1+\zeta x)^2}; \quad (5.2a)$$

$$\varphi_2(\zeta) = \frac{9\sqrt{3}}{18\pi}\int_0^\infty \frac{x^3 K_{2/3}(x)\,dx}{(1+\zeta x)^4}.$$

We can express these integrals in the form of hypergeometric series for the cases $\zeta > 1$ and $\zeta < 1$. For $\zeta < 1$, the series are semi-convergent. For $\zeta > 1$, the series will converge. The latter series defines an analytic function of ζ on all planes of the complex variable ζ. However, the hypergeometric series, as well as the closed formula for the total radiation energy, are difficult to work with. Thus,it is difficult, starting from the latter hypogeometric series, to obtain an asymptotic expansion of the total radiation energy for $\zeta \ll 1$. Therefore, we calculate these integrals in such a form that that formula which is obtained can be used in practice as a closed formula for all values of ζ.

Writing the integral for $\varphi_1(z)$ in the form

$$\varphi_1(\zeta) = -\frac{9\sqrt{3}}{16\pi}\frac{\partial}{\partial\zeta}\int_0^\infty \frac{xK_{5/2}(x)\,dx}{1+\zeta x} = -\frac{9\sqrt{3}}{16\pi}\frac{\partial}{\partial\zeta}Q(\zeta), \quad (5.3)$$

we can, with the aid of the Mellin transformation theory, represent the expansion for $Q(\zeta)$ in the form of an integral

$$Q = -\frac{\pi}{4}\frac{1}{\zeta^2}\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\Gamma\left(\frac{s}{2}-\frac{5}{6}\right)\Gamma\left(\frac{s}{2}+\frac{5}{6}\right)}{\sin\pi s}(2\zeta)^s\,ds, \quad \frac{5}{3} < k < 2. \quad (5.4)$$

Introducing the notation

$$\Phi_\nu(z) = i^{-\nu}[J_\nu(z) - \mathbf{J}_\nu(z)] + i^\nu[J_{-\nu}(z) - \mathbf{J}_{-\nu}(z)], \quad (5.5)$$

where $J_\nu(z)$ is the Bessel function and $\mathbf{J}_\nu(z)$ is an Anger function, and closing the contour of integration in Eq. (5.4) at infinity on the left, we get

$$Q(\zeta) = -\frac{2}{3}\pi^2\zeta^{-2}\left[\Phi_{5/3}\left(\frac{i}{\zeta}\right) + \frac{\sqrt{3}}{2\pi}\zeta\right]. \quad (5.6)$$

The integral $\varphi_2(\zeta)$ can be computed similarly. Carrying out the calculation, we obtain the following expression for $\varphi(\zeta)$:

$$\varphi(\zeta) = \frac{3\sqrt{3}}{8}\pi\left\{\frac{\partial}{\partial\zeta}\frac{\Phi_{5/3}(i/\zeta)}{\zeta^2} - \frac{\zeta^2}{3}\frac{\partial^3}{\partial\zeta^3}\frac{\Phi_{2/3}(i/\zeta)}{\zeta} - \frac{\sqrt{3}}{2\pi}\frac{1}{\zeta^2}\right\}. \quad (5.7)$$

The functions on the right side of Eq. (5.7) are well known. In the calculation of Eq. (5.2) this expression gives a closed formula for the total radiation energy.

For $\zeta \gg 1$, use can be made of series expansions for the Bessel and Anger functions, keeping the desired number of terms. The principal term for the total radiation energy in this case has the form

$$W^{(\infty)} \approx \frac{32}{27}\frac{\Gamma(2/3)}{3^{4/3}}\frac{ce^2}{R^2}\left(\frac{Rmc}{h}\right)^{4/3}\left(\frac{E}{mc^2}\right)^{4/3}. \quad (5.8)$$

To find the asymptotic expression for the total energy of radiation for $\zeta \ll 1$, we can use the well-known asymptotic expansion (see reference 6)

$$\mathbf{J}_\nu(z) \approx J_\nu(z) - \frac{\sin\pi\nu}{\pi z}\left[\frac{\nu}{z} - \frac{\nu(2^2-\nu^2)}{z^3} + \frac{\nu(2^2-\nu^2)(4^2-\nu^2)}{z^5} - \dots\right] + \frac{\sin\pi\nu}{\pi z}\left[1 - \frac{1^2-\nu^2}{z^2} + \frac{(1^2-\nu^2)(3^2-\nu^2)}{z^4} - \dots\right], \quad (5.9)$$

from which it follows that in this case

$$\Phi_\nu(z) \approx 2\cos\frac{\nu}{2}\pi\,\frac{\sin\nu\pi}{\pi z}\left[\frac{\nu}{z} - \frac{\nu(2^2-\nu^2)}{z^3} + \frac{\nu(2^2-\nu^2)(4^2-\nu^2)}{z^5} - \dots\right] + 2i\sin\frac{\nu}{2}\pi\,\frac{\sin\nu\pi}{\pi z}\left[1 - \frac{1^2-\nu^2}{z^2} + \frac{(1^2-\nu^2)(3^2-\nu^2)}{z^4} - \dots\right]. \quad (5.10)$$

Upon consideration of Eq. (5.10), it follows from Eq. (5.7) that for $\zeta \ll 1$ the asymptotic expression

[6] G.I. Watson, *Theory of Bessel Functions.*

$$W \approx W_{Cl}\left(1 - \frac{55\sqrt{3}}{24}\zeta + \frac{64}{3}\zeta^2 - \frac{8855\sqrt{3}}{108}\zeta^3 + \frac{89600}{81}\zeta^4 - \ldots\right), \quad (5.11)$$

holds. This coincides with the asymptotic expansion found in the direct calculation[5] of the total radiation energy in the asymptotic sense.

It was pointed out above that in the extreme relativistic case, one can, for the study of almost the whole spectrum, make use of the asymptotic formula (3.15). With the help of this formula, we can also compute the principal term of the total radiation energy. The result coincides with Eq. (5.8). In comparison with the total radiation energy given in the classical theory, a decrease of many orders of magnitude is observed in the extreme relativistic case. However, such a decrease comes about chiefly from the fact that in this case, classical theory takes into account mainly the frequencies which ought not to radiate. If, in the calculation of the total radiation energy by the formulas of classical theory, we restrict ourselves in the integration only to the region of frequencies which actually radiate, $\omega \leq E/h$ (it must be emphasized that it is impossible to incorporate such a limitation of the range of integration into the framework of the classical theory), we then get the following expression for the total energy:

$$W_{Cl}^{(\infty)} \approx 3^{7/6}(4\pi)^{-1}(ce^2/R^2)\,\Gamma(2/3)\,(Rmc/h)^{4/3}(E/mc^2)^{4/3}. \quad (5.12)$$

The constant h enters into the formula as a result of the upper limit of integration.

In this case the classical theory describes almost the entire spectrum sufficiently well (with a relative error not exceeding 10%).

Translated by R. T. Beyer
16