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The Solution of Inhomogeneous, One-Dimensional Motion Problems in Magnetic Gasodynamics

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An account is given of particular solutions, describing the motion of gases in a magnetic field for the case of variable entropy. At the same time the motions of a gas are investigated for the instance when the coefficient of proportionality between the field intensity and the gas density depends on the entropy.

IN Ref. 1 it is demonstrated that if the solution of any problem of uniform motion in gas dynamics is known, it is easily possible to obtain a solution of the corresponding problem in magnetic gasodynamics, i.e., in the theory of motion of an electroconductive gas in a magnetic field. Such a generalization is possible as a result of the existence in this case of the integral:

$$H/\rho = Hv = b, \tag{1}$$

where the factor b, which remains in force during the motion, is determined by the initial conditions. Here H is the intensity of the magnetic field, ρ is the density of the gas, and v is its specific volume. As a result of Eq. (1), the equations of uniform motion in magnetic gasodynamics can be related to the classical equations of uniform motion in gasodynamics, where, however, instead of the gas pressure ρ_r it is necessary to use the full pressure p:

$$p = p_r + (\mu/8\pi)H^2 = p_r + (\mu/8\pi)b^2/v^2, \tag{2}$$

by means of which the aforementioned generalization is obtained. In the light of Eq. (1), the total pressure for polytropic motions (including adiabatic or isothermic motions) as well as the gas pressure is a single-valued function of the specific volume or density.

However, the solutions of the equation of uniform motion in magnetic gasodynamics examined in Ref. 1, as well as the corresponding solutions of the equation in classical gasodynamics, can be applied in practical computations only in cases of uniform distribution of the gas parameters at the initial moment of time, i.e., in those cases where the density, entropy, velocity of the gas, intensity of the magnetic field, and so forth, in their initial condition, are constant in regions separated from each other by initial discontinuities. In other words, the solutions of Eq. (1) can be employed only in those cases, where the magnitude b in (1) does not change upon transition from one element of mass to the other, i.e., is constant (however, b can change abruptly in the initial discontinuities). However, the necessity frequently arises for the solution of such equations also, in which the factor b, which is preserved for the given element of mass during its motion, changes during the transition of one element of mass into another. We will call such problems homogeneous.

The general method of solving inhomogeneous problems of one-dimensional motion in classical gasodynamics has been adequately demonstrated in Ref. 2. (Examination shows that Ref. 2 refers to those instances in which the inhomogeneity is due to inequality of the value of entropy, density, and other such parameters in various elements of mass under the initial conditions.) In the present work, this method is employed for the solution of inhomogeneous problems of magnetic gasodynamics.

Let us first convert the equations of one-dimensional motion in magnetic gasodynamics[1] to Lagrangian form. As one may easily see, they have the form:

$$u_t + \left(p_r + \frac{\mu}{8\pi}H^2\right)_h = 0, \quad u_h = v_t; \tag{3}$$

$$S_t = 0, \quad (vH)_t = 0 \quad \text{or} \quad S = S(h),$$

$$Hv = b(h).$$

Here, as usual, the derivatives are designated by a subscript; u_x is the velocity of the gas, t is the time, S is entropy, and $h = \int_0^x \rho\, dx$ is the Lagrange coordinate (the mass of the gas contained between the sections x=0 and the current section). As can be seen from the third equation in (3), the factor b in (1) is, as was to have been expected, a function of h. In Ref. 1 the case was examined

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[1]S. A. Kaplan and K. P. Staniukovich, Dokl. Akad. Nauk SSSR 95, 769 (1954).

in which $b(h)$=constant and $S(h)$=constant. Here we will consider them as given functions; in the particular case, one of them may be a constant.

The precise solution of (3) is as yet unknown. It is possible to obtain an approximate solution, however, if the total pressure is approximated by an expression of the form:

$$p = p_r + \frac{\mu}{8\pi}\frac{b^2(h)}{v^2} = \left[\frac{A^2k}{v-\sigma(h)}\right]^k\left(\frac{h_0}{h+h_0}\right)^{3k-1}, \quad (4)$$

where A, k and h_0 are certain constants, and $\sigma(h)$ is a function selected, like these constants, from conditions of optimum approximation of the total pressure. Since (4) contains three arbitrary constants and a single arbitrary function, the accuracy of this approximation is entirely satisfactory for the solution of many problems of magnetic gasodynamics. The further solution of (3) simultaneously with (4), according to the method in Ref. 2 is accurate. A particularly important part is played by the constant h_0, which characterizes the degree of inhomogeneity. For large h_0, the initial distribution approaches uniformity; at small h_0, it is strongly inhomogeneous.

Transforming (3) and (4) to new variables with the aid of the substitutions of Ref. 2

$$\tau = (h+h_0)^{-1}, \quad (5)$$

$$z = p\tau, \quad w = u/\tau + \int p\,dt,$$

we obtain the system:

$$w_t = z_\tau, \quad w_\tau = A^2 h_0^{(3k-1)/k} z^{-(k+1/k)} z_t, \quad (6)$$

the solution of which determines both the motion of the gas and the change in the magnetic field in the problems examined.

In order to find the special solution describing a uniformly propagating wave (analogous to the Riemann solution), we multiply the first equation in (6) by

$$A h_0^{(3k-1)/2k} z^{-(k+1)/2k} \quad (6A)$$

or add (or subtract) it to (from) the second. We then obtain:

$$A h_0^{(3k-1)/k} z^{-(k+1)/2k}\left[w \pm \frac{2kA}{k-1} h_0^{(3k-1/k)} z^{(k-1/2k)}\right]_t + \left[w \pm \frac{2kA}{k-1} h_0^{(3k-1)/2k} z^{(k-1)/2k}\right]_\tau = 0\,. \quad (7)$$

from this we find that the values of the factors

$$w \pm \frac{2kA}{k-1} h_0^{(3k-1)/2k} z^{(k-1)/2k} = \text{const} \quad (8)$$

and that they are propagated with a speed

$$\frac{dx}{dt} = \frac{(h+h_0)^2}{A\rho} \cdot \frac{z^{(k+1)/2k}}{h_0^{(3k-1)/2k}} = \frac{p^{(k+1)/2k}}{A\rho}\left(\frac{h+h_0}{h_0}\right)^{(3k-1)/2k}, \quad (9)$$

i.e., Eq. (9) is the velocity of propagation of a disturbance in the problems under consideration.

Integrating (7) with the aid of (8), we obtain:

$$t = \pm A h_0^{(3k-1)/2k} z^{-(k+1)/2k}\tau + F(z), \quad (10)$$

where F is an arbitrary function. Returning to the old variables:

$$t = \pm\left(\frac{h_0}{h+h_0}\right)^{(k-1)/2k}\frac{Ah_0}{p^{(k+1)/2k}} + F\left(\frac{p}{h+h_0}\right). \quad (11)$$

This is then the special solution, describing in implicit form the dependence of the total pressure on time and h. The dependence of the velocity of the gas u on time and h is found by the substitution of (11) in the first equation of (3):

$$u = \pm\left(\frac{h_0}{h+h_0}\right)^{(3k-1)/2k} \times A p^{(k-1)/2k} + \Phi\left(\frac{p}{h+h_0}\right), \quad (12)$$

where the function $\Phi(z)$ satisfies the condition

$$d\Phi/dz = -z\,dE/dz. \quad (12a)$$

As an example, let us examine the case in which F=0, which is the analogue of self-simulated motion in inhomogeneous problems. From (11) and (12) we then find:

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[2] K. P. Staniukovich, Dokl. Akad. Nauk SSSR 96, 441 (1954).

$$p = \left(\frac{h_0}{h+h_0}\right)^{(k-1)/2k} \left(\frac{Ah_0}{t}\right)^{2k/(k+1)}, \tag{13}$$

$$u = u_0 - \left(\frac{h_0}{h+h_0}\right)^{2k} A \left(\frac{Ah_0}{t}\right)^{(k-1)/(k+1)}$$

$$u_0 = \text{const.}$$

The initial values of p and u are found at some value $t=t_0$. The formulas in (13) make it possible to examine qualitatively the peculiarities of motion in inhomogeneous problems. In particular, as may be deduced from (13) and (12), the total pressure decreases with time according to the law $t^{-2k(k+1)}$; whereas the velocity increases from its zero value to

$$u_{\max} = \left(\frac{h_0}{h+h_0}\right)^{(3k-1)/2k} A p_0^{(k-1)/2k}, \tag{14}$$

where p_0 is the initial value of the full pressure. Thus the maximum velocity of propagation depends, generally, on h. This circumstance can lead to the formation of rarefaction shock waves, described in Eq. (13).

In fact, since $u=(\partial t/\partial t)_h$ then, integrating the second equation of (13), we arrive at a relationship which is determinant for x:

$$x = -\frac{k+1}{2}\left(\frac{h_0}{h+h_0}\right)^{2k} \tag{15}$$

$$\times A\left(\frac{Ah_0}{t}\right)^{(k-1)/(k+1)} t + u_0 t + \varphi(h),$$

or

$$x = \frac{k+1}{2}(u-u_0)t + u_0 t + \varphi(h) \tag{16}$$

$$= \frac{k+1}{2}ut - \frac{k-1}{2}u_0 t + \varphi(h),$$

where

$$t = t_0,\ x = x_0(h), \tag{17}$$

$$p_0 = \left(\frac{h_0}{h+h_0}\right)^{(k-1)/(k+1)} \left(\frac{Ah_0}{t_0}\right)^{2k/(k+1)},$$

$$u = u_0 - \left(\frac{h_0}{h+h_0}\right) A \left(\frac{Ah_0}{t_0}\right)^{(k-1)/(k+1)};$$

then from (4) we find $v=v_0=dx_0/dh$, from which we determine $h=h(x_0)$ and $\varphi(h)$. However, since $\sigma(h)$ can be arbitrary in the given problem, then we will on the contrary set $t=t_0$ $\varphi(h)$, from which we determine $h=h(x_0)$. Therefore, let $\varphi(h)$ be given.

The conditions for the formation of the shock wave in Lagrange coordinates can be described extremely simply. As a matter of fact, the trajectories of the various particles must intersect, and consequently, knowing that $x=x(h,t)$, we require that

$$\left(\frac{\partial x}{\partial h}\right)_t = A(k+1)\left(\frac{h_0}{h+h_0}\right)^{2k+1} \tag{18}$$

$$\times\left(\frac{t}{h_0}\right)^{2/(k+1)} + \frac{\partial\varphi}{\partial h} = 0.$$

As an example, let us suppose that

$$\varphi = B\left(\frac{h_0}{h+h_0}\right)^{\alpha}; \tag{18a}$$

then we arrive at the following expression

$$(k+1)A^{2k/(k+1)}\left(\frac{t}{h_0}\right)^{2/(k+1)}\left(\frac{h_0}{h+h_0}\right)^{2k+1} \tag{19}$$

$$= \frac{B\alpha}{h_0}\left(\frac{h_0}{h+h_0}\right)^{\alpha+1}$$

This condition relates the Lagrange coordinate h with the time of the beginning of the formation of the shock wave for the given particle. In the particular case, when $d=2k$, we arrive at the conclusion that the shock wave forms at the moment of time

$$t = \left(\frac{2k}{k+1} B A^{-2k/(k+1)} h_0^{-(k-1)/(k+1)}\right)^{(k+1)/2}, \tag{20}$$

i.e., simultaneously over the entire region of existence of the wave. As we see, the influence of a strong magnetic field on the motion of a gas can be extremely significant.

In conclusion we may note that if k satisfies the condition $k=(2n+3/(2n+1)$, where $n=-1, 0, 1, 2, \ldots$, then it is also possible to obtain a general solution for the system (6), depending on two arbitrary functions. Referring the reader to Ref. 2 for details, we immediately write the final result:

$$t = \frac{\partial^{n+1}}{\partial\theta^{2(n+1)}}[F_1(\theta+w) + F_2(\theta-w)], \tag{21}$$

$$\theta = \frac{2kA}{k-1} h_0^{(3k-1)/2k}\ z^{(k-1)/2k}$$

$$= \frac{2kh_0}{k-1}\left[\frac{h_0 p}{h+h_0}\right]^{(k-1)/(k+1)}.$$

With the aid of this solution it is possible to investigate the motion of a gas in magnetic gasodynamics after the formation of shock waves.

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65