The Compton Effect at High Energies

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The Compton effect is investigated at high energies. Those terms are analyzed which arise from diagrams giving the maximum power of $\ln(\omega/m)$ with each power of e^2. Such terms are of order $[e^2\ln^2(\omega/m)]^n$. Formulas are given for single and multiple Compton effect, for the distribution of secondary photons in energy and angle, and for the energy distribution of secondary electrons. At small angles a peculiar change in the scattering is found, which leads to a change in the refractive index of energetic photons in matter.

IN an earlier paper[1] the scattering of an electron by an external field was analyzed, and it was found that at high energies the leading terms in each order of perturbation theory were those containing with every power of e^2 the square of a large logarithm, for example $\ln^2(E/m)$. This was true so long as we were not concerned with the radiation of exceedingly soft photons for which $\ln(m/\omega) > \ln(E/m)$. Here we study in the same approximation the behavior of the Compton effect at high energies. We shall see that the behavior is no longer described, as it was for the scattering problem, by a decrease of the simple effect compensated by an increase of multiple processes.

1. THE COMPTON EFFECT AT NOT TOO SMALL ANGLES

We begin with the simple Compton scattering of a photon by an electron. We use the following notations: p_1 and p_2 are the 4-momenta of the initial and final electron, l_1 and l_2 of the initial and final photon. The momenta satisfy the relation

$$p_1 + l_1 = p_2 + l_2, \tag{1}$$

and we write

$$p = p_1 - p_2 = l_2 - l_1. \tag{2}$$

From Eq. (1) we deduce

$$(p_1l_1) = (p_1p_2) + (p_1l_2) - m^2 = (p_1l_2) - {}^1/_2 p^2. \tag{3}$$

In the electron rest system the scalar products become

$$(p_1l_1) = m\omega_1,\ (p_1p_2) = mE_2,\ (p_1l_2) = m\omega_2,$$

where ω_1 and ω_2 are the energies of initial and final photon, and E_2 the final electron energy. Thus (p_1l_1) is in all cases larger than the other two scalar products. For the relative magnitude of the other two products, we have two extreme cases to consider.

$$\text{I)}\ (p_1p_2) \approx (p_1l_1) \gg (p_1l_2) \tag{4}$$

or

$$\text{II)}\ (p_1l_2) \approx (p_1l_1) \gg (p_1p_2). \tag{5}$$

In Case I the photon transfers almost all its energy to the electron, and in Case II the electron receives only a small part of the energy. If $\omega_1 \gg m$, then case II corresponds to very small angle scattering. Equation (1) implies

$$\omega_1\omega_2(1-\cos\theta) = m(\omega_1-\omega_2), \tag{6}$$

where θ is the photon scattering angle in the rest system of the electron. Case II requires $\omega_1 - \omega_2 \ll \omega_1$ or $\theta \ll \sqrt{m/\omega} \ll 1$. Since the total cross section is proportional to a logarithmic integral, the small-angle region cannot make a significant contribution to it.

We consider first Case I, which is all that is needed for the total cross section. The matrix element for the Compton effect in zero-order approximation is proportional to

$$\frac{\gamma_\sigma(\hat{p}_1+\hat{l}_1)\gamma_\tau}{2(p_1l_1)} - \frac{\gamma_\tau(\hat{p}_1-\hat{l}_2)\gamma_\sigma}{2(p_1l_2)}. \tag{7}$$

We always omit m in the numerator, because the terms in m are negligible at high energies. The first term arises from the diagram in which the photon l_1 is first absorbed and the photon l_2 is emitted afterwards, the second term for the diagram in which the order is inverted. Because we are considering Case I, the second term is much larger than the first. Thus the fundamental Feynman diagram for the Compton effect is the one shown in Fig. 1*a*. We shall not go further into the calculation of

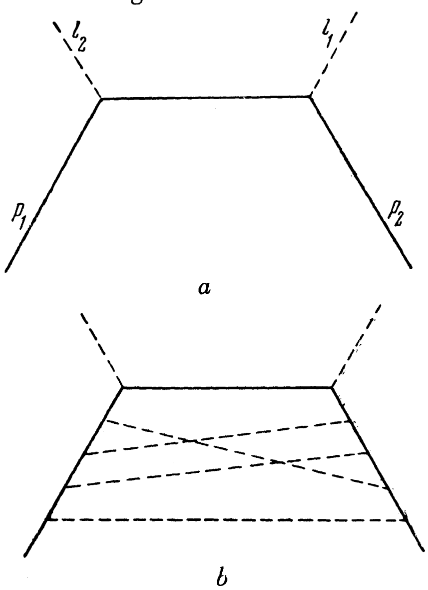

FIG. 1

the single Compton effect. We mention only the fact that, if the total cross section is expressed as an integral over ω_2, the important range of integration is $m << \omega_2 << \omega_1$. In this range the integration over ω_2 behaves logarithmically.

We now introduce virtual lines. Since the radiative corrections to electron and photon propagators contain only a single large logarithm, these corrections need not be considered. The vertex parts in this diagram have one large electron momentum (in the sense that the square of the 4-momentum is large)and twosmall momenta corresponding to the free electron and photon. In the important range of integration, $(p_1 - l_2)^2 \approx -2(p_1 l_2) = 2m\omega_2 >> m^2$. Sudakov[2] has proved that such vertex parts do not give doubly-logarithmic terms. Hence we need only consider diagrams of the kind shown in Fig. 1*b*.

We begin with the first-order radiative correction. This is described by an integral which we multiply on the left by $\hat{p}_1$ and on the right by $\hat{p}_2$ to obtain the result

$$J = \frac{e^2}{\pi i} \int \frac{\hat{p}_1 \gamma_\mu (\hat{p}_1 - \hat{k}) \gamma_\tau (\hat{p}_1 - \hat{l}_2 - \hat{k}) \gamma_\sigma (\hat{p}_2 - \hat{k}) \gamma_\nu \hat{p}_2 d_{\mu\nu}(k) d^4k}{[(p_1 - k)^2 - m^2]\,[(p_1 - l_2 - k)^2 - m^2]\,[(p_2 - k)^2 - m^2]\,k^2}. \quad (8)$$

It differs from the integral considered previously[1] by having an extra factor in the denominator. We decompose k into components along the small vectors p_1 and p_2 and in the plane perpendicular to p_1 and p_2, as in Reference 1,

$$k = p_1\left(\frac{a^2u - av}{a^2 - 1}\right) + p_2\left(\frac{a^2v - au}{a^2 - 1}\right) + k_\perp, \quad x = -k_\perp^2,$$

with $a = (p_1 p_2/m^2)$. Then

$$(p_1 - l_2 - k)^2 - m^2 \approx -2(p_1 l_2) - 2(k_\perp, (p_1 - l_2)_\perp) + k^2.$$

We used the fact that in the important range u and v are small. The extra factor in the denominator now depends on the angle in the plane of $k_\perp$. Averaging over this angle, we find

$$\overline{[(p_1 - l_2 - k)^2 - m^2]^{-1}} = \{[k^2 - 2(p_1 l_2)]^2 + 4x(p_1 - l_2)_\perp^2\}^{-1/2}. \quad (9)$$

Here $(p_1 - l_2)_\perp$ is the component of $(p_1 - l_2)$ in the plane of $k_\perp$. The square of it is $(p_1 - l_2)_\perp^2 = (p_1 - l_2)^2 - (p_1 - l_2)_\parallel^2$. Since $(p_1 - l_2)_\parallel^2 << (p_1 - l_2)^2$, we write $(p_1 - l_2)_\perp^2 \approx (p_1 - l_2)^2$. Therefore the square root in Eq. (9) is $-2(p_1 l_2)$ when $x << (p_1 l_2)$, and is k^2 when $x >> (p_1 l_2)$.

In the first case we are reduced to the same integral. which was evaluated for the vertex parts,[2] though the range of the variables is different, and there is the extra constant denominator $-2(p_1 l_2)$. In the second case we have an extra k^2 in the denominator, and so the terms in the numerator containing $\hat{k} \ldots \hat{k}$ become important. But in this case we find $(p_1 p_2)$ instead of $(p_1 l_2)$ in the denominator. Since $(p_1 p_2) >> (p_1 l_2)$, the contribution from the range $x >> (p_1 l_2)$ turns out to be negligible.

Thus we are dealing with an integral similar to the one considered earlier.[1] We may therefore neglect $\hat{k}$ in the numerator and change $d_{\mu\nu}(k)$ into $\delta_{\mu\nu}$. The doubly-logarithmic integral is obtained after taking the residue of the x-integration at the point $k^2 = 0$. To determine the ranges of integration of the variables u and v, we use the results of the earlier paper.[1] We shall not consider secondary photons with energy less than m in the laboratory system. This means that any number of such soft photons may be emitted during the primary process.

In general, to find the cross section, we ought to sum over processes involving any number of real photons, integrating over each photon momentum up to $(pk) \sim m^2$. But we showed before[1] that such a summation gives the same result as putting

$$(p_1^2 - m^2)/2m \sim m, \quad (p_2^2 - m^2)/2m \sim m/(p_1 p_2) \tag{10}$$

in our integrals. The limit of the x-integration is set, as we have seen, by the condition

$$x \ll (p_1 l_2). \tag{11}$$

Since we are taking the residue at the point $k^2 = 0$ or $x = (p_1 p_2)\, uv$, Eq. (11) reduces to

$$k^2 = 0 \quad \text{or} \quad x = (p_1 p_2)\, uv: \qquad |uv| \ll (p_1 l_2)/(p_1 p_2). \tag{12}$$

In terms of the variables $\lambda = -\ln v$, $\mu = -\ln u$, the region of integration has the form shown in Fig. 2. The integral is proportional to the shaded area and is equal to

$$J = -\frac{\hat{p}_1 \gamma_\tau (p_1 - p_2) \gamma_\sigma \hat{p}_2}{2(p_1 l_2)} \times \left[-\frac{e^2}{2\pi}\left(\frac{3}{2}\ln^2\frac{(p_1 p_2)}{m^2} - \frac{1}{2}\ln^2\frac{(p_1 p_2)}{(p_1 l_2)}\right)\right]. \tag{13}$$

This result differs from the zero-order approximation (7) only by the factor in square brackets.

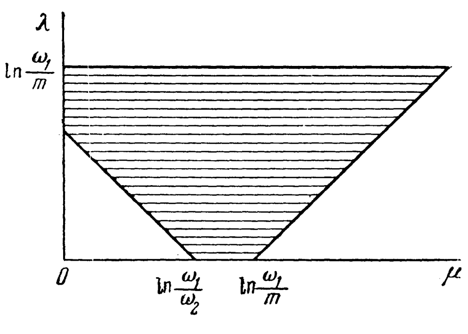

FIG. 2

Let there now be two virtual photons. In this case the extra denominator in the matrix element becomes

$$(p_1 - l_2 - k_1 - k_2)^2 - m^2 \approx -2(p_1 l_2) - 2(k_{1\perp}, (p_1 - l_2)_\perp) - 2(k_{2\perp}, (p_1 - l_2)_\perp) + k_1^2 + k_2^2.$$

After averaging over all the angles, we find that the condition (12) applies to the momentum of each photon. Hence the range of integration of each photon is the same. In this way we can carry out the summation over all diagrams of the type of Fig. 1b with any number of virtual lines, using the method of Ref. 1. The result differs from the zero-order approximation by an exponential factor, the exponent being the expression in square brackets in Eq. (13). Hence the cross section for the single Compton-effect (meaning that only one photon is emitted with $\omega > m$) at high energies is

$$d\sigma = d\sigma_0 \exp\left[-\frac{e^2}{\pi}\left(\frac{3}{2}\ln^2\frac{\omega_1}{m} - \frac{1}{2}\ln^2\frac{\omega_1}{\omega_2}\right)\right], \tag{14}$$

where $d\sigma_0$ is the cross section in zero-order approximation

$$d\sigma_0 = (\pi e^4/m\omega_1)\, d\omega_2/\omega_2. \tag{15}$$

We have used the conditions (10) to replace the procedure of introducing fictitious values for $(p_1^2 - m^2)$ and $(p_2^2 - m^2)$ and calculating consistently the radiation of real photons with energy less than m in the electron rest system. We now investigate the radiation of secondary photons of higher energy. First we look at the double Compton effect in zero-order approximation. We can insert into the generalized Feynman diagram[1] three kinds of lines representing real photons. Lines connecting equal electron momenta, as we saw earlier,[1] give singly-logarithmic contributions when $p^2 - m^2 \ll m^2$ and otherwise give no logarithmic contributions at all. If a real photon line connects an electron line with momentum p_1 to a line with momentum $(p_1 - l_2)$, then the integral is the same as it would be for a virtual photon line overlapping one vertex, and this does not give doubly-logarithmic contributions either. A doubly logarithmic integral arises only when a real photon line connects electron lines with momenta p_1 and p_2, just as in the case of a virtual photon. The momenta of real photons satisfy the conditions

$$(p_1 k) \ll (p_1 p_2), \quad (p_2 k) \ll (p_1 p_2), \tag{16}$$

$$a(p_1k) > (p_2k) > (p_1k)/a, \qquad (17)$$

with $a = (p_2 p_2)/m^2$. In terms of the variables λ and μ this means $\lambda,\mu > 0$, $\lambda - \ln a < \mu < \lambda + \ln a$. The condition $\omega > m$ gives $\lambda < \ln a$. These conditions on the ranges of integration for real photons are the same as for virtual photons.

It remains to see whether a condition analogous to equation (12) will exist. This condition arose from the electron line with momentum $(p_1 - l_2)$. In the formula for the transition probability, these lines give a factor $[(p_1 l_2)(p_2 l_1)]^{-1}$. We now have $(p_2 l_2) \neq (p_1 l_2)$ because the conservation of momentum gives

$$p_1 + l_1 = p_2 + l_2 + k, \qquad (18)$$

instead of equation (1). A detailed analysis of the new denominator, which we shall not reproduce here, shows that the condition (12) is the same for real as it was for virtual photons. Hence the ranges of integration for real and virtual photons are identical.

It is now wasy to write down the cross section for the n-fold Compton effect. As in Ref. 1, it follows a Poisson distribution

$$d\sigma = d\sigma_0 e^{-\bar{n}} (\bar{n})^n / n!, \qquad (19)$$

in which the average multiplicity $\bar{n}$ is given by

$$\bar{n} = \frac{e^2}{\pi}\left(\frac{3}{2}\ln^2\frac{\omega_1}{m} - \ln^2\frac{\omega_1}{\omega_2}\right). \qquad (20)$$

2. PROPERTIES OF THE SECONDARY PHOTONS

We now examine the physical meaning of the results we have obtained. The important range of integration for the momenta of both virtual and real photons is defined by the conditions (16), (17) and (12). We write condition (12) in the form

$$(p_1k)(p_2k) \ll (p_1p_2)(p_1l_2). \qquad (21)$$

Within the important range, the cross section for double Compton effect in the zero-order approximation has the form

$$d\sigma = d\sigma_0 \frac{e^2}{\pi}\frac{du}{u}\frac{dv}{v}. \qquad (22)$$

We can express u and v in terms of the energy and angle of the photon in the laboratory system,

$$u = \omega/\omega_1, \quad v = (\omega/m)(1 - \beta\cos\theta), \qquad (23)$$

where $\beta = \sqrt{E_2^2 - m^2}/E_2 \approx 1$
$- m^2/2E_2^2 \approx 1 - m^2/2\omega_1^2$

is the velocity of the outgoing electron. Then Eq. (22) takes the form

$$d\sigma = d\sigma_0 \frac{2e^2}{\pi}\frac{d\theta}{\theta}\frac{d\omega}{\omega}. \qquad (24)$$

Here we used the fact that $v \ll 1$, which implies that also $\theta \ll 1$.

Equation (24) coincides with the result of a purely classical treatment. This becomes understandable if we write conditions (16) and (17), which limit the important range of integration, in the center of mass system. Equation (16) then implies

$$\omega_{cm} \ll \omega_{1\,cm}, \qquad (16')$$

which means that the secondary photon has a much smaller energy than the primary particles. Equation (21) gives

$$\omega_{cm} \sin\theta_{cm} \ll \omega_{2\,cm}\theta_{2\,cm}. \qquad (21')$$

This means that the transverse momentum of the secondary photon is much smaller than the transverse momentum of the primary. This is not a consequence of (16′); in fact $(p_1 l_2) \ll (p_1 l_1)$ implies $\theta_{2CM} \ll 1$, and so the transverse momentum of photon l_2 is much smaller than the longitudinal momentum.

Equations (16′) and (21′) show that in the important range of values of the momentum of the secondary photon, the radiation of that photon has a negligible reaction on the momenta of the primary particles, and so the secondary radiation can be treated classically. The same circumstance explains the fact which we discovered earlier, that the radiation of successive secondary photons is statistically independent. The probability that a certain number of successive independent events will occur follows a Poisson distribution, as we have found by actual calculation. Thus quantum electrodynamics here leads to the result that the important range of integration is the range where classical theory is valid. A correct treatment of the virtual processes leads to a Poisson distribution. Knowing this result, we can use the classical theory of radiation for practical calculations, the conditions for validity of classical methods being easily stated in the center of mass system.

We now collect some formulas which describe the secondary radiation. We have seen that the radiation of different secondary photons is independent. Therefore the coefficient of $d\sigma_0$ in Eq. (24) gives the distribution of secondary photons in energy and angle in the laboratory system. Conditions (16), (17) and (21) give the following limits to the distribution in the laboratory system,

$$\theta \gg m/\omega_1, \quad (25)$$

$$\theta \ll \sqrt{m/\omega} \quad \text{for} \quad \omega < \omega_2, \quad (26)$$

$$\theta \ll \sqrt{m\omega_2/\omega^2} \quad \text{for} \quad \omega > \omega_2, \quad (27)$$

$$m \ll \omega \ll \omega_1. \quad (28)$$

If we are interested only in the energy distribution, then Eq. (24) should be integrated over the range of angles defined by Eqs. (25)-(28). The result is

$$dn(\omega) = \frac{2e^2}{\pi}\frac{d\omega}{\omega}\ln\frac{\omega_1}{\sqrt{m\omega}} \quad \text{for} \quad \omega < \omega_2,$$

$$dn(\omega) = \frac{2e^2}{\pi}\frac{d\omega}{\omega}\ln\frac{\omega_1}{\omega}\sqrt{\frac{\omega_2}{m}} \quad \text{for} \quad \omega > \omega_2. \quad (29)$$

If we are only interested in the angular distribution of photons with energy greater than m, then Eq. (24) must be integrated with respect to ω. This gives

$$dn(\theta) = \frac{2e^2}{\pi}\frac{d\theta}{\theta}\ln\frac{\omega_1}{m} \quad \text{for} \quad \frac{m}{\omega_1} < \theta < \frac{\sqrt{m\omega_2}}{\omega_1},$$

$$dn(\theta) = \frac{2e^2}{\pi}\frac{d\theta}{\theta}\ln\frac{1}{\theta}\sqrt{\frac{\omega_2}{m}} \quad (30)$$

$$\text{for} \quad \frac{\sqrt{m\omega_2}}{\omega_1} < \theta < \sqrt{\frac{m}{\omega_2}},$$

$$dn(\theta) = \frac{2e^2}{\pi}\frac{d\theta}{\theta}\ln\frac{1}{\theta^2} \quad \text{for} \quad \theta > \sqrt{\frac{m}{\omega_2}}.$$

The average number of secondary photons for given ω_2 is obtained by integrating Eq. (29) or (30) with respect to ω or θ, respectively. The result agrees with Eq. (20).

We consider next the following question. From Eq. (29) we see that, although in the center-of-mass system the secondary photons are much softer than the primary outgoing photon; in the laboratory system the reverse may be true. We shall calculate the probability for a given energy loss of the outgoing electron, compared with the primary photon energy. We denote this energy loss of the electron by ϵ; it is equal to the sum of the energies of the outgoing photons. Since the integrals over the photon energies are always logarithmic, we may suppose that one of the outgoing photons carries a much larger energy than the others. So we have to deal with an integral over the energies and angles of outgoing photons, for a given value of the energy of the most energetic among them.

In the expression for the cross section there are two terms. One gives the probability that all the secondary photons have energy less than the primary outgoing photon. This is obtained by multiplying the zero-order cross section by the probability for the absence of secondary photons of energy greater than ω_2. From the Poisson law it follows that this probability is $e^{-\bar{n}}$, where $\bar{n}$ is the average number of secondary photons of energy greater than ω_2. To obtain $\bar{n}$ we integrate Eq. (29) with respect to ω from ω_2 to ω_1. This gives

$$d\sigma' = d\sigma_0 \exp\left(-\frac{e^2}{\pi}\ln\frac{\omega_1}{m}\ln\frac{\omega_1}{\omega_2}\right). \quad (31)$$

In the first term of the cross section, the energy removed by photons is the energy of the primary outgoing photon, so that we put $\omega_2 = \epsilon$ in Eq. (31).

The second term in the cross section gives the probability that one of the secondary photons has energy greater than ω_2. Hence ϵ is the energy of the hardest secondary photon. The probability that a secondary photon has energy in the interval $d\epsilon$ is

$$\frac{2e^2}{\pi}\frac{d\varepsilon}{\varepsilon}\ln\frac{\omega_1}{\varepsilon}\sqrt{\frac{\omega_2}{m}}.$$

For this to be the hardest photon it is necessary that photons of higher energy be absent. The corresponding probability is $e^{-\bar{n}}$, where $\bar{n}$ is obtained by integrating Eq. (29) with respect to ω from ϵ to ω_1, thus

$$\left(\bar{n} = \frac{e^2}{\pi}\ln\frac{\omega_1\omega_2}{\varepsilon m}\ln\frac{\omega_1}{\varepsilon}\right).$$

Since the primary outgoing photon has energy small compared with ϵ, we are not interested in the distribution of its energy and we integrate the cross section at once with respect to ω_2 from m to ϵ. The final expression for the cross section is obtained by adding this integral to equation (31).

We calculate the probability $dw(\epsilon)$ that the energy loss of the electron will lie in the interval $d\epsilon$. This is obtained by dividing $d\sigma$ by the total cross section, which is in turn the integral of $d\sigma$ with respect to ϵ from m to ω_1. The total cross section is equal to the zero-order expression for the Compton scattering

$$\sigma = (\pi e^4/m\omega_1)\ln(\omega_1/m). \quad (32)$$

This must be so, because the Poisson distribution is normalized to unity. The probability for an energy loss in the interval $d\epsilon$ is then

$$dw(\varepsilon) = \frac{1}{\ln(\omega_1/m)} \frac{d\varepsilon/\varepsilon}{(e^2/\pi)\ln^2(\omega_1 \varepsilon)} \Big[\Big(\frac{2e^2}{\pi}\ln^2\frac{\omega_1}{\varepsilon}+1\Big) \times \exp\Big(-\frac{e^2}{\pi}\ln^2\frac{\omega_1}{\varepsilon}\Big) + \Big(\frac{e^2}{\pi}\ln\frac{\omega_1}{\varepsilon}\ln\frac{\omega_1}{m}+1\Big) \exp\Big(-\frac{e^2}{\pi}\ln\frac{\omega_1}{\varepsilon}\ln\frac{\omega_1}{m}\Big)\Big]. \quad (33)$$

Equations (19), (20), (29), (30) apply unchanged to the process of multiple-photon annihilation of an energetic positron with an electron (again we consider only photons with energy greater than m). The diagram for this process is identical with the Compton-effect diagram. The difference is only that the most energetic particle is the positron instead of the incident photon; hence, we should replace ω_1 by the position energy E in the final results. Also, for the annihilation process case II cannot occur, and so the calculations we have made for case I apply directly to the positron cross section.

Our calculations of multiple radiation, although immediately applicable only to the two specific processes considered, contain in fact many features which are common to all processes involving secondary radiation. The two most important features are the classical behavior of the radiation, and the doubly-logarithmic order of magnitude of the radiation probability.

It is of interest to discuss the various practical conditions under which one of these processes might be observed. For definiteness we shall consider the multiple-photon annihilation of a positron. This process was studied by Gupta,[3] who suggested it as an explanation of the famous photon shower of Schein.[4] Gupta's preliminary calculation led to a completely incorrect result

$$d\sigma^{(n+2)} = d\sigma^{(2)}\Big(\frac{e^2}{\pi}\ln^2\frac{E}{m}\Big)^n,$$

quite different from Eq. (19). Gupta concluded that multiple-photon annihilation becomes highly probable at energies of the order of 10^{14} ev, comparable with the total energy of the Schein event. In fact, however, Eq. (20) shows that in the most favorable case 10^{-13} ev are necessary for the production of a single secondary photon. To produce on the average 20 photons, the number observed in the Schein event, we should need at least 10^{38} ev, an energy far beyond the range of validity of contemporary electrodynamics. In any case the positron spectrum in cosmic rays falls rapidly with energy, and particles of such high energy are certainly not present. Such a star might arise more easily from a multiple-photon fluctuation in the annihilation of a positron of much lower energy. Suppose $E = 10^{14}$ ev. The average number of secondary photons for a positron of this energy is about 1. The probability for creating 20 photons is then 10^{-18} times the probability for creating one.

Our equations thus show that with positrons of practically attainable energy the number of observable secondary photons will be quite small. This number may become even smaller as a result of the material in which the process occurs. The material produces this effect in two ways; by its refractive index, and by multiple scattering.

We consider first the effect of refractive index. The importance of this effect was pointed out by Ter-Mikaelyan.[5] It works as follows. In Eq (20), strictly speaking, there stands $[d\cos\theta(-\beta\cos\theta)]$, which for angles in the range $1 \gg \theta \gg (1-\beta)^{1/2}$ can be approximated by $[2d\theta/\theta]$. Here we took β to be equal to its vacuum value $\beta_0 = (E^2-m^2)^{1/2}/E$. If the material has a refractive index n, then the true value of β is $\beta_0 n$. For photons of high energy

$$n^2 = 1 - (4\pi e^2 N/m\omega^2),$$

where N is the number of electrons per cubic centimeter. So we obtain in addition to Eq. (25) a new lower bound for θ,

$$\theta \gg \omega^{-1}\sqrt{4\pi e^2 N/m}. \quad (34)$$

The second effect of the material was noticed by Landau and Pomeranchuk.[6] It takes place as follows. Multiple radiation is a process extended over a certain time τ. During this time the radiating particle is multiply scattered, and this changes the probability of radiation. The result is another condition of the form $\theta \gg \theta_p$ limiting the range in which the cross section behaves logarithmically.

The mean scattering angle θ_p is given by[7]

$$\overline{\theta_p^2} = xE_s^2/E^2\beta^2,$$

where

$$E_s = m\sqrt{4\pi/e^2} = 21\ \text{eV},$$

and x is the distance in radiation lengths. Since $\beta = 1$, and the time entering into the process is of the order of magnitude

$$\tau \sim (\omega - kv)^{-1} \approx 2/\omega\theta^2,$$

the lower bound for the angle becomes

$$\theta \gg (2E_s^2/E^2X_0\omega)^{1/4}, \quad (35)$$

where X_0 is the radiation length of the material. In Fig. 3 the area of integration is shown in terms of the variables $\ln(\omega/m)$ and $\ln(l/\theta)$. In the lower right corner are lines representing the conditions (34) and (35). They are given by the equations

$$\ln\frac{1}{\theta_1} = \ln\frac{\omega}{m} + a, \quad \ln\frac{1}{\theta_2} = \frac{1}{4}\ln\frac{\omega}{m} + b,$$

$$a = \frac{1}{2}\ln\frac{m^3}{4\pi e^2 N}, \quad b = \frac{1}{4}\ln\frac{mX_0E^2}{2E_s^2}.$$

These lines cut a certain part out of the shaded area. The quantity $\bar{n}$ is proportional to the remaining area.

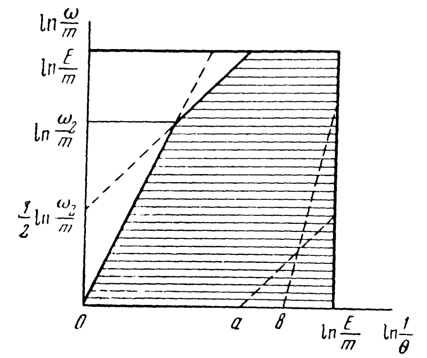

FIG. 3

The numerical value of the reduction in the area depends on the energy of the process and on the material. Thus, for the annihilation of a positron of energy $E=10^{14}$ ev in air, the most favorable case is $\omega_2=\omega_1$. In this case condition (34) removes 0.057 of the area, and condition (35) removes 0.066. The two conditions together reduce $\bar{n}$ by 0.083 of its original value. The effect of conditions (34) and (35) can be seen more clearly by looking at the energy which corresponds to a given $\bar{n}$. If these conditions are ignored, then $E= 10^{14}$ ev corresponds to $\bar{n}=1.27$. When the conditions are taken into account, the same value of $\bar{n}$ corresponds to $E= 2.2\times10^{14}$ ev.

3. THE COMPTON EFFECT AT SMALL ANGLES

Although the region of small angles [case II or equation (5)] is unimportant for calculating the total cross section, we shall study it in detail because an interesting effect occurs in it. We now deal with momenta satisfying the inequality $(p_1 l_2) \sim (p_1 l_2) \gg (p_1 p_2)$. In this case both the Compton effect diagrams are of the same order of magnitude. We consider again an integral of the form of equation (8). In it there is a doubly-logarithmic region of the same kind as we found in Section 1, but without any limit on $|uv|$. The situation in this case looks at first glance identical with the situation in the scattering problem.[1]

However, this is not the only way to obtain a doubly-logarithmic integral. We make the substitution $k \to p_1 - k$. Then the integral takes the form

$$J = \frac{e^2}{\pi i}$$

$$\times \int \frac{\hat{p}_1\gamma_\mu\hat{k}\gamma_\tau(\hat{k}-\hat{l}_2)\gamma_\sigma(-\hat{p}+\hat{k})\gamma_\nu\hat{p}_2 d_{\mu\nu}(p_1-k)\,d^4k}{(k^2-m^2)[(k-l_2)^3-m^2][(k-p)^2-m^2](k-p_1)^2}.$$

This integral can give a doubly-logarithmic contribution only when a factor k^2 appears in the denominator to cancel the small quantity $\hat{k}$ in the numerator. We resolve $k_{\parallel}$ into components along the small vectors l_2 and p_1; thus $k = p_1 u + l_2 v + k_\perp$. An extra power of k can come only from the factor ($[(k-p)^2 - m^2]$

We now examine this term in detail. If $u, v \ll 1$, it becomes

$$(k-p)^2 - m^2$$
$$= k^2 - m^2 + p^2 - 2k_\perp p_\perp \cos\varphi, \quad (36)$$

and averaging over φ gives

$$\overline{[(k-p)^2 - m^2]^{-1}}$$
$$= [(p^2 + k^2 - m^2)^2 - 4x' p^2|]^{-1/2}.$$

To obtain a factor k^2 out of this, we must have

$$|uv| \gg |p^2|/(p_1 l_2). \quad (37)$$

The situation here is just the opposite of what happened in the preceding paragraphs. In the x-integration the important contribution is the residue of the pole at $k^2 = m^2$. In order that $x > 0$ at the pole, we must have $(p_1 l_2)\,uv > m^2$. Together with Eq. (37), this condition may be written in the form

$$|uv| \gg (p_1 p_2)/(p_1 l_2). \quad (38)$$

We now return to Eq. (36). If the ϕ-integration is carried out in the complex plane, the main contribution comes from the residue of the pole at which $[(p-k)^2 - m^2]$ vanishes. In the subsequent integration we set $k^2 = m^2$ and $|p^2| << x$, so that the pole occurs at a real value of ϕ. Thus we come to the conclusion that the values of k which are important in the integral are those at which both electron propagators have poles.

Since the small-angle scattering is almost coherent, we may assume the vectors p_1 and p_2 in the laboratory system to be small compared with l_1 and l_2. Consequently, the second factor k in the numerator can only come from the term $(-\hat{p} + \hat{k})$. The factors $\hat{k} \ldots \hat{k}$ can then be replaced by $\hat{k}_\perp \ldots \hat{k}_\perp$, since the remainder is an expression with uncompensated small factors in the numerator. For the same reason, when we average over the direction of k, it is sufficient to keep only the term $-1/2\,(\gamma_\mu \ldots \gamma_\mu)\,x$. After this we carry out the calculation in exactly the same way as before, and obtain

$$J = \frac{e^2}{4\pi} \frac{\hat{p}_1 \gamma_\tau (-\hat{l}_2) \gamma_\sigma \hat{p}_2}{-2(p_1 l_2)} \ln^2 \frac{(p_1 l_2)}{(p_1 p_2)}. \quad (39)$$

Next we look at the matrix element corresponding to a diagram with two virtual photon lines. Suppose the two lines do not cross. Then the denominator is

$$[(p_1 - k_1)^2 - m^2]\,[(p_1 - k_1 - k_2)^2 - m^2]$$
$$\times\ [(p_1 - l_2 - k_1 - k_2)^2 - m^2]$$
$$\times\ [(p_2 - k_1 - k_2)^2 - m^2]$$
$$\times\ [(p_2 - k_1)^2 - m^2]\, k_2^2 k_1^2.$$

We make the substitution

$p_1 - k_1 - k_2 \to k_2,\ \ p_1 - k_1 \to k_1$. This gives

$$(k_1^2 - m^2)\,(k_2^2 - m^2)$$
$$\times\ [(k_2 - l_2)^2 - m^2]\,[(k_2 - p)^2 - m^2]$$
$$\times\ [(k_1 - p)^2 - m^2]\,(k_1 - k_2)^2 (k_1 - p_1)^2.$$

The integral over k_2 is now of exactly the same kind as the one considered previously. We first resolve $k_{2\parallel}$ into components along k_1 and l_2, and then find the k_1-integration has also the same form as before. But when the two photon lines cross, this does not happen.

We now examine various other arrangements of virtual lines which might give doubly-logarithmic terms. First we remark that the kind of doubly-logarithmic term obtained above is unique and does not arise from other types of diagram. However, the possibilities for obtaining infra-red integrals (with $k^2 = 0$) are greatly increased by inserting electron lines with $p^2 = m$. Suppose there is a diagram with one virtual photon line, giving an integral of the kind considered earlier (in Fig. 4 the heavy lines form such a diagram.) If we now add virtual lines with $k^2 = 0$, as shown in Fig. 4*a* by the lines labelled *1* and *2*, the leading terms will contain factors of the form $\ln\,[m^2/(k^2 - m^2)]$. But in the subsequent integration these lines do not give any contribution of the required order. Thus, if we begin with the diagram formed by the heavy lines, the addition of lines with $k^2 = 0$, either lying between the heavy lines or closer to the original vertices, does not give any new doubly-logarithmic terms.

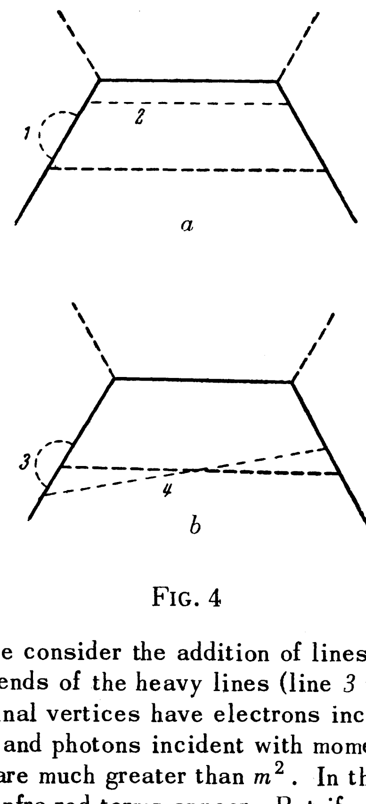

FIG. 4

Next we consider the addition of lines overlapping the ends of the heavy lines (line *3* in Fig. 4*b*). The original vertices have electrons incident with $p^2 = m^2$, and photons incident with momenta whose squares are much greater than m^2. In these circumstances infra-red terms appear. But if we add a line with $k^2 = 0$ intersecting the heavy line (line *4* in Fig. 4*b*), then it almost exactly cancels the infra-red term from the line *3* enclosing the vertex. In fact it is easy to verify that line *4* has no effect on the heavy line. The integral arising from line *4* is very similar to the integral arising from line *3*, only it has the opposite sign. In Ref. 1 we proved that integrals from infra-red virtual lines are proportional to the area of the shaded strip in Fig. 1 of Ref. 1. When we add the contributions from lines *3* and *4*, the result is proportional to the difference between the areas of two strips with width ln (kp_1/m^2) and ln $[p_1(k-p)/m^2]$. So the sum is proportional to

$$\ln\frac{(kp_1)}{(p_1(k-p))} = -\ln\left[1-\frac{p^2}{2(p_1k)}\right]^{-1} \approx \frac{p^2}{2(p_1k)} \ll 1.$$

This is small by virtue of Eq. (38) and the fact that $v \ll 1$.

We are therefore left only with infra-red virtual lines which follow after the heavy lines. Also, if we consider generalized diagrams and put in real photon lines, we must study the effects of real lines corresponding to virtual lines of the types *3* and *4* (lines *1* and *2* in Fig. 5). But these real lines give small terms, just like the virtual lines. Therefore there remain only the real and virtual lines of the kind shown in Fig. 5*b* (lines *3* and *3'*). These lines represent the radiation of photons with $(p_1k) \ll (p_1p_2)$, which means that the photon energies are small compared with the electron recoil energy, which is itself very small in this case. Such soft photons are not of interest, and we do not wish to set any limit to the number of them which may be radiated. This implies that we take into account only the integrals arising from heavy lines.

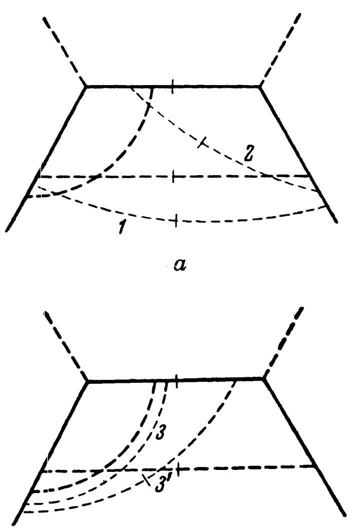

FIG. 5

Since the heavy lines do not cross, we can sum them by means of an integral equation. The equation

$$\hat p_1 M_{\tau\sigma}(p_1, p_2; p_1 - l_2)\hat p_2 = \frac{\hat p_1\gamma_\tau(\hat p_1-\hat l_2)\gamma_\sigma\hat p_2}{-2(p_1l_2)} + \frac{e^2}{\pi i}\hat p_1\int\gamma_\mu\frac{1}{\hat k-m}M_{\tau\sigma}(k, k-p; k-l_2) \times \frac{1}{\hat k-\hat p-m}\gamma_\mu\frac{d^4k}{(k-p_1)^2}\hat p_2 \quad (40)$$

is satisfied by the sum of all matrix elements of the form (10). Neglecting some small terms, we look for a solution $M_{\tau\sigma}$ of the form

$$M_{\tau\sigma}(p_1, p_2; p_1 - l_2) = \frac{\gamma_\tau\hat l_2\gamma_\sigma}{2(p_1l_2)} f\left(\frac{(p_1l_2)}{(p_1p_2)}\right). \quad (41)$$

We chose the argument of the function f so as to agree with equation (39). We can greatly simplify the equation for f, using the same method which we used to calculate the first approximation to $M_{\tau\sigma}$. We also introduce the logarithmic variables $(-\ln u) = \lambda$ and $(-\ln v) = \mu$. The range of integration is then a triangle bounded by the coordinate axes and by the line $\lambda + \mu = \alpha = \ln[(p_1l_1)/(p_1p_2)]$. The equation for f reduces to

$$f(\alpha) = 1 + \frac{e^2}{2\pi}\int^{\alpha} d\lambda\int_0^{\alpha-\lambda} d\mu f(\alpha-\lambda), \quad (42)$$

or

$$f(\alpha^2) = 1 + \frac{e^2}{4\pi}\int_0^{\alpha^2} f(x)\,dx.$$

The solution is $f = \exp\left\{\frac{e^2}{4\pi}\alpha^2\right\}$.

When we sum the matrix elements of the other class, in which the photon l_1 is absorbed first and the photon l_2 emitted afterwards, the result differs from that obtained above only in the spinor factor. Since $(p_1l_2) \sim (p_1l_2)$, the function f is the same. Hence the cross section is given by multiplying the usual expression by the square of the f-factor.

The result is of interest because, unlike the cases considered earlier, it shows a doubly-logarithmic factor which is not simply a normalization of the cross section to compensate for multiple processes. The deviation of the cross section with emission of any number of secondary photons from the zero-approximation cross section occurs only at small angles, so that the effect resembles a diffraction phenomenon. The small-angle angular distribution in the laboratory system is

$$d\sigma = \frac{2\pi e^4}{m^2}\,\theta\, d\theta \exp\left(\frac{e^2}{2\pi}\ln^2\frac{m}{\omega_1\theta^2}\right) \qquad (43)$$

$$\text{при } \sqrt{m/\omega_1} \gg \theta \gg m/\omega_1,$$

$$d\sigma = \frac{2\pi e^4}{m^2}\,\theta\, d\theta \exp\left(\frac{e^2}{2\pi}\ln^2\frac{\omega_1}{m}\right) \qquad (44)$$

$$\text{при } \theta \ll m/\omega_1.$$

The most important effect here is the change in the coherent scattering, for which the amplitude is multiplied by $\exp\left(\frac{e^2}{4\pi}\ln^2\frac{\omega_1}{m}\right)$. This gives the following formula for the refractive index at high frequencies:

$$1 - n^2 = \frac{4\pi N e^2}{m\omega^2}\exp\left(\frac{e^2}{4\pi}\ln^2\frac{\omega}{m}\right). \qquad (45)$$

In conclusion I take the opportunity to thank Academician L. D. Landau for his valuable advice.

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Translated by F. J. Dyson
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