On the Effect of Radio Waves on the Properties of Plasma (Ionosphere)

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A successive method is developed for determining the electron-velocity distribution function in plasma in an alternating electric field and a constant magnetic field, assuming the collisions between the electrons and molecules or ions to be elastic. Expressions are derived and analyzed for the mean electron energies, for the conductivity and the dielectric constant of the plasma, and for the effective number of electron collisions.

1. INTRODUCTION

If an electromagnetic wave is propagated in the ionosphere, the intense alternating electric field of the wave may change substantially the energy of the electrons, and consequently the conductivity and the dielectric constant of the plasma. To evaluate these changes, one must first determine the electron-distribution function.

Considering only the elastic collisions between the electrons and the heavy particles (molecules or ions) of the plasma we know that with each collision the electron loses only a small portion of its energy [on the order of $\delta = 2m/M$, where m is the electron mass and M the molecule (or ion) mass]. It is therefore possible to obtain the distribution function in a simpler manner by representing the collision integral in the Boltzmann equation in differential form. Furthermore, since δ is small, the mean square velocity of the electrons is many times greater than the mean directed velocity even in strong electric fields. The symmetrical portion (which depends only on the modulus of the velocity) of the distribution function is correspondingly much greater than its asymmetric portion. Taking these into account, Davydov[2] simplified the kinetic equation considerably and solved it for the case of stationary electric and magnetic fields.

In nonstationary fields the electron energy is changed directly only by the electric field. Thus, depending on the ratio between the time t_E required to change substantially the electric field (for example, if $E = E_0 \cos \omega t$, $t_E \sim 1/\omega$), and the relaxation time of the electron energy ($t_r \approx 1/\delta\nu$, where ν is the collision frequency of the electron), it is possible to distinguish among cases in which the fields vary slowly ($t_E \gg t_r$) or rapidly ($t_E \ll t_r$). In the former (quasi-stationary) case the solution obtained in Ref. 1 for a time-dependent electric field intensity E is correct. In a rapidly-varying field, when $t_r \gg t_E$ the mean electron energy cannot change as fast as the electric field; the symmetrical portion of the distribution function should therefore be stationary, to a zero-order approximation. Let us also note that since the current relaxation time t_j is much smaller than the electron relaxation time t_r, because $t_j \sim 1/\nu \ll \delta\nu \sim t_r$, the ratio between t_E and t_j affects the nature of the distribution in rapidly-varying fields ($t_E \ll t_r$).

The case where the electric field varies periodically with time is discussed in Refs. 2 and 3 (in the absence of a magnetic field) and in Ref. 4 (assuming a constant magnetic field). However, owing to incorrect solution methods, some of the results obtained of References 3 and 4 are in error (cf. below).* An analogous problem was also solved in an analysis of the cross-modulation effect of radio waves in the ionosphere, where corrections were derived for the Maxwell distribution function for the case of a relatively weak alternating electric field, when the energy delivered by the field to the electrons is much less than their thermal energy (Refs. 6 and 7 and Sec. 64 of Ref. 8).

The purpose of this article is to solve the kinetic equations, assuming an alternating electric field and a constant magnetic field (without the above-mentioned restrictions on the field intensity), and to analyze the expressions for the conductivity and the dielectric constant of the plasma. Section 2 discusses the method for calculating the distribution function, really an expansion into a series in powers of the parameter t_E / t_r (for rapidly varying fields) or t_r / t_E (for slowly-varying fields). A zero-order approximation is obtained for the distribution function. In Sec. 3 we obtain and analyze expressions for the average electron energy and for the conductivity and dielectric constant of the plasma. In Sec. 4 we investigate the possibility of using in our calculations the values of ϵ and σ obtained from elementary-theory equations.

*The expression for the distribution function, derived in Ref. 4, was corrected by its authors in a later work[5] (see footnote 3).

We show that in the case of collisions with molecules the use of these equations leads to but a slight discrepancy with the results of the kinetic theory. Generally speaking, this discrepancy is more significant in the case of collisions with ions.

Estimates show that in many cases the incident wave may change substantially the average electron energy and the properties of the plasma.

2. CALCULATION OF DISTRIBUTION FUNCTION

Consider a spatially-homogeneous plasma located in an electric and magnetic field. We assume the collisions between the electrons and the molecules or ions of the gas to be elastic. The electron distribution function

$$f(\mathbf{v}, t) = f_0(v, t) + \frac{\mathbf{v}}{v}\mathbf{f}_1(v,t) + \chi(\mathbf{v}, t)$$

obeys in this case the following approximate system of equations, derived in Ref. 1:

$$\frac{\partial f_0}{\partial t} + \frac{e}{3mv^2}\frac{\partial}{\partial v}(v^2\mathbf{E}\mathbf{f}_1) - \frac{1}{v^2}\left\{\nu\frac{v^2kT}{M}\frac{\partial f_0}{\partial v} + \nu\frac{v^3m}{M}f_0\right\} = 0, \tag{1}$$

$$\frac{\partial \mathbf{f}_1}{\partial t} + \frac{e\mathbf{E}}{m}\frac{\partial f_0}{\partial v} + \frac{e}{mc}[\mathbf{H}\mathbf{f}_1] + \nu\mathbf{f}_1 = 0, \tag{1'}$$

wherein terms on the order of χ were neglected in the derivation of Eq. (1). (As shown in Ref. 1, $\chi \sim \sqrt{8}\,|f_1| \sim \delta f_0$). The normalization condition for the distribution function is

$$4\pi\int_0^\infty f_0(v, t)\,v^2\,dv = 1. \tag{2}$$

In Eq. (1) e is the electron charge, k the Boltzmann constant, T the plasma temperature, E and H the electric and magnetic field intensities, and $\nu(v)$ the frequency of collisions between the electron and the molecules (or ions). In the case of collisions with molecules, we have

$$\nu(v) = \pi a^2 N_M v, \tag{3}$$

where a is the "radius" of the molecule and N_M the molecule concentration; in the case of collisions with ions, on the other hand, we have (see Ref. 8)

$$\nu(v) = \frac{2\pi N_i e^4}{m^2v^3}\ln\left(1 + \frac{\rho_m^2 m^2}{e^4}v^4\right), \tag{3'}$$

where N_i is the ion concentration, ρ_m is the maximum collision parameter, which can be assumed to equal the Debye radius[8] (at frequencies that are not too high).

Let us now assume that the electric field varies rapidly, i.e., $t_E << t_r$. In this case the solution of Eq. (1) can be solved by successive approximation:

$$f_0(v, t) = f_{00} + f_{01} + \dots, \tag{4}$$

$$\mathbf{f}_1(v, t) = \mathbf{f}_{10} + \mathbf{f}_{11} + \dots$$

Neglecting in the zero-order approximation the variation of the distribution function due to the collisions (on the order of f/t_r) as compared with the first term of the equation ($\partial f/\partial t \sim f/t_E$), we have $\partial f_{00}/\partial t = 0$ or

$$f_{00} = f_{00}(v), \tag{5}$$

i.e., in the zero-order approximation the symmetrical portion of the distribution function is independent of time (see Introduction). On the other hand, the dependence of f_{00} on v is determined, as it should be, not by the equation for the zero-order approximation (4), but by the requirement that the boundary conditions for the next (first-order) approximation be satisfied (see below).

Inserting now (5) into (1′), we obtain f_{10} (the electric field is assumed for simplicity periodic, $E = E_0\cos\omega t$, and the magnetic field H is assumed constant):

$$\mathbf{f}_{10} = -\frac{eE_0}{2m}\frac{df_{00}}{dv}\left\{\frac{(i\omega+\nu)^2(\mathbf{E}_0/E_0) + \omega_H^2\cos\beta\,(\mathbf{H}/H) + \omega_H(i\omega+\nu)[\mathbf{H}\mathbf{E}_0]/HE_0}{(i\omega+\nu)[\omega_H^2 + (i\omega+\nu)^2]}e^{i\omega t}\right.$$
$$\left. + \frac{(-i\omega+\nu)^2(\mathbf{E}_0/E_0) + \omega_H^2\cos\beta\,(\mathbf{H}/H) + \omega_H(-i\omega+\nu)[\mathbf{H}\mathbf{E}_0]/HE_0}{(-i\omega+\nu)[\omega_H^2 + (-i\omega+\nu)^2]}e^{-i\omega t}\right\}, \tag{6}$$

where $\omega_H = eH/mc$ is the gyromagnetic frequency, and β the angle between H and E_0.

Next, inserting f_{00} and f_{10} into the equation for the first approximation f_{01} and integrating it with respect to time, we obtain in expression for f_{01} the following term

$$\Delta f_{01} = \frac{\delta}{2v^2}\frac{\partial}{\partial v}\left\{\nu v^2\left(\frac{e^2E_0^2}{3m^2\delta}\varphi(v)\frac{df_{00}}{dv} + \frac{kT}{m}\frac{df_{00}}{dv} + vf_{00}\right)\right\}t,$$

which increases without limit as $t\to\infty$. Stipulating the existence of the distribution function, i.e., the boundedness of f_{01} at $t\to\infty$, we set the expression in the braces equal to zero, and obtain

$$f_{00} = C\exp\left\{-\int_0^v \frac{mv\,dv}{kT + (e^2E_0^2/3m\delta)\varphi(v)}\right\}. \quad (7)$$

Here

$$\varphi(v) = \frac{\cos^2\beta}{\omega^2+\nu^2} + \frac{\sin^2\beta}{2}\left(\frac{1}{(\omega+\omega_H)^2+\nu^2} + \frac{1}{(\omega-\omega_H)^2+\nu^2}\right), \quad (7')$$

and the constant C is defined by the normalization condition (2). Thus a unique solution to the system of equations (1) is obtained only by satisfying the supplementary condition that the solution be bounded at $t\to\infty$.

Integrating the equations for f_{01} and f_{11} with respect to time, we can obtain the following approximation (with accuracy to within a time-independent function, which is determined by specifying that f_{02} be bounded at $t\to\infty$), etc. Comparing the successive terms of expansion (4) we find that in case of collisions with molecules the resultant expressions are correct provided the following condition is satisfied

$$\frac{1{,}5\,\delta\nu_0}{\omega}\left(1+\frac{eE_0}{\nu_0\sqrt{6kTm\delta}}\mu\right)^{1/2} \approx \frac{1{,}2\cdot10^{-14}N_M\sqrt{T}}{\omega}\left(1+1{,}4\cdot10^{16}\frac{E_0}{N_MT}\mu\right)^{1/2} \ll 1, \quad (8)$$

where $\nu_0 = \nu$ and $\mu \lesssim 1$, namely:

$$\mu = \begin{cases} \cos\beta + \sin^2\beta\,\dfrac{\sqrt{2}\nu_0eE_0}{\omega_H^2\sqrt{3kTm\delta}} & \text{if } \omega_H^2 \gg \nu_0^2 + \dfrac{\sqrt{2}eE_0\nu_0}{\sqrt{3kTm\delta}}, \\ 1 & \text{if } \omega_H^2 \ll \nu_0^2 + \dfrac{\sqrt{2}eE_0\nu_0}{\sqrt{3kTm\delta}}. \end{cases}$$

In the numerical coefficient we assumed $\delta = 3.4\times10^{-5}$ and $\pi a^2 = 4.3\times10^{-16}$ (see Ref. 8); E_0 is in millivolts per meter.

In case of slowly-varying fields, reversing the inequality sign in (8) we obtain for f_{00} and f_{10} the expressions similar to (6) or (7) but with E_0 being replaced by $\sqrt{2}E_0\cos\omega t$ in f_{00}. Thus we have

$$f_{00} = C(t)\exp\left\{-\int_0^v \frac{mv\,dv}{kT + (2e^2E_0^2/3m\delta)\varphi(v)\cos^2\omega t}\right\}, \quad (9)$$

where the constant $C(t)$ is determined by the normalization condition (2). For $\omega t\to 0$, i.e., for $E\to E_0$, the distribution function (9) agrees with that obtained in Ref. 1 for the stationary case. At $H\to 0$ the distribution function (7) agrees with that obtained in Refs. 2 and 3. However, (9) does not agree with the function obtained under the same conditions in Ref. 3 (Section 7). In connection with this let us note that Refs. 2 and 3 attempted to solve Eqs. (1) by expanding them into a Fourier series, and although this led to separation of the variables, it also resulted in an infinite system of interrelated differential equations. Since general analysis of such a system is exceedingly difficult (the termination method was used in Reference 3, but the authors of this article did not investigate fully the conditions under which this method is applicable,* and therefore the results they obtained for the case of low-frequency fields (Ref. 3, Sec. 7) are not quite correct, even though the distribution obtained in Refs. 2 and 3 for rapidly varying fields does not agree with (7). Let us note that Ref. 3 investigated not the approximate system of Eqs. (1), but a complete infinite system of equations, equivalent to the Boltzmann equation. The method developed in this article can also be used to solve that problem. The distribution function (7) does not agree with that obtained in Ref. 4.**

In the case when the principal role is played by collisions between electrons and ions, expres-

*The attempt to approximate c, made in Sec. 5 of Ref. 3 (from Sec. 3 of the same reference) is not sufficiently consistent: instead of function f_{00}^c which should be obtained in this approximation, function f_{00}^b was used (approximation c, Sec. 3), which is incorrect within the framework of the method employed.

**Recently the authors of that reference derived a correct expression for the distribution function,[5] agreeing with (7). The same expression was derived for the distribution function by Fain.[9] The kinetic equations were solved in these investigations as in Ref. 3. Fain also investigated the applicability of this method and arrived at a condition that is equivalent to (8) (see also Ref. 10).

sion (7) is valid for the distribution function if the following condition is satisfied:

$$\left(\sqrt{2/3}eE_0/\omega\sqrt{kTM}\right)+\delta\nu_{0i}/\omega\ll 1. \quad (10)$$

Neither the question of obtaining the distribution function in case of collisions with ions when conditions (10) are not satisfied (stationary and quati-stationary fields), nor gyro-resonance are considered in this article.*

The solution above is for a plane-polarized electric field ($\mathbf{E}=\mathbf{E}_0\cos\omega t$). However, in anisotropic plasma ($H\neq 0$), the plane electromagnetic wave breaks up into two elliptically-polarized waves. It is therefore interesting to determine the electron distribution function for the case of an elliptically-polarized electric field E. Resolving E along the three principal polarization axes we have

$$\mathbf{E}=\mathbf{E}_{\parallel 0}\cos\omega t+\mathbf{E}_{\perp 0}^{+}e^{i\omega t}+\mathbf{E}_{\perp 0}^{-}e^{-i\omega t}$$

($\mathbf{E}_{11}$ is the plane-polarized field with $\mathbf{E}_{11}\parallel \mathbf{H}$; $\mathbf{E}_{\perp}^{+}$ and $\mathbf{E}_{\perp}^{-}$ are the circularly-polarized fields in a plane perpendicular to H and rotating in the same (minus) or opposite (plus) direction as the electron in the magnetic field), it is easy to show that the same expression (7) is valid for the distribution function, provided that $E_0^2\ \varphi(v)$ is replaced by

$$E_0^2\varphi(v)\rightarrow\frac{E_{\parallel 0}^2}{\omega^2+\nu^2}+\frac{2E_{\perp 0}^{+2}}{(\omega+\omega_H)^2+\nu^2}+\frac{2E_{\perp}^{-2}}{(\omega-\omega_H)^2+\nu^2}. \quad (11)$$

In particular, if the field is circularly polarized in a plane perpendicular to H, it can be seen from (11) that the effect of the magnetic field is actually equivalent to an increase (or decrease) in the frequency of the electric field, a physically understandable effect.

3. MEAN ELECTRON ENERGY, PLASMA CONDUCTIVITY AND DIELECTRIC CONSTANT

Using the expression obtained above for the distribution function, let us compute the mean electron energy $\overline{\mathcal{E}}$, and the plasma conductivity σ and dielectric constant ϵ. Integrating the expressions for the mean energy and for the electric-current density with respect to the angular variables and employing the orthogonality of the functions f_0, f_1, and χ we obtain[1]

$$\overline{\mathcal{E}}=2\pi m\int_0^\infty v^4 f_0 dv; \quad (12)$$

$$\mathbf{j}=\left(\sigma+i\omega\frac{\varepsilon-1}{4\pi}\right)\mathbf{E}=\frac{4\pi eN}{3}\int_0^\infty v^3\mathbf{f}_1 dv.$$

Using the Hermite properties of the tensors ϵ and σ and aligning the z axis with the magnetic field, we obtain from (12) and (7) the following expression for the components of the tensors ϵ and σ:

$$\sigma_{zz}=\frac{8}{3\sqrt{\pi}}\frac{e^2N}{m}\int_0^\infty F\frac{\nu}{\omega^2+\nu^2}du;$$

$$\sigma_{xx}=\sigma_{yy}=\frac{8}{3\sqrt{\pi}}\frac{e^2N}{m}\int_0^\infty F\nu du\frac{1}{2}\left\{\frac{1}{(\omega-\omega_H)^2+\nu^2}+\frac{1}{(\omega+\omega_H)^2+\nu^2}\right\};$$

$$\sigma_{xy}=-\sigma_{yx}=i\frac{8}{3\sqrt{\pi}}\frac{e^2N}{m}\int_0^\infty F\nu du\frac{1}{2}\left\{\frac{1}{(\omega+\omega_H)^2+\nu^2}-\frac{1}{(\omega-\omega_H)^2+\nu^2}\right\};$$

$$\frac{\varepsilon_{zz}-1}{4\pi}=-\frac{8}{3\sqrt{\pi}}\frac{e^2N}{m}\int_0^\infty Fdu\frac{1}{\omega^2+\nu^2}; \quad (12')$$

$$\frac{\varepsilon_{yy}-1}{4\pi}=\frac{\varepsilon_{xx}-1}{4\pi}=-\frac{8}{3\sqrt{\pi}}\frac{e^2N}{m}\int_0^\infty Fdu\frac{1}{2\omega}\left\{\frac{\omega-\omega_H}{(\omega-\omega_H)^2+\nu^2}+\frac{\omega+\omega_H}{(\omega+\omega_H)^2+\nu^2}\right\};$$

$$\frac{\varepsilon_{xy}}{4\pi}=-\frac{\varepsilon_{yx}}{4\pi}=-i\frac{8}{3\sqrt{\pi}}\frac{e^2N}{m}\int_0^\infty Fdu\frac{1}{2\omega}\left\{\frac{\omega-\omega_H}{(\omega-\omega_H)^2+\nu^2}-\frac{\omega+\omega_H}{(\omega+\omega_H)^2+\nu^2}\right\};$$

$$\sigma_{xz}=\sigma_{zx}=\sigma_{yz}=\sigma_{zy}=0;$$

$$\varepsilon_{xz}=\varepsilon_{zx}=\varepsilon_{yz}=\varepsilon_{yz}=0,$$

where a dimensionless variable $u=v\sqrt{2kT/m}$ was introduced;

$$u=v\sqrt{2kT/m};\ F=f_{00}\frac{u^4}{1+(e^2E_0^2/3kTm\delta)\varphi};$$ f_{00} and φ

*The many difficulties arising in the analysis of the distribution function in stationary fields in case of collisions with ions were pointed out to the author by V. L. Ginzburg. I take advantage of this opportunity to thank him for attentive examination of the results of this investigation.

are given by expressions (7) and (7′). In the absence of a magnetic field, it can clearly be seen from (12′) that $\sigma_{xy} = \sigma_{yx} = 0$ and $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma$, with analogous relationships for ϵ.

a) Magnetic field $H = 0$. In the absence of a magnetic field (for collisions with molecules) we obtain from (12):

$$\overline{\mathcal{E}} = \frac{3}{2} kT (\gamma + q^2)^{1/2} \frac{W_{\gamma/2-3/4;\ \gamma/2+5/4}(\gamma + q^2)}{W_{\gamma/2-1/4;\ \gamma/2+3/4}(\gamma + q^2)};$$
$$\sigma = \frac{8}{3\sqrt{\pi}} \frac{e^2 N}{m\nu_0} (\gamma + q^2)^{1/2} \frac{W_{\gamma/2-3/4;\ \gamma/2+1}(\gamma + q^2)}{W_{\gamma/2-1/4;\ \gamma/2+3/4}(\gamma + q^2)};$$
$$\frac{\varepsilon - 1}{4\pi} = -\frac{e^2 N}{m\nu_0^2} \frac{W_{\gamma/2-5/4;\ \gamma/2+3/4}(\gamma + q^2)}{W_{\gamma/2-1/4;\ \gamma/2+3/4}(\gamma + q^2)}. \quad (13)$$

Here $W_{\mu,\lambda}(z)$ is the Whittaker function of order μ, λ; to simplify the notation we again introduce the dimensionless parameters

$$\gamma = \frac{e^2 E_0^2}{3kTm\delta\nu_0^2} \approx \frac{3{,}4 \cdot 10^{32} E_0^2}{N_M^2 T^2}; \quad (13')$$
$$q = \frac{\omega}{\nu_0}; \quad \nu_0 = \sqrt{\frac{2kT}{m}}\, \pi a^2 N_M \approx 2{,}36 \cdot 10^{-10} N_M \sqrt{T},$$

where γ is a quantity on the order of the square of the ratio of the energy delivered to the electron by the constant field E_0 to its thermal energy; ν_0 is a frequency on the order of the number of electron collisions in the absence of an electric field. In the numerical coefficients we assumed $\delta = 3.4 \times 10^{-5}$ and $\pi a^2 = 4.3 \times 10^{-16}$ cm^2; E_0 is in millivolts per meter. If $\gamma \gg 1$ we have from (12):

$$\overline{\mathcal{E}} = 3kTx (1 + \gamma/q^2) I_{3/2}(x) / I_{1/2}(x),$$
$$\sigma = \frac{16}{\sqrt{3\pi}} \frac{eN}{E_0} \sqrt{\frac{kT\delta x}{m}} (1 + \gamma/q^2)^{1/2} \frac{I_2(x)}{I_{1/2}(x)} \left(1 - \frac{6}{\sqrt{2\gamma}} \frac{I_3(x)}{I_2(x)}\right); \quad (14)$$
$$\frac{\varepsilon - 1}{4\pi} = -\sqrt{6} \frac{eN}{E_0 \nu_0} \sqrt{\frac{kT\delta}{m}} \frac{I_{3/2}(x)}{I_{1/2}(x)} \left(1 - \frac{5}{\sqrt{2\gamma}} \frac{I_{5/2}(x)}{I_{3/2}(x)}\right),$$

where

$$I_p(x) = \frac{1}{\Gamma(p+1)} \int_0^\infty e^{-t^2 - 2tx} t^p\, dt \quad (14')$$
$$= \frac{e^{x^2/2}}{2^{(p+1)/2}} D_{-p-1}(\sqrt{2x}),$$

if $x \gg 1$

$$I_p(x) = (2x)^{-p-1} \{1 - (p+1)(p+2)/(2x)^2\}$$

(here $D_h(z)$ are the parabolic-cylinder functions). and

$$x = \sqrt{3}\, \omega^2 m \sqrt{\delta} / 2\pi a^2 N_M e E_0 \approx 660\, \omega^2 / E_0 N_M$$

(x is on the order of $(\omega/\nu)^2$, where ν is the electron collision frequency in a strong electric field). The functions $I_p(x)$ are plotted in Fig. 1. According to Eqs. (14) and (14′), $\overline{\mathcal{E}}$ increases monotonically with E_0 and T and decreases monotonically with ω and N_M. If $x \gg 1$ (high frequencies) we obtain from (14) and (14′):

$$\overline{\mathcal{E}} = \frac{3}{2} kT (1 + \gamma/q^2),$$
$$\sigma = \frac{8}{3\sqrt{\pi}} \frac{e^2 N \nu_0}{m\omega^2} (1 + \gamma/q^2)^{1/2},$$
$$\frac{\varepsilon - 1}{4\pi} = -\frac{e^2 N}{m\omega^2}.$$

In this case the distribution function is Maxwellian with an effective electron temperature

$$T_{эф} = T(1 + \gamma/q^2) = T(1 + e^2 E_0^2 / 3kTm\delta\omega^2).$$

If $x \ll 1$ (low frequency):

$$\overline{\mathcal{E}} = \frac{\Gamma(5/4)}{\sqrt{3}\,\Gamma(3/4)} \frac{eE_0}{\sqrt{\delta}\,\pi a^2 N_M}, \quad (15)$$
$$\sigma = \frac{\sqrt{\pi}}{2^{1/2} 3^{3/4} \Gamma(3/4)} \frac{N e^{3/2} \delta^{1/4}}{\sqrt{\pi a^2 N_M m E_0}} \approx 1{,}5 \cdot 10^{10} \frac{N}{\sqrt{N_M E_0}},$$
$$\frac{\varepsilon - 1}{4\pi} = -\frac{2\Gamma(5/4)}{\sqrt{3}\,\Gamma(3/4)} \frac{eN\delta^{1/2}}{\pi a^2 N_M E_0}$$
$$\approx -1{,}67 \cdot 10^{11} \frac{N}{N_M E_0}.$$

In this case the distribution function is of the Druyvesteyn type with an effective field $E_{ef.} = E_0/\sqrt{2}$, as it should be, for if $x \ll 1$, i.e., if $\omega \ll \nu$, the alternating electric field acts on the average as a constant field $E = E_{ef} = E_0/\sqrt{2}$. Let us also introduce corrections for the expressions for $\overline{\mathcal{E}}$, σ, and ϵ in a weak field ($\gamma \ll 1$). If $\omega^2 \gg \nu_0^2$ we have:

$$\overline{\mathcal{E}} = \frac{3}{2} kT (1 + \gamma/q^2);$$
$$\sigma = \frac{8}{3\sqrt{\pi}} \frac{e^2 N \nu_0}{m\omega^2} (1 + \gamma/q^2)^{1/2};$$
$$(\varepsilon - 1)/4\pi = -e^2 N / m\omega^2.$$

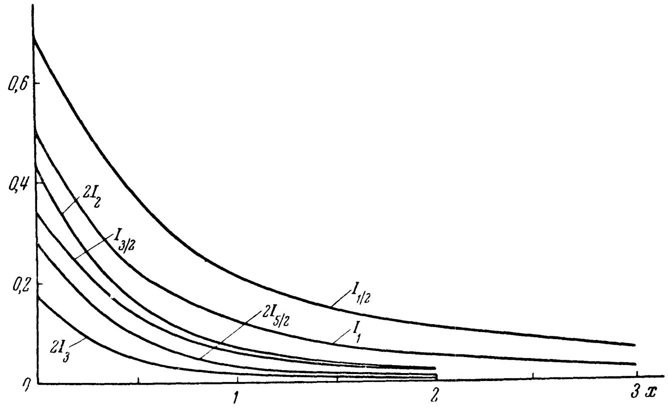

FIG. 1

If $\omega^2 \ll \nu_0^2$

$$\overline{\mathcal{E}} = \frac{3}{2}kT\,(1 + {}^2/_3\,\gamma);$$

$$\sigma = \frac{4}{3\sqrt{\pi}}\frac{e^2N}{m\nu_0}\,[1 - 2\gamma\,(1 - \ln 2)];$$

$$I_p(x) = (2x)^{-p-1}\,\{1 - (p+1)(p+2)/(2x)^2\}$$

Figures 2, 3 and 4 show plots of $\overline{\mathcal{E}}$, σ, and $(\epsilon - 1)/4\pi$) (related to their values at $E \to 0$) vs. $q = \omega/\nu_0$ for various values of γ. It is clear from the figures that at $\gamma \gg 1$ the mean electron energy and plasma conductivity and dielectric constant depend strongly on the electric field intensity. It

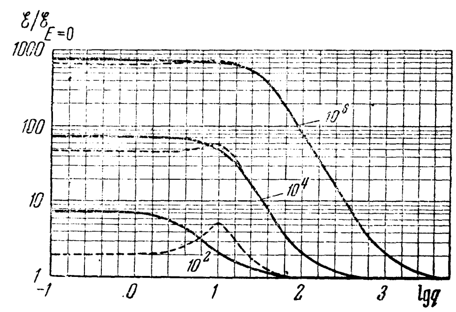

FIG. 2. Dependence of $\mathcal{E}/\mathcal{E}_{E=0}$ on $q = \omega/\nu_0 = 4.24 \times 10^9\,\omega/N_M\sqrt{T}$ for the values of γ indicated on the curves, and for $\omega_H/\nu_0 = 10$. Solid lines are for $\beta = 0$, dotted for $\beta = \pi/2$.

is interesting to note that at low frequencies ($x \ll 1$), σ decreases with increasing E_0 and the mean electron energy increases, and with it the energy delivered the wave to the plasma molecules (see Eq. 15). If we assume here that the absorption of radio waves is proportional to the conductivity, an apparent contradiction arises: the energy delivered by the wave to the plasma molecules increases with the field in a strong low-frequency field, and at the same time the absorption of the wave decreases. This, however, does not contradict the law of conservation of energy, for the energy of the wave increases as the square of E_0, while the mean energy of the plasma electrons is proportional to E_0 [(from (15)]. Consequently, even though the total energy delivered by the wave to the plasma molecules increases with E_0, its relative value decreases; therefore, the absorption also decreases, and exhibits a relative decrease in the amplitude of E on the given section of the path.

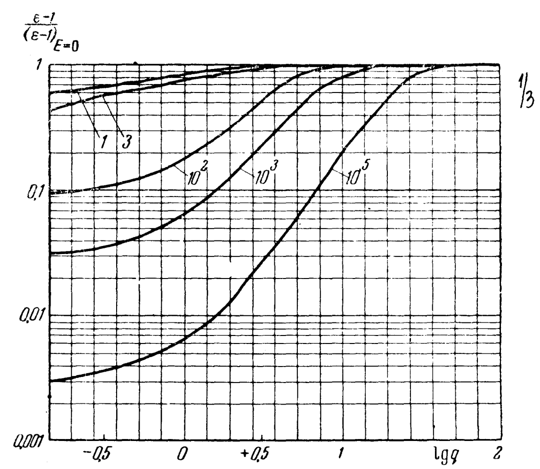

FIG. 3. Dependence of $(\epsilon - 1/(\epsilon - 1)_{E=0}$ on q for collisions with molecules at $H = 0$ for various values of γ as marked on the curves.

Let us also note that the probability of the electron having an energy greater than the prescribed value of $\mathcal{E}_0$ is given by the following equation

$$P\,(\mathcal{E} \geqslant \mathcal{E}_0) = \int_{v_0}^{\infty} v^2 f_{00}\,dv,$$

where $v_0 = \sqrt{2\bar{\mathcal{E}}/m}$. It is now easy to obtain an expression for $P\ (\mathcal{E} \geq \mathcal{E}_0)$ for various limiting cases. In particular, for $\gamma >> 1$ we have

$$P(\mathcal{E} \geqslant \mathcal{E}_0) = I_{1/2}(t_0, x) / I_{1/2}(x), \qquad (16)$$

$$I_{1/2}(t_0, x) = \frac{2}{\sqrt{\pi}} \int_{t_0}^{\infty} t^{1/2} \exp\{-t^2 - 2tx\}\, dt; \qquad (16')$$

$$t_0 = \frac{\mathcal{E}_0}{2kTx(1+\gamma/q^2)}.$$

It is clear from (16) and (16′) that in a strong field the electron energy is less likely to deviate substantially from its mean value $\bar{\mathcal{E}}$ at low frequencies than at high frequencies.

b) Magnetic field $H \neq 0$. If the effect of the constant magnetic field is taken into account, the computation of $\bar{\mathcal{E}}$, σ, and ϵ is in general much more complicated; however, in some particular instances it is possible to convert the expressions for $\bar{\mathcal{E}}$, σ, and ϵ to those discussed above by introducing effective parameters E_0', ω', which take the magnetic field into account.

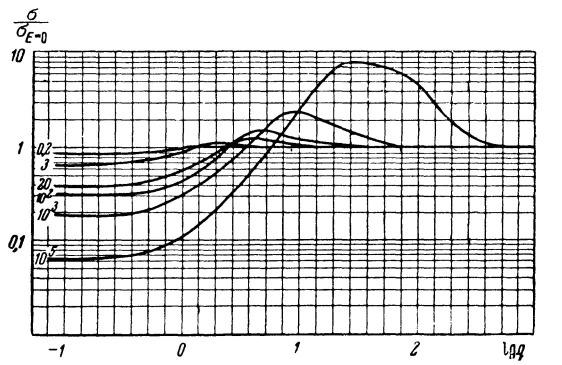

FIG. 4. Dependence of $\sigma/\sigma_{E\to 0}$ on q for collisions with molecules (at $H=0$) for the various values of γ marked on the curves.

It is evident that if $\beta = 0$ (β is the angle between E and H) the magnetic field does not affect the symmetrical portion of the distribution function. Consequently, the average electron energy, and also the components of the conductivity and dielectric constant tensors (σ_{zz}, ϵ_{zz}), which are parallel to H are given by the expressions analyzed previously ($E_0' = E_0$; $\omega' = \omega$). For the remaining components of tensors σ and ϵ we obtain the somewhat more complicated expressions (12). In several limiting cases they can be simplified. For example, if $\gamma >> 1$, $x >> 1$, and $x(\omega - \omega_H)^2/\omega^2 >> 1$ the expressions for σ_{xx} etc. are the same as obtained in the weak-field theory [with the temperature replaced by its effective value T_{ef}, as given in (20)], namely;

$$\sigma_{xx} = \sigma_{yy} = \frac{8}{3\sqrt{\pi}} \frac{e^2 N \nu_0}{m} \frac{1}{2} \qquad (17)$$

$$\times \left(\frac{1}{(\omega-\omega_H)^2} + \frac{1}{(\omega+\omega_H)^2}\right)(1+\gamma/q^2)^{1/2};$$

$$\sigma_{xy} = -\sigma_{yx} = -i\frac{8}{3\sqrt{\pi}} \frac{e^2 N \nu_0}{m} \frac{1}{2}$$

$$\times \left(\frac{1}{(\omega-\omega_H)^2} - \frac{1}{(\omega+\omega_H)^2}\right)(1+\gamma)/q^2)^{1/2};$$

$$\frac{\varepsilon_{xx}-1}{4\pi} = \frac{\varepsilon_{yy}-1}{4\pi} = -\frac{e^2 N}{m(\omega^2-\omega_H)^2};$$

$$\frac{\varepsilon_{xy}}{4\pi} = -\frac{\varepsilon_{yx}}{4\pi} = -i\frac{e^2 N}{m(\omega^2-\omega_H^2)} \frac{\omega_H}{\omega}.$$

The magnetic field affects the electron distribution most when $\beta = \pi/2$. In this case we have for $\gamma >> 1$ and $x(\omega + \omega_H)^2/\omega^2 >> 1$

$$\bar{\mathcal{E}} = \bar{\mathcal{E}}_{H=0}(E_0', \omega'); \qquad (18)$$

$$\sigma_{xx} = \sigma_{yy} = \frac{1}{2}\left(1 + \left(\frac{\omega-\omega_H}{\omega+\omega_H}\right)^2\right)\sigma_{H=0}(E_0', \omega'),$$

$$\sigma_{xy} = -\sigma_{yx} = -i\frac{2\omega\omega_H}{(\omega+\omega_H)^2}\sigma_{H=0}(E_0', \omega');$$

$$\frac{\varepsilon_{xx}-1}{4\pi} = \frac{\varepsilon_{yy}-1}{4\pi}$$

$$= -\sqrt{6}\frac{eN}{\nu_0 E_0'}\sqrt{\frac{kT\delta}{m}} \frac{I_{3/2}(x')}{I_{1/2}(x')}\left\{\frac{\omega^2-\omega_H^2}{(\omega+\omega_H)^2}\right.$$

$$\left. + \frac{(\omega-\omega_H)^2}{(\omega+\omega_H)^2}\frac{5}{2x'}\frac{I_{5/2}(x')}{I_{3/2}(x')}\right\},$$

$$\frac{\varepsilon_{xy}}{4\pi} = -\frac{\varepsilon_{yx}}{4\pi}$$

$$= -i\sqrt{6}\frac{eN}{\nu_0 E_0'}\sqrt{\frac{kT\delta}{m}}\frac{\omega_H}{\omega}\frac{I_{3/2}(x')}{I_{1/2}(x')}\left\{\frac{\omega_H^2-\omega^2}{(\omega+\omega_H)^2}\right.$$

$$\left. + \frac{(\omega-\omega_H)^2}{(\omega+\omega_H)^2}\frac{5}{2x'}\frac{I_{5/2}(x')}{I_{3/2}(x')}\right\},$$

$$E_0' = \frac{E_0}{\sqrt{2}}\sqrt{1+\left(\frac{\omega-\omega_H}{\omega+\omega_H}\right)^2}; \qquad (18')$$

$$\omega' = |\omega - \omega_H|; \quad x' = \frac{\sqrt{3}}{2}\frac{\omega'^2 m\sqrt{\delta}}{\pi a^2 N_M e E_0'}$$

If, on the contrary, $x(\omega_1+\omega_H)^2/\omega^2 \ll 1$, the effect of the magnetic field is insignificant; for the mean energy of the electrons and for the diagonal components of tensors σ and ϵ ($\sigma_{xx}=\sigma_{yy}=\sigma_{zz},\ldots$) we have $E'_0=E_0$; $\omega'=\omega$. The remaining components have the following form:

$$\sigma_{xy}=\sigma_{yx}=-i\frac{2\sqrt{2\pi}}{3^{1/4}\Gamma(3/4)}\frac{\omega\omega_H m^{1/2}\delta^{3/4}e^{1/2}N}{(E_0\pi a^2 N_M)^{1/2}}; \quad (19)$$

$$\frac{\varepsilon_{xy}}{4\pi}=-\frac{\varepsilon_{yx}}{4\pi}=i\frac{2\Gamma(5/4)}{\sqrt{3}\,\Gamma(3/4)}\frac{\omega_H}{\omega}\frac{eN\sqrt{\delta}}{E_0\pi a^2 N_M}.$$

Figures 2 and 5 show plots of the functions ϵ and σ_{xx} (at $\beta=\pi/2$) vs. $q=\omega/\nu_0$ for $\omega_H/\nu_0=10$ and for various values of γ. The mean electron energy $\mathcal{E}/\mathcal{E}_{E=0}$, is seen clearly from (18) and (18′) to increase resonantly in the vicinity of the gyro-frequency [$\omega\sim\omega_H$, compare (13)]; this resonance is observed when the following conditions $\gamma\gg 1$; $x(\omega_H/\omega)^2\gg 1$, are satisfied, as can also be seen from Fig. 2. Under these conditions $\sigma/\sigma_{E=0}$ decreases resonantly (see Fig. 5), while at $\omega=\omega_H$ increases, and $\mathcal{E}$ decreases with increasing E_0, the same as at low

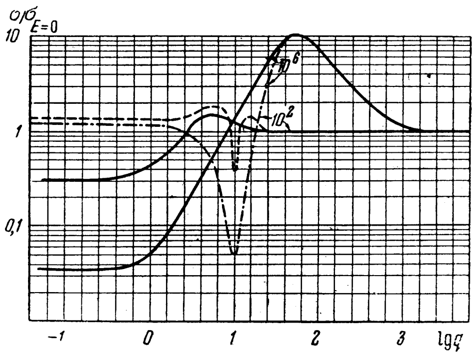

FIG. 5. Dependence of $\sigma/\sigma_{E=0}$ on q (Collisions with molecules for $\omega/\nu_0=10$ and for various values of γ marked on the curve). Solid curves are for $\beta=0$, dotted for $\beta=\pi/2$.

frequencies in the absence of a magnetic field (see Eq. 15).

Let us also note that the case $\beta=\pi/2$ also occurs for longitudinal propagation of the electromagnetic wave. The wave breaks up here into two waves — ordinary and extraordinary — circularly polarized in a plane perpendicular to H. For each of these waves, as can be clearly seen from (11), $E'_0=E_0\sqrt{2}$; ; $\omega'=|\omega\pm\omega_H|$. Comparing (19) with (18) and (18′) we see that in first approximation both circularly-polarized waves, which make up a plane polarized wave, exert an equal effect of the electrons in all cases except gyro-resonance ($\omega\sim\omega_H$, $x(\omega_H/\omega)^2\gg 1$). In the case of gyro-resonance, however, the increase in electron energy is due only to the extraordinary wave.

If angle β is arbitrary and the following conditions are satisfied

$$x/\cos^2\beta\gg 1;\quad (2x/\sin^2\beta)(\omega-\omega_H)^2/\omega^2\gg 1,$$

the distribution function is Maxwellian with an effective temperature

$$T_{\text{эф}}=T\left\{1+\frac{e^2E_0^2}{3kTm\delta}\left[\frac{\cos^2\beta}{\omega^2}+\frac{\sin^2\beta\,(\omega^2+\omega_H^2)}{(\omega^2-\omega_H^2)^2}\right]\right\}, \quad (20)$$

i.e., the same expressions hold for $\overline{\mathcal{E}}$, σ, and ϵ as for weak fields, except that T is replaced by T_{ef} (see Eq. 17). Let us note that the electron gas also has this effective temperature (20) in case of collisions with ions, provided the following conditions are satisfied.

$$\omega\gg\nu_{0i};\quad |\omega-\omega_H|\gg\nu_{0i}. \quad (21)$$

Here ν_{0i} is on the order of the frequency of collisions between electrons and ions in the absence of an electric field. If $x(\omega+\omega_H)^2\ll 1$, and $y\gg 1$, the distribution function (in case of collisions with molecules is of the Druyvesteyn type with an effective field $E_{ef}=E_0/\sqrt{2}$; expressions (15), (19) are valid for $\mathcal{E}$, ϵ, and σ.

4. EFFECTIVE FREQUENCY OF ELECTRON COLLISIONS

The expressions for σ and ϵ obtained above with the aid of the kinetic theory are in general quite complicated. It becomes therefore advantageous to investigate the possibility of calculating σ and ϵ from the elementary theory equations:

$$\sigma_{(el)}=\frac{e^2N}{m}\frac{\nu_{\text{эф}}}{\omega^2+\nu_{\text{эф}}^2}; \quad (22)$$

$$\varepsilon_{(el)}=1-\frac{4\pi e^2N}{m}\frac{1}{\omega^2+\nu_{\text{эф}}^2},$$

For this purpose it is necessary to determine the effective electron-collision frequency (ν_{ef}) and to compare the values of the conductivity and dielectric constant (σ_{el}, ϵ_{el}) from Eqs. (22), with those (σ_k, ϵ_k), obtained from kinetic theory.

Let us first consider the case of a weak electric field ($e^2E_0^2/3kTm\delta(\omega^2+\nu_0^2 \ll 1$), when the effect of the wave of the plasma can be neglected.* Here, as is well known, it is possible to determine the effective electron-collision frequency ν_{ef0} at high frequencies ($\omega \gg \nu_0$) in such a way that

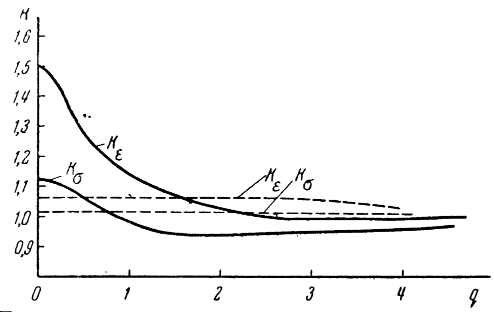

FIG. 6. Dependence of K_σ and K_ϵ on q for collisions with molecules. The dotted lines show curves for strong electric fields (at $\gamma = 10^5$).

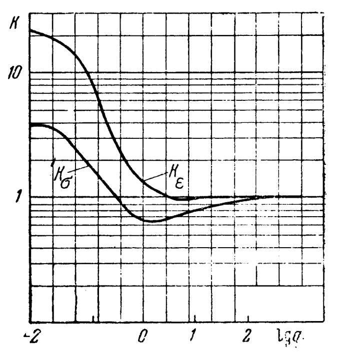

FIG. 7. Dependence of K_σ and K_ϵ on log q_i for collisions with ions.

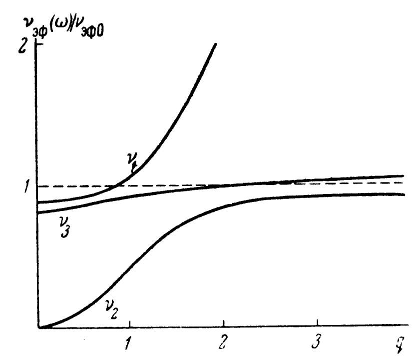

FIG. 8. Dependence of $\nu_{ef}(\omega)/\nu_{ef0}$ on q for collisions with molecules.

the conductivity and the dielectric constant, calculated from the kinetic theory and from Eq. (22), are in agreement (see Refs. 11-13):

$$\nu_{\text{эф}\,0} = \frac{8}{3\sqrt{\pi}}\nu_0 \qquad (22')$$

$$= \frac{8\sqrt{2}}{3\sqrt{\pi}}\sqrt{\frac{kT}{m}}\pi a^2 N_M \quad \text{(for collisions with molecules)}$$

$$\nu_{\text{эф}\,0i} = \frac{4}{3\sqrt{\pi}}\nu_{0i}$$

$$= \frac{2\sqrt{2\pi}}{3}\frac{e^4N_i}{m^{1/2}(kT)^{3/2}}\ln\left\{\frac{(kT)^3}{2\pi N_i e^6}\right\} \quad \text{(for collisions with ions).}$$

If the condition $\omega \gg \nu_0$ is not satisfied then σ_{el} and ϵ_{el} no longer equal σ_k and ϵ_k. It is therefore necessary to introduce correction coefficients K_σ and K_ϵ, which are determined in the following manner:

$$\sigma_{\text{к}} = K_\sigma \sigma_{\text{эл}}; \quad \varepsilon_{\text{к}} - 1 = K_\varepsilon(\varepsilon_{\text{эл}} - 1).$$

The coefficients K_σ and K_ϵ plotted in Figs. 6 and 7 are functions of the frequency. In the case of collisions with molecules (Fig. 6) the correction coefficients K_σ and K_ϵ are nearly equal to unity; on the other hand, in the case of collisions with ions (Fig. 7) they may differ considerably from unity (at $\omega \lesssim 10\,\nu_{0i}$),* and this circumstance must be taken into account in the calculation of the conductivity and of the dielectric constant.

In a strong electric field, K_{ef} can be obtained from effective electron gas temperature T_{ef}. In the case of collisions with molecules we have here $\nu_{ef} = \nu_{ef0}\sqrt{T_{ef}/T}$, where T_{ef} is determined by the following relationship:

---

*Let us note that the weak-field condition specified here is valid in the case of collisions with ions only if $\omega \gtrsim \nu_{0i}$.

*It must be noted that in general the parameter ν_{ef} can be defined in such a way as to make σ_{el} and ϵ_{el} equal to σ_k and ϵ_k; for this purpose it is necessary to examine $\nu_{ef}(\omega)$, as was done earlier (see Ref. 8, Sec. 61). However, analysis shows that such determination of the effective collision frequency leads to three expressions for $\nu_{ef}(\omega)$, as shown in Figs. 8 (collisions with molecules) and 9 (collisions with ions). As can be seen from the figures, the indicated three expressions for $\nu_{ef}(\sigma)$ differ considerably even at the same value of ω and vary substantially with ω. Therefore ν_{ef} loses in this case its physical significance of an electron collision frequency ($\nu_{ef2} \to 0$ at $\omega \to 0$). In addition, the requirement that ϵ_{el} and $\varkappa_{el}$ be equal to ϵ_k and σ_k calls for the use of at least two substantially-varying functions $\nu_{ef}(\omega)$. There is no sense in doing this and it is more convenient to employ the coefficient K_σ and K_ϵ. In strong fields the use of ν_{ef} leads to even more serious difficulties.

$$\frac{T_{(ef)}}{T} = 1 + \frac{e^2 E_0^2}{3kTm\delta} \left\{ \frac{\cos^2\beta}{\omega^2 + \nu^2_{(ef)^0} T_{(ef)}/T} + \frac{\sin^2\beta}{2} \left[\frac{1}{(\omega - \omega_H)^2 + \nu^2_{(ef)^0} T_{(ef)}/T} + \frac{1}{(\omega + \omega_H)^2 + \nu^2_{(ef)^0} T_{(ef)}/T} \right] \right\}. \quad (23)$$

Comparing now the values of the conductivity and dielectric constant calculated with the simple equations (22), (22′), (23) and those obtained with the kinetic-theory equations (Sec. 3), we see that in strong fields (as well as weak ones) the coefficients K_σ and K_ϵ are nearly equal to unity as before: at $x \ll 1$ (low frequencies), $K_\sigma \approx 1.03$ and $K_\epsilon \approx 1.07$ (see, for example, the dotted curve on figures 6), and at $x \gg 1$ (high frequencies) $K_\sigma \to 1$; $K_\epsilon \to 1$.

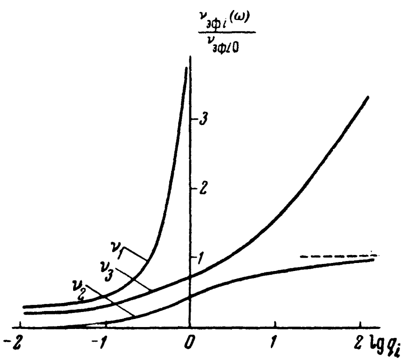

FIG. 9. Dependence of $\nu_{ef}(\omega)/\nu_{ef0}$ on $q_i = \omega/\nu_{0i}$ for collisions with ions.

In the case of collisions with ions, if we restrict ourselves to high frequencies ($\omega \gtrsim 10\,\nu_{0i}$), Eqs. (22) and (22′), used in conjunction with the effective electron temperature T_{ef} as defined by equation (20), result in strongfield values of σ and ϵ that are even in closer agreement with the results of the kinetic theory than the weak-field values.**

In conclusion the author thanks Ia. L. Al' pert for guidance in this investigation and for many advises and comments.

**The latter circumstance follows from the fact that the effective frequency of collisions between electrons and ions (22) decreases with increasing T_{ef}, i.e., the condition $\omega \gg \nu_{efi}$ is better satisfied in a strong field than in a weak one. Let us also recall that if the electron temperature T_{ef} differs from the ion temperature T, a somewhat different expression is used to calculate ν_{ef0i} than (22) (see Ref. 8, Sec. 61).

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