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On the Singularity of the Electromagnetic Potential in the Higher Approximations of Perturbation Theory

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It is shown that the singularity at the origin in the potential of the electromagnetic interaction of two electrons (or of an electron with a positron), taken in any, arbitrarily high, approximation of perturbation theory, involving the nth power of the charge, is not of higher order than $(1/r)\ln^{(n-2)/2}r$.

IT is known that the potential of the electromagnetic interaction* of two spinor particles in the second approximation of perturbation theory behaves like $1/r$ (r is the three-dimensional distance between the particles) at the origin, which theoretically allows the possibility of bound stationary states of a system of two oppositely charged particles, as actually observed in nature (for example, the hydrogen atom, and also the recently discovered positronium). The question naturally arises as to the character of the singularity at the origin in the potential in higher approximations of the theory.

Calculations with nonrelativistic quantum electrodynamics give a strengthening of the singularity in the potential (to $1/r^3$) in the fourth approximation. Such a strong singularity of the potential is brought about, for example, by the term $e^2[A_\mu^2(\mathbf{x}_1) + A_\mu^2(\mathbf{x}_2)]$, which appears on the introduction of the electromagnetic field into the Schröedinger equation for two particles ($\mathbf{x}_1$ and $\mathbf{x}_2$ are the space coordinates of the particles). Calculations with relativistic nonquantum theory[1] give an analogous strengthening (to $1/r^3$) of the singularity of the potential in fourth approximation.

With regard to relativistic quantum electrodynamics, here, because of the well-known fundamental difficulties with infinities, up until recent years it has not appeared possible, without violating the consistency of the theory, to obtain the potential in an approximation higher than the first nonvanishing one (i.e., the second), and the question of the singularity of the potential in higher approximations has remained open. At the present time, owing to the appearance of consistent methods for the removal of infinities, relativistic quantum electrodynamics can to a certain extent be regarded as a completed, consistent theory, and the answer to this question would be of great interest. It it were to turn out that also in the relativistic quantum electrodynamics the singularity of the potential at the origin were essentially strengthened (becoming stronger than $1/r^2$) in higher approximation, as occurs in the nonrelativistic and nonquantum approximations, this would mean a new difficulty in principle, as the possibility of bound stationary states, which follows from the second approximation, would become fictitious--the theory would obviously be in contradiction with observed facts.

The purpose of the present paper is to show that the interaction potential of two spinor particles (for concreteness we shall call them electrons), calculated in relativistic quantum electrodynamics with known methods of removing infinities, has a singularity at the origin not higher than $1/r^{1+\epsilon}$ (with ϵ an arbitrarily small positive number), in any arbitrarily high (but of course finite) order of perturbation theory.

In this paper "natural" units are used ($\hbar = c = 1$). In addition, in applying the summation convention, the fourth term of a sum over a "dummy" index is taken with the minus sign.

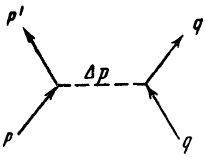
It is well known that the singularity at the origin in the interaction potential of two particles can be determined from the asymptotic behavior of the S-matrix for the scattering of the particles at high momenta. For this prupose it suffices to take into account in the calculation of the S-matrix only those of its elements that correspond to Feynman diagrams that form an irreducible representation (see Bethe and Salpeter[2]). Besides this, the S-matrix must be calculated without use of the law of conservation of energy for the initial and final states.

The essence of the proof consists of showing that in passing from the nth approximation, which is assumed reduced to finite form, to the next [obviously, the $(n + 2)$nd], after the removal of all infinities in the $(n + 2)$nd approximation, the asymptotic approach to zero of the S-matrix with

* Here and in what follows we shall have in mind the ordinary electrodynamics with the interaction part of the Hamiltonian not containing derivatives of the field operators.

increasing momenta is weakened by a factor not stronger than logarithmic.

If we denote the initial momenta of the electrons by p_μ and q_μ, and the final momenta by p'_μ and q'_μ, then the element of the S-matrix in the first nonvanishing (the second) approximation, corresponding (apart from "exchange") to the single diagram in this approximation



(I)

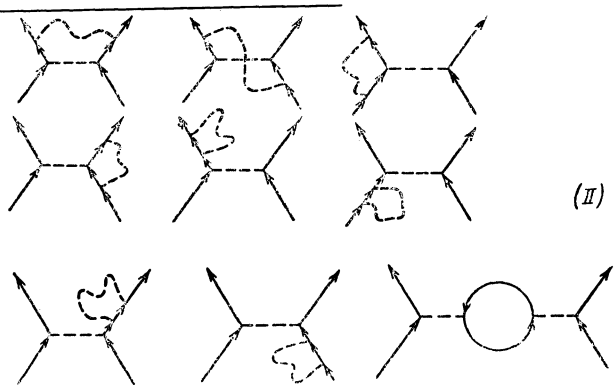
is given by

$$S_2 = -\frac{(2\pi)^4 ie^2}{2} \qquad (1)$$

$$\times \frac{1}{(\Delta p_\mu)^2} \cdot \Psi^+_{p'}\gamma_\mu\Psi_p \cdot \Psi^+_{q'}\gamma_\nu\Psi_q \cdot \delta(p_\nu + q_\nu - p'_\nu - q'_\nu).$$

Here $\Delta p_\mu \equiv p'_\mu - p_\mu$; Ψ_p, Ψ_q and so on denote the amplitudes of plane waves with the indicated momenta; $\Psi^+ \equiv \Psi^* \gamma_4$, where the sign * denotes the complex conjugate and transposed quantity. It follows from (1) that for $\Delta p_\mu \to \infty$, S_2 behaves like $1/(\Delta p_\mu)^2$, which corresponds to a potential with a $1/r$ singularity at the origin.

We shall show that the passage from any arbitrary approximation to the following one can be accomplished either by the "connecting" of any two electron lines (solid lines in the diagram), which may in particular be coincident lines, by a photon (dotted) line, or else by the "insertion" in any photon line of a closed electron loop (consisting of two electron lines). In this way one can immediately obtain from diagram (I) all the diagrams of the fourth approximation, of which only the first is reducible:



(II)

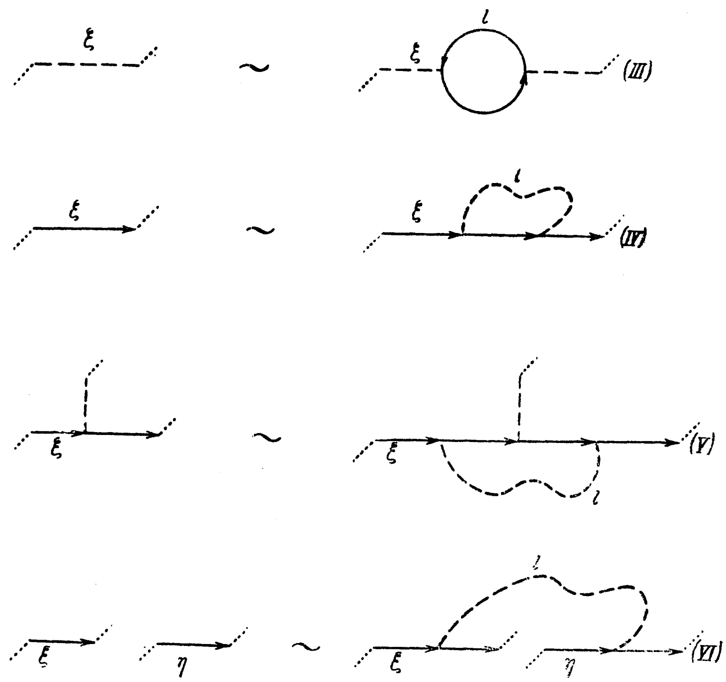
Attention must be drawn here to the following circumstance. It is well known that besides the divergences of virtual particles at high momenta, removable by the relativistically invariant method of renormalization and regularization, the S-matrix of electrodynamics also contains integrals over the momenta l_μ of virtual photons that diverge at zero (the "infrared catastrophe"). This divergence can be removed in two well-known ways: 1) "cutting off" near zero in the l-space, or 2) assigning to the photon a certain finite rest mass*. It is true that, even if we omit consideration of the ambiguity of these methods, they cannot be regarded as fully consistent in view of the fact that the first method destroys the relativistic invariance of the S-matrix and the second destroys the gauge invariance. The fact is, however, that the difficulty with the infrared catastrophe is not one of principle and, as is well known, is brought about by the illegitimacy of the expansion of the S-matrix in the series of perturbation theory. The exactly calculated S-matrix is free from this difficulty.

In what follows it will be assumed that all occurring integrals over virtual momenta have, if divergent at zero, been brought to finite form by one of the known methods; we shall suppose, however, that no inconsistency in principle is thus introduced.

We consider the following types of passages from the nth approximation to the $(n + 2)$nd, shown in

* It must be noted that the possibility of removing the infrared catatrophe by the introduction of real photons is excluded by the very statement of our problem.

diagrams (*III*)-(*VII*) [or to the $(n+4)$th, in case (*VIII*)].



The diagonally directed series of dots represent certain entirely arbitrary parts of the diagram. In case (*VI*) it is assumed that 1) if the ξ and η lines belong to one succession of electron lines, then between the ends of the line l (along the succession of electron lines) there are at least two points, and 2) if one (or both) of the lines ξ and η belongs to a closed loop of electron lines, then in the left-hand part of (*VI*) this loop must consist of not fewer than four (electron)lines.

The necessity of special consideration of cases (*VII*) and (*VIII*), passages from closed loops of two electron lines directly to loops of four lines, is occasioned by our wish to exclude from consideration diagrams with loops of three electron lines, because of the unremovable infinities appearing in the element of the S-matrix corresponding to each such diagram. The exclusion of these cases is, of course, permissible, as by Furry's theorem[3] the sum of the elements of the S-matrix corresponding to all such diagrams is equal to zero.

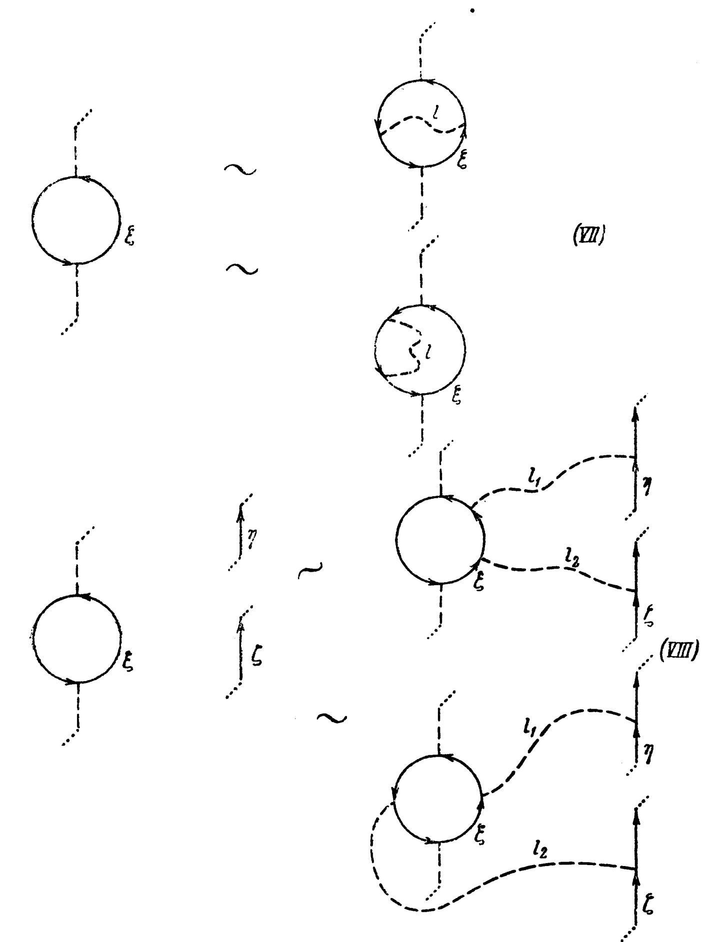
It is obvious that by means of alterations (*III*)-(*VIII*) one can obtain from the collection of diagrams of any arbitrary approximation all possible diagrams (with nonvanishing total contributions) of any higher approximation.

We note that ξ_μ, η_μ and ζ_μ are algebraic sums of some number (which may, in particular, be zero) of virtual momenta l_μ, over which integration is to be performed, and one of the terms $-\Delta p_\mu$, $\Delta p_\mu - p_\mu$, $\Delta p_\mu - q_\mu$, p_μ, q_μ (or of virtual momenta only), i.e., symbolically

$$\xi \equiv \xi(p,\ l),\ \ \eta \equiv \eta(p,\ l),\ \ \zeta \equiv \zeta(p,\ l).$$

In the passage from the nth approximation, assumed reduced to finite form, to the $(n+2)$nd, or in case (*VIII*) to the $(n+4)$th, there occurs in the integrand of the expression for the S-matrix a replacement of a factor $M_i(\xi_\mu)$* by another: $M_i(\xi_\mu) \to M_i'(\xi_\mu)$; the latter is in general infinite, but reducible to finite form: $M_i'(\xi_\mu) \to M_i''(\xi_\mu)$. Comparison of these two factors M_i and M_i'' makes possible a decision about the change of asymptotic behavior of the S-matrix with increasing order of approximation. First of all we can convince ourselves that in all of cases (*III*)-(*VII*) the asymptotic behavior of $M_i''(\xi_\mu)$ for $\xi_\mu \to \infty$ can differ from that of $M_i(\xi_\mu)$ by nothing stronger than a

* For case (*VI*) $M_i(\xi_\mu, \eta_\mu)$, and for case (*VIII*) $M_i(\xi_\mu, \eta_\mu, \zeta_\mu)$, respectively.



logarithmic factor, and in case (*VIII*) by nothing stronger than its square, i.e.,

$$M_i''(\xi_\mu) \underset{\xi_\mu\to\infty}{\sim} M_i(\xi_\mu)\cdot\Lambda_i(\xi_\mu), \tag{2}$$

where for $\xi_\mu \to \infty$, $\Lambda_i(\xi_\mu)$ approaches infinity in cases (*III*)-(*VII*) no more strongly than $\ln \xi_\mu^2$, and in case (*VIII*) not more strongly than $\ln^2 \xi_\mu^2$. (Incidentally, as is easily seen, there follows from this the full renormalizability of electrodynamics.)

We present as an example the factors M_1 and M_1'' for case (*III*):

$$M_1=\delta_{\mu\nu};\quad M_1''=\frac{e^2}{4(2\pi)^2}(\xi_\mu\xi_\nu-\xi^2\delta_{\mu\nu}) \times\int_0^1 dx\,\frac{x^2(1-{}^1/_3x^2)}{m^2+(\xi^2/_4)(1-x^2)}; \tag{3}$$

$$\underset{\xi_\mu\to\infty}{M_1\sim\text{const}};\quad \underset{\xi_\mu\to\infty}{M_1''\sim\text{const}\ln(\xi_\mu^2/m^2)}. \tag{4}$$

In deriving the asymptotic behavior of the factors M_i'' in the more complicated cases (*VI*) and (*VIII*) it is necessary to use the general formulas (for arbitrary integers l, n, k, m):

$$\frac{1}{a_1\ldots a_l}=(l-1)!\int_0^1 dx_1\ldots \times\int_0^{x_{l-2}} dx_{l-1}\cdot[a_1x_{l-1}+a_2(x_{l-2}-x_{l-1})+\ldots +a_l(1-x_1)]^{-l}; \tag{5}$$

$$\int_{(\infty)} d^4l\cdot\overbrace{l_\mu l_\xi\ldots l_\eta l_\nu}^{2n}\cdot f(l^2) = 4^{-n}\int_{(\infty)} d^4l\,(l_\mu)^{2n}f(l^2)\cdot\sum\overbrace{\delta_{\mu\xi}\ldots\delta_{\alpha\nu}}^{n} \tag{6}$$

[for half-integral n integral (6) is equal to zero],

$$\int_{(\infty)} d^4 l \; \frac{l^{2k}}{(l^2+\lambda)^m} \tag{7}$$

$$= \frac{\pi^2 i}{\lambda^{m-k-2}} \Big[\frac{1}{(m-k-2)\,(m-k-1)}$$

$$- C_1^k \frac{1}{(m-k-1)(m-k)}$$

$$(m-k-2 \geqslant 1).$$

$$+ \ldots + (-1)^k \frac{1}{(m-2)(m-1)}\Big].$$

The summation in (6) is taken over all permutations of the indices μ, ξ, . . . , α, ν. Equation (6) can be proved from general considerations of relativistic invariance.

Furthermore, on the basis of (2) it is not difficult to prove the following assertion: if the integral

$$S_n = \int_{(l)} f[l, p, \xi(p, l)]\, M[\xi(p, l)] \quad \text{for} \quad p_\mu \to \infty$$

approaches zero (just this approach to zero is essential) like some function $J_n(p_\mu)$, then

$$S_{\substack{n+2\\n+4}} = \int_{(l)} f[l, p, \xi(p, l)]\, M''[\xi(p, l)] \;\text{for}\, p_\mu \to \infty$$

will approach zero in cases (*III*)-(*VII*) no more weakly than $J_n(p_\mu)\ln(p_\mu)^2$ and in case (*VIII*) no more weakly than $J_n(p_\mu)\ln^2(p_\mu)^2$. In other words, in the passage from the *n*th approximation to the $(n+2)$nd the asymptotic approach to zero of the scattering *S*-matrix for $p_\mu \to \infty$ is weakened by nothing stronger than a logarithmic factor; and this means that the singularity at the origin in the (electromagnetic) interaction potential of two electrons, and consequently, of the system electron-positron, in any arbitrarily high *n*th (in powers of the charge) approximation of perturbation theory cannot be stronger than*

$$(1/r)\ln^{(n-2)/2} r.$$

Furthermore, it is not hard to see that in nonrelativistic approximation for $p_\mu \to \infty$ the main term of the *S*-matrix in an arbitrary approximation will be, generally speaking, of the nature of a constant, which corresponds to a potential with a singularity of the form r^{-3}. But this need not appear strange, as a consideration of the interaction potential at the origin (which is equivalent to a consideration of the scattering *S*-matrix at large momenta) in the nonrelativistic approximation (i.e., in the approximation of small momenta) is internally self-contradictory.

In conclusion, the writer expresses his gratitude to M. A. Markov for suggesting the problem and for his aid in the work.

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* This result of course does not mean that the asymptotic behavior of the sum of the whole infinite series of perturbation theory will differ from $1/r$ by only a (finite) power of the logarithm of r. A discussion of the question arising in this connection, and also of the question in general of the convergence of the whole series of perturbation theory, which has given rise to doubts, particularly for large momenta (i.e., for small r), is beyond the scope of this paper.

[1] Zh. S. Takibaev, Dissertation, Phys. Inst. Acad. Sci. USSR, 1952.

[2] E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951).

[3] W. H. Furry, Phys. Rev. 51, 125 (1937).

Translated by W. H. Furry
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