Letters to the Editor

The Two-Dimensional Schrödinger Equation and Representations of the Group of Plane Motions

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WE have found the irreducible unitary infinite-dimensional representations of the group of plane motions. The infinitesimal operators of the representations, $H_+ = iA_1 - A_2$, $H_- = iA_1 + A_2$, $H_3 = iA_3$ satisfy the commutation relations

$$[H_+H_-] = 0; \ [H_+H_3] = -H_+, \ [H_-H_3] = H_-, \quad (1)$$

where

$$A_1 = -\partial/\partial x, \ A_2 = -\partial/\partial y, \ A_3 = -x\partial/\partial y + y\partial/\partial x.$$

In the infinite basis of the normalized eigen-vectors f_m of H_3 the operators H_+, H_-, H_3 are given by matrices of the following form:

$$H_+ = \left\| \begin{matrix} \ddots & & & \\ \ddots & 0 & & 0 \\ & \beta & 0 & \\ 0 & & \beta & \ddots \\ & & & \ddots \end{matrix} \right\| \quad (2)$$

$$H_- = \left\| \begin{matrix} \ddots & \ddots & & \\ & \beta & & 0 \\ & 0 & \ddots & \\ & & \beta & \\ 0 & & 0 & \ddots \\ & & & \ddots \end{matrix} \right\|$$

$$H_3 = \left\| \begin{matrix} \ddots & & & \\ & m-1 & & 0 \\ & & m & \\ & & & m-1 \\ 0 & & & & \ddots \end{matrix} \right\|$$

The weight β of the representation is given by an equation of second order

$$\Delta f_m^\beta + \beta^2 f_m^\beta = 0, \quad (3)$$

where Δ is the scalar operator of the group (Laplace operator) commuting with all operators of the representations.

We realize the representation we have obtained in function space. For this purpose we write H_+, H_-, and H_3 in polar coordinates:

$$H_+ = e^{i\varphi}\left\{-i\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\varphi}\right\};$$

$$H_- = e^{-i\varphi}\left\{-i\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial\varphi}\right\}; \ H_3 = -i\frac{\partial}{\partial\varphi}.$$

Then Eq. (2) takes the form:

$$\frac{\partial^2 f_m^\beta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f_m^\beta}{\partial\varphi^2} + \frac{1}{r}\frac{\partial f_m^\beta}{\partial r} + \beta^2 f_m^\beta = 0. \quad (4)$$

From Eqs. (2) and (4) it follows that $f_m^\beta(r,\varphi) = e^{im\varphi}R_m^\beta(r)$, where R_m^β satisfies Bessel's equation

$$\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) + \left(\beta^2 - \frac{m^2}{r^2}\right)\right] R_m^\beta(r) = 0. \quad (5)$$

From the form of the matrices (2) one can obtain the well-known recurrence relations for Bessel functions[1].

Thus if the representations of the rotation group are realized in the space of spherical functions[2], then the representations of the group of plane motions are realized in the space of Bessel functions. It is easy to see that in the representations we have obtained maximal vectors are absent, in consequence of which the weight β of a representation is not connected with m. An analogous situation occurs in the theory of the representations of the Lorentz group, in the case of the unitary representations. In the latter case of the two numbers giving a representation, that one k_1 which for finite-dimensional representations gives the maximum value of the moment k is not connected with k in the infinite-dimensional case[3]. The problem of decomposing the Kronecker product $R_{\beta_1} \times R_{\beta_2}$ of irreducible representations of the group of plane motions into irreducible representations is solved by the reduction to diagonal form of the operator Δ, written in the basis $f_{m_1} g_{m_2}$[4].

The results obtained are of interest in the investigation of the isotropy of a quantum-mechnical Hamiltonian in the plane

$$\hat{H} = -(\hbar^2/2\mu)(\partial^2/\partial x^2 + \partial^2/\partial y^2) + U(x, y).$$

The condition of isotropy can be written in the form $[A_i, H] = 0$. We note that the components of the quantum-mechanical angular momentum coincide with A_i for Z = const. Using Eqs. (2) and (4), one can carry out a classification of the quantum states of a two-dimensional system according to the representations of the group of plane motions, by expanding the Ψ-function in terms of Bessel functions. Since all of the representations we have found are irreducible, the quantum states of the system turn out to be nondegenerate. Thus β appears as a new quantum number, giving the state of a two-dimensional quantum-mechanical system.

In conclusion, we regard it as our pleasant duty to express gratitude to M. L. Tsetlin and F. A. Ermakov for fruitful discussions.

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Radiographic Study of X-Ray Photoelectric Emission

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A RADIOGRAPHIC method has been applied to the study of the dependence of x-ray photoelectric emission on the atomic number of an element. We used a variant of the "reflection" method[1,2] in the arrangement described below. Samples of various substances were carefully ground and polished. Then a few chosen samples were spread out on a glass plate and paraffin was poured over them. When the paraffin had cooled, the glass plate was removed. This procedure gave a block of samples, the upper surfaces of which were located in a single plane. This block was then pressed against the light-sensitive surface of a photographic plate, placed in an aluminum cassette and the entire system was placed in a beam of hard x-rays (~200 kv) in such a way that the x-rays passed through the photographic plate and fell on the surfaces of the samples which were pressed against the photographic emulsion. Due to the great penetrability of hard x-rays, short exposures do not produce appreciable blackening of the emulsion. On the other hand, the photoelectrons and the secondary electrons ("reflected") connected with them produce a significant blackening, which is especially noticeable when electron-sensitive photographic plates are used.

The blackening of the photographic plate was comparatored (on a microphotometer of type MF-2) at various places in the neighborhood of each of the impressed samples, and the previously measured background blackening was subtracted. Graphs indicated that the blackening of the photographic plate approached the straight line region of the blackening curve.

The following combinations of substances were studied:

1) Cr, Mn, Fe, Co, Ni, Cu, Zn, Ga, Ge, Se;
2) Mo, Pd, Ag, Cd, Sn, Sb;
3) Ta, W, Pt, Au, Hg, Pb, Bi;
4) Cu, Ag, Au;
5) Zn, Cd;
6) Si, Sn, Pb;
7) Cr, Se, Mo, W;
8) Ni, Pd, Pt.

1)-3) are "horizontal" groups, while 4)-8) are "vertical" groups. In other words, in the first case the block of specimens is made up of elements which increase in atomic number as one goes along a period of the Mendeleev chart, while in the second case the block is made up of elements from a single group of the periodic system. Some of the results of the measurements are given in the accompanying figures (the atomic number of the radiator-element is plotted as abscissa and the blackening, in relative units, is plotted as ordinate.

It is clear from Fig. 1 that the blackening, which characterizes the intensity of emission of photoelectrons and secondary electrons, increases almost linearly with increase in atomic number, as has already been pointed out in the literature[1,2]. Figure 2 shows that there is a sharp decrease in the intensity of the electron emission when one passes from the elements of the first group to those of the second. This result occurs for a transition from copper to zinc, from silver to cadmium, etc., that is, for elements such that