

## The Theory of Galvanomagnetic Effects in Metals

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The theory of galvanomagnetic effects in metals in large magnetic fields is developed, without any restrictive assumptions concerning the dispersion law of electrons or the collision integral<sup>1</sup>.

A LARGE number of experimental and theoretical studies have been devoted to the behavior of metals in electric and magnetic fields. After the discovery of Kapitza<sup>2</sup> of a linear increase of resistivity with field in a number of metals, it was shown by Justi<sup>3</sup> that all metals can be divided into two groups. In the first group (Cu, Na, In, Al,..) the resistivity in large magnetic field tends to a saturation value; in the second group (Bi, Be, Zn, Mg, ...) it grows without limit as the square of the magnetic field ( $\rho \sim H^2$ ). Borovik<sup>3</sup> moreover showed that there is a characteristic difference in the behavior of the Hall field between these two groups. He also showed<sup>3</sup> that the linear growth of resistivity (Kapitza law) is observed in the transition region between two regions of quadratic growth.

In previous theories of the galvanomagnetic effects, it has been usual to start with some specific assumption about the form of the electron energy spectrum and of the metal and about the electron-lattice interactions. In order to explain an unlimited growth of resistivity, it has been usual to introduce "holes" (positively charged current carriers) somewhat artificially as well as electrons (negatively charged current carriers), and the precise physical meaning of the holes was obscure.

In the present work, a theory of galvanomagnetic effects is developed which involves no special assumptions beyond the use of the Fermi distribution for the electrons in the metal.

### 1. CONDUCTION ELECTRONS IN A MAGNETIC FIELD

The current carriers in a metal--the elementary excitations usually called metallic electrons or conduction electrons--are characterized by quasi-momentum  $p$ , energy  $\epsilon$ , charge  $e$  and spin  $\frac{1}{2}$ . The periodicity of the crystalline lattice leads to a periodic dependence of the energy  $\epsilon$  on the quasi-momentum  $p$  with the period of the reciprocal lat-

tice<sup>4</sup>.  $\epsilon = \epsilon(p)$  is a many-valued function, so that more exactly we have  $\epsilon = \epsilon_n(p)$ , where  $n$  denotes the number of the energy band ("zone"); in what follows we shall omit the subscript  $n$ . Close to points of minimum or maximum energy, the constant-energy surfaces are clearly closed, and moreover, in the immediate neighborhood of the extremal points they are ellipsoidal. Any (closed) constant-energy surface close to an energy minimum encloses (in momentum space) a region where the energy is smaller than its value on the surface ( $\nabla_p \epsilon$  directed out of the region), while any closed surface close to an energy maximum encloses a region where the energy is greater than its value on the surface ( $\nabla_p \epsilon$  is directed into the region).

Since  $\epsilon(p)$  is a periodic function, the surfaces described above will be repeated periodically throughout the reciprocal lattice. Between these surfaces, which are topologically simple, there must occur more complex ones--self-intersecting and open surfaces. We call open surfaces those which pass continuously through the whole reciprocal lattice. Thus Fig. 1 shows the energy surfaces in two dimensions given by the equation\*

$$\epsilon = A_1 \cos k_x a_x + A_2 \cos k_y a_y.$$

It will be seen that if  $A_1 = A_2$ , a single open surface exists (Fig. 1a), but that if  $A_1 \neq A_2$ , there exists a whole family of them (Fig. 1b).

In the real three-dimensional case, a great variety of open surfaces may arise, which may be simply or multiply connected (some examples of open surfaces are shown in Figs. 2-5).

The energy levels of a solid, the structure of which we have described above, are filled by elec-

\* The corresponding equation in three dimensions gives the dependence of energy on momentum according to the tight-binding approximation<sup>5</sup>.

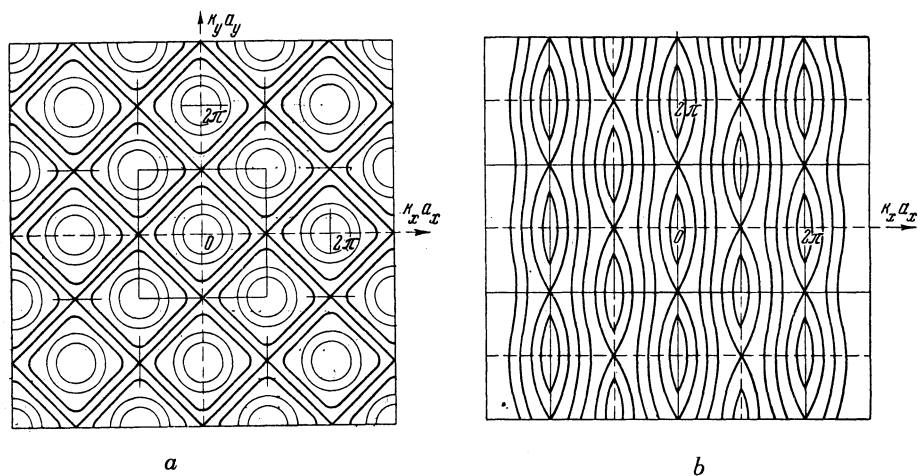


FIG. 1

trons in a way which is determined ( at absolute zero ) by the number of electrons per atom and by the disposition of the zones ( i.e., whether or not they overlap). In what follows, we shall be interested only in the zones which are only partly filled ( clearly, such zones always exist in a metal<sup>4</sup> ). In the general case, there will be several such zones, so that the limiting Fermi surface,  $\epsilon(\mathbf{p}) = \zeta$ , will in general consist of several separate surfaces,

each repeated periodically because of the periodicity of the function  $\epsilon(\mathbf{p})$ .

Consider now the motion of a conduction electron in a constant and uniform magnetic field  $\mathbf{H}$  directed along the  $z$  axis ( $H_x = H_y = 0; H_z = H$ ). We shall use the classical equations of motion, i.e., we neglect quantization of motion in the magnetic field. For magnetic fields satisfying  $\mu H \ll \zeta$

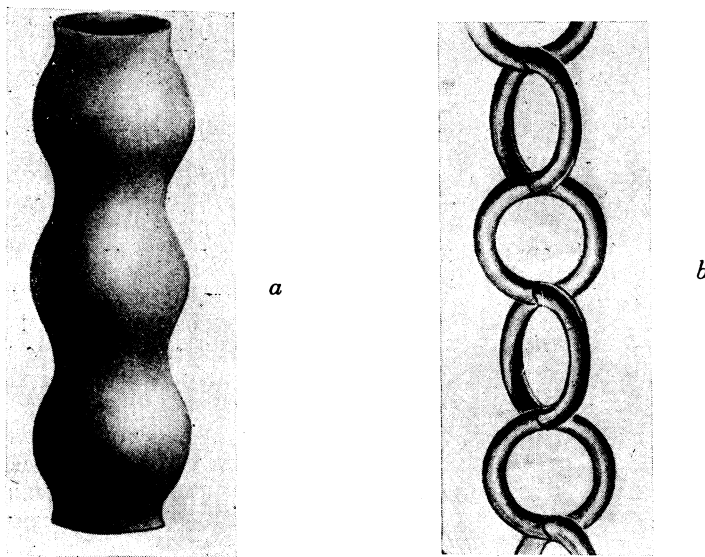


FIG. 2

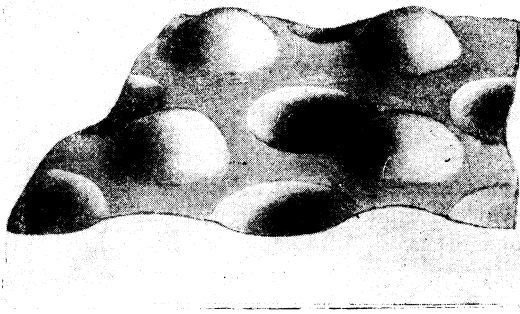


FIG. 3

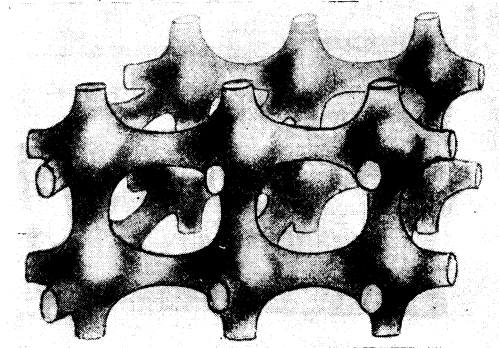


FIG. 4

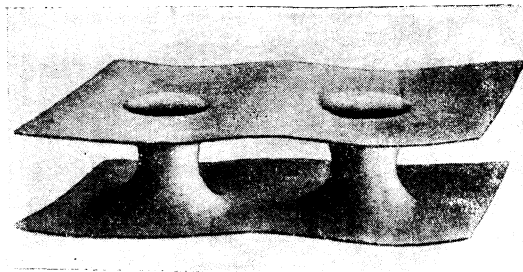


FIG. 5

(which is perfectly well satisfied for all attainable fields), this is clearly permissible if we are not interested in the small quantum oscillations<sup>6</sup> superposed on the monotonic rise of resistivity and

Hall field with magnetic field\*.

When moving in a magnetic field  $H$ , the conduction electrons conserve their energy  $\epsilon$  and  $p_z$ , the component of the momentum  $p$  along the  $z$  axis. Thus the trajectory of the electrons in momentum space is given by the curve

$$\epsilon(\mathbf{p}) = \text{const}; \quad p_z = \text{const}. \quad (1)$$

The character of the motion along the trajectory in momentum space depends essentially on whether the curve (1) is closed, i.e., breaks up into a number of closed curves each lying within a single cell of the reciprocal lattice\*\*, or open, i.e., running continuously through the whole reciprocal lattice.

For closed energy surfaces, every trajectory in a magnetic field is a closed curve. For open energy surfaces, both open and closed trajectories may occur. [Suppose, for instance, that the constant-energy surface is an "undulating cylinder" (Fig. 2a); then if the magnetic field is perpendicular to the axis of the cylinder, the trajectory is open, but for all other field directions it is closed.] In investigating the properties of conduction electrons in a magnetic field, it is convenient to classify open surfaces as follows:

1) Surfaces which contain open orbits only for unique directions of the magnetic field (or for none at all);

2) Those for which there is a one-dimensional infinity (two-dimensional angle) of directions of magnetic field leading to open trajectories;

3) Those for which there is a two-dimensional infinity (solid angle) of directions of magnetic field leading to open trajectories.

The simplest example of the first class of surface is that shown in Fig. 2b; of the second class, the

\* More exactly, in order for the classical equations of motion, and in particular, the concept of a trajectory in phase space, to be valid, two conditions must be satisfied:

$$\lambda = \hbar / p_0 \ll r = cp_0 / eH, \quad a \ll r;$$

where  $a$  is the lattice constant. It is easy to see that the first condition is identical with that given above (assuming  $\epsilon_0 = p_0^2 / 2m_0$ ), and the second condition is then evidently satisfied, since  $\hbar / p_0 \approx n^{1/3}$ , where  $n$  is the density of electrons, of the order of, or less than, one electron per atom (i.e.,  $\hbar / p_0 \geq a$ ).

\*\* Note that if the  $(x, y)$  plane does not coincide with one of the crystallographic planes, these closed curves will not all be identical.

“undulating cylinder”(Fig. 2a; the two-dimensional angle is here  $2\pi$ ); of the third class, the “undulating plane”(Fig. 3; the solid angle is here  $4\pi$ , excluding only the direction perpendicular to the “plane”). A variety of multiply-connected surfaces can occur, all of which can be classified in this way. It should be noted that for surfaces of more complicated topology (as illustrated in Figs. 4 and 5) open trajectories are only encountered extremely rarely.

In the present work we shall examine in more detail the properties of conduction electrons in a magnetic field in those cases where the basic role (see below) is played by closed trajectories. The investigation of other cases involving more complicated topologies of the energy surfaces will be the subject of a separate communication.

The position of an electron on the curve (1) is determined by the time of rotation  $t$ , reckoned from some (arbitrary) point on the trajectory. From the equations of motion

$$d\mathbf{p}/dt = (e/c)[\mathbf{vH}]; \quad \mathbf{v} = \nabla_{\mathbf{p}}\varepsilon \quad (2)$$

we obtain, noting that  $\mathbf{v}_\perp$  (with components  $v_x$  and  $v_y$ ) is normal to curve (1),

$$dl/dt = -(eH/c)v_\perp; \quad v_\perp = \sqrt{v_x^2 + v_y^2}.$$

Here  $dl$  is the element arc of the curve (1), taken in the direction of motion along the trajectory in momentum space. Integrating the above equation, we obtain

$$t = -(c/eH) \int dl/v_\perp. \quad (3)$$

If the curve (1) is closed, then the period of rotation around the curve is

$$T = -\frac{c}{eH} \oint \frac{dl}{v_\perp} = -\frac{c}{eH} \frac{\partial S}{\partial \varepsilon} \quad (4)$$

Here  $S = S(\varepsilon, p_z)$  is the area of the plane  $p_z = \text{constant}$  intersected by the surface  $\varepsilon(\mathbf{p}) = \text{constant}$ . The last equation is easily obtained, noticing that

$$S = \iint dp_x dp_y = \int d\varepsilon \oint dl/v_\perp.$$

It is natural to call the quantity  $m^* = (1/2\pi) \times \partial S/\partial \varepsilon$  the effective mass of the electron in a magnetic field. For free electrons,

$$\varepsilon = p^2/2m_0, \quad S = \pi p_\perp^2 = \pi(2m_0\varepsilon - p_z^2),$$

so that  $\partial S/\partial \varepsilon = 2\pi m_0$ . We note that  $m^* = m^*(\varepsilon, p_z)$  is a function of  $\varepsilon$  and  $p_z$ --quantities which are conserved in a magnetic field. The usual definition of the effective mass tensor--

$$(1/m)_{ik} = \partial^2 \varepsilon / \partial p_i \partial p_k$$

is inconvenient in a magnetic field, since on this definition the mass varies around the trajectory. The connection between period and effective mass,

$$T = -2\pi c m^* / eH \quad (5)$$

is the same as for free electrons.

The sign of the effective mass  $m^*(\varepsilon, p_z)$  is determined by whether the energy inside the surface  $\varepsilon(\mathbf{p}) = \varepsilon$  is greater or less than  $\varepsilon$ . In the first case  $m^*$  is positive for all values of  $p_z$ ; in the second case, negative.

If the curve (1) is self-intersecting, it is not possible to establish such a simple criterion for the sign of the effective mass.

It should be emphasized that for open trajectories the concept of effective mass, naturally connected with the period of rotation around the orbit, cannot be introduced.

## 2. KINETIC EQUATION FOR CONDUCTION ELECTRON IN A MAGNETIC FIELD

So far, we have investigated the motion of a single conduction electron in a magnetic field. We now consider the electron gas in a metal in constant and uniform electric and magnetic fields. To describe the states of the electrons it is now natural to use as variables  $\varepsilon$ ,  $p_z$  and the dimensionless quantity

$$\tau = t/T_0 = (1/2\pi m_0) \int dl/v_\perp,$$

specifying the position of the electron along the trajectory in momentum space (cf. Eq. (3)). Here

$$T_0 = -2\pi c m_0 / eH, \quad (5a)$$

where  $m_0$  is a characteristic mass introduced only for convenience; since in the final formulas  $m_0$  disappears, we need not specify it more precisely.

The kinetic equation for the distribution function  $f = f(\epsilon, p_z, \tau)$  in the variables chosen has the form:

$$\frac{\partial f}{\partial \tau} \dot{\tau} + \frac{\partial f}{\partial p_z} \dot{p}_z + \frac{\partial f}{\partial \epsilon} \dot{\epsilon} + \left( \frac{\partial f}{\partial t} \right)_{\text{CT}} = 0. \quad (6)$$

We consider the stationary case; the values of  $\dot{\tau}$ ,  $\dot{p}_z$  and  $\dot{\epsilon}$  are obtained from the equations of motion

$$d\mathbf{p}/dt = (e/c)[\mathbf{vH}] + e\mathbf{E}, \quad (7)$$

where  $\mathbf{E}$  is the electric field strength.  $(\partial f/\partial t)_{\text{coll}} \equiv t_0^{-1} W\{f\}$  is the collision integral, the form of which will not concern us except in Sec. 6;  $t_0$  is a characteristic relaxation time. Using Eqs. (2) and (7), we find

$$\begin{aligned} \dot{\epsilon} &= e\mathbf{vE}; & \dot{p}_z &= eE_z; \\ \dot{\tau} &= \frac{1}{T_0} \left( 1 - \frac{c}{v_{\perp}^2 H} [\mathbf{v}_{\perp} \mathbf{E}]_z \right). \end{aligned} \quad (8)$$

In what follows we shall, as is customary, assume that the energy acquired by an electron between collisions is not only very small compared with the Fermi energy  $\zeta$ , but that it is also small compared with  $k\theta$  ( $\theta$  = temperature of the electron gas). This permits linearization of the kinetic equation with respect to the electric field  $E$ . Thus we suppose that

$$f = f_0 - et_0 \psi_i E_i, \quad f_0 = [e^{(\epsilon - \zeta)/k\theta} + 1]^{-1}, \quad (9)$$

where  $f_0$  is the equilibrium Fermi function. For  $\psi_i$  we easily get a linear equation, on neglecting terms quadratic in the electric field,

$$\partial \psi_i / \partial \tau + \gamma_0 \hat{W} \{ \psi_i \} = \gamma_0 f_0'(\epsilon) v_i. \quad (10)$$

Here  $\gamma_0 = H_0/H$ , where  $H_0$  is the magnetic field for which the period of the Larmor precession  $T_0$  is equal to the relaxation time  $t_0$  [cf. (5a)]. The role of boundary condition for Eq. (10) is played for closed orbits by the periodicity condition (with period  $T/T_0$ ) on the function  $\psi_i$ , and for open trajectories by the requirement that  $\psi_i$  shall remain finite.

Averaging Eq. (10) over  $\tau$ , we obtain

$$\overline{\hat{W} \{ \psi_i \}} = f_0'(\epsilon) \bar{v}_i. \quad (11)$$

The bar denotes an average with respect to the "time"  $\tau$ ;

$$\bar{u} = \frac{T_0}{T} \int_0^{T/T_0} u d\tau.$$

In the case of open curves the average must be taken in the limit as  $T \rightarrow \infty$ .

For the closed curves (1)

$$\bar{v}_\alpha = 0, \quad (\alpha = x, y). \quad (12)$$

Indeed, the dependence of  $\mathbf{v}$  on  $\tau$  in the linearized equation is given by the equations of motion (2):

$$v_x = -\frac{1}{2\pi m_0} \frac{\partial p_y}{\partial \tau}; \quad v_y = \frac{1}{2\pi m_0} \frac{\partial p_x}{\partial \tau}. \quad (13)$$

The averaging of the derivatives with respect to  $\tau$  for closed trajectories leads naturally to the result (12). The mean value of the  $z$ -component of velocity  $v_z$  is always different from zero, because in the direction of the magnetic field the motion is unbounded.

As remarked above, a detailed study of the various types of open trajectory will be published separately. To show that a change in the topology of the energy surfaces leads to new results, we consider briefly here the simplest case\*, in which the curves (1) are sections of the "undulating cylinder" by the planes  $p_z = \text{constant}$ . Suppose the magnetic field to be perpendicular to the cylinder axis. Choosing the  $x$  direction to be along the cylinder axis, we find, on averaging (13),

$$\bar{v}_x = 0; \quad \bar{v}_y \neq 0; \quad \bar{v}_z \neq 0. \quad (14)$$

In the case of arbitrary open curves,  $\bar{v}_\alpha \neq 0$ .

### 3. SOLUTIONS OF THE KINETIC EQUATIONS. STRONG MAGNETIC FIELD

In what follows we shall be interested in magnetic fields large compared with  $H_0$ , i.e.,  $\gamma_0 \ll 1$ . We look for a solution of Eq. (10), satisfying condition (11), in the form of a power series in  $\gamma_0$ :

$$\psi_i = \sum_{k=0}^{\infty} \gamma_0^k \psi_i^{(k)}. \quad (15)$$

\* The major problem in treating surfaces of the second and third type (cf. Sec. 1) is to determine the variation of physically interesting quantities with field direction. In the present communication this problem is not solved.

Substituting (15) in (10), we obtain a set of equations for the functions  $\psi_i^{(k)}$ :

$$\begin{aligned} \frac{\partial \psi_i^{(0)}}{\partial \tau} &= 0; \quad \frac{\partial \psi_i^{(1)}}{\partial \tau} + \hat{W} \{\psi_i^{(0)}\} = f'_0(\varepsilon) v_i; \\ \frac{\partial \psi_i^{(k)}}{\partial \tau} + \hat{W} \{\psi_i^{(k-1)}\} &= 0; \quad (k = 2, 3, \dots). \end{aligned} \quad (16)$$

From these,

$$\psi_i^{(k)} = C_i^{(k)} - \tilde{\psi}_i^{(k)}; \quad \tilde{\psi}_i^{(0)} = 0; \quad (17)$$

$$\begin{aligned} \tilde{\psi}_i^{(1)} &= \int_0^\tau \{\hat{W}(C_i^{(0)}) - f'_0(\varepsilon) v_i\} d\tau; \\ \tilde{\psi}_i^{(k)} &= \int_0^\tau \hat{W} \{\psi_i^{(k-1)}\} d\tau \quad (k = 2, 3, \dots). \end{aligned}$$

Here the  $C_i^{(k)}$  are functions only of  $\varepsilon$  and  $p_z$ , found from Eq. (11):

$$\hat{W}_0(C_i^{(0)}) = f'_0(\varepsilon) \bar{v}_i; \quad (18)$$

$$\hat{W}_0(C_i^{(k)}) = \overline{\hat{W}(\tilde{\psi}_i^{(k)})} \quad (k = 1, 2, \dots);$$

$$\hat{W}_0 = \frac{T_0}{T} \int_0^{T/T_0} \int_0^{T/T_0} \hat{W}(\varepsilon, p_z, \tau; \varepsilon', p_z', \tau') d\tau d\tau'.$$

The expressions (17) and (18) enable us in principle to calculate the functions  $\psi_i^{(k)}$ . For explicit calculations, it is necessary to know the form of the operator  $\hat{W}$ . However, many important results (on the variation of conductivity, Hall field, etc. with  $H$  in strong magnetic fields) can be obtained without explicit calculation. We shall see that it is possible to deduce the asymptotic behavior of the experimentally measurable quantities simply from a knowledge of the topology of the constant-energy surfaces in the neighborhood of the limiting Fermi energy  $\zeta$ .

Two cases may be distinguished (noting that because of the factor  $f'_0(\varepsilon)$  on the right-hand side of Eq. (10), only energies close to the Fermi energy are important]:

1) In the interval  $\delta\varepsilon \sim k\theta$  where the Fermi function is changing, there are no open trajectories for the field direction considered;

2) In the interval  $\delta\varepsilon \sim k\theta$  where the Fermi function is changing, open trajectories exist (in particular, the trajectories on the Fermi surface itself are

open)\*.

In the first case, we may use the fact that in the whole of the relevant energy interval,  $\bar{v}_\alpha = 0$  ( $\alpha = x, y$ ). We then have  $C_\alpha^{(0)} = 0$ \*\* so that [cf (17) and (13)]:

$$\begin{aligned} \psi_x &= \gamma_0 (-p_y f'_0 / 2\pi m_0 + C_x^{(1)}) + \gamma_0^2 \psi_x^{(2)} + \dots, \\ \psi_y &= \gamma_0 (p_x f'_0 / 2\pi m_0 + C_y^{(1)}) + \gamma_0^2 \psi_y^{(2)} + \dots; \\ \psi_z &= C_z^{(0)} + \gamma_0 \psi_z^{(1)} + \dots \end{aligned} \quad (19)$$

The important result emerges that  $\psi_x$  and  $\psi_z$  have different asymptotic behaviors in high fields, i.e., for  $\gamma_0$  tending to zero.

In the second case (where open trajectories exist in the interval  $\delta\varepsilon \sim k\theta$ ) zero-order terms appear in the expansion for all  $\psi_i$ :

$$\psi_i = C_i^{(0)} + \gamma_0 \psi_i^{(1)} + \dots \quad (20)$$

For the particular case of a magnetic field perpendicular to the axis of the "undulating cylinder", we obtain from (14) and (18):

$$\begin{aligned} \psi_x &= \gamma_0 (-p_y f'_0 / 2\pi m_0 + C_x^{(1)}) + \gamma_0^2 \psi_x^{(2)} + \dots; \\ \psi_{y,z} &= C_{y,z}^{(0)} + \gamma_0 \psi_{y,z}^{(1)} + \dots \end{aligned} \quad (21)$$

#### 4. THE CONDUCTIVITY TENSOR

To construct the conductivity tensor we use the expression for the current density:

$$j_i = \frac{2e}{h^3} \int v_i f(d\mathbf{p}). \quad (22)$$

Here the integral should be understood in the following sense:

$$\int \dots(d\mathbf{p}) = \lim_{G \rightarrow \infty} \frac{1}{G} \int_{(GV_0)} \dots(d\mathbf{p}), \quad (22')$$

where  $G$  is the number of cells in the reciprocal lattice and  $V_0$  the volume of one cell of the reciprocal lattice.

In the case where only closed trajectories are present, the range of integration in (22') must be

\* Various cases can then be distinguished, as was discussed above.

\*\* The first of Eqs. (18) has in this case only the trivial solution  $C_\alpha^{(0)} = 0$ . This may be easily deduced from the fact that otherwise all the remaining equations for the  $C_\alpha^{(k)}$  would have no solution.

chosen so that each orbit lies wholly within it. In particular, if these trajectories belong only to closed surfaces (more exactly, if only such trajectories are of importance), the integration over all  $p$ -space can be reduced to a sum of integrations over separate cells of the reciprocal lattice (summed over the different zones), where in each integral (i.e., in each zone) the cell must be chosen so that the whole closed surface lies within it.

Substituting for  $f$  in terms of the functions  $\psi_i$  and comparing the expression obtained with Chm's law  $j_i = \sigma_{ik} E_k$ , we obtain

$$\sigma_{ik} = -2e^2 t_0 h^{-3} \int v_i \psi_k (d\mathbf{p}). \quad (23)$$

We note that  $\sigma_{ik}(H) = \sigma_{ki}(-H)$  in virtue of the principle of symmetry of kinetic coefficients (Ref. 7, Sec. 118). These relations are easily obtained by making direct use of the kinetic equations for the function  $f$  and the Hermitian property of the collision operator.

Substituting the value of  $\psi_k$  from (19) into (23), we obtain in the *first case* (no open trajectories in the interval where the Fermi function is changing) a conductivity tensor of the form:

$$\hat{\sigma}_{ik}(H) = \begin{pmatrix} \gamma_0^2 a_{xx} & \gamma_0 a_{xy} & \gamma_0 a_{xz} \\ \gamma_0 a_{yx} & \gamma_0^2 a_{yy} & \gamma_0 a_{yz} \\ \gamma_0 a_{zx} & \gamma_0 a_{zy} & a_{zz} \end{pmatrix}. \quad (24)$$

The expansion of the matrix elements  $a_{ik}$  in powers of  $\gamma_0$  in general begins with the zero order term. It should be noted that in special cases some terms may be zero from symmetry requirements (for instance,  $a_{xz} = a_{yz} = a_{zx} = a_{zy} = 0$  in the isotropic case)\*.

All the matrix elements  $a_{ik}$  are functionals of  $C_i^{(k)}$  and consequently depend on the collision integral (cf. below, Sec. 5). An exception is  $a_{xy}^{(0)} (= -a_{yx}^{(0)})$ —the zero-order term in the expansion of  $a_{xy}$  in powers of  $\gamma_0$ . Indeed, from (23) and (19),

$$a_{xy}^{(0)} = -\frac{2e^2 t_0}{k^3} \int v_x \left( \frac{p_x}{2\pi m_0} + C_y^{(0)} \right) f_0'(\varepsilon) (d\mathbf{p}).$$

changing the variables  $\varepsilon, p_z$  and  $\tau$ , and noting that the Jacobian of the transformation is

$$\partial(p_x, p_y, p_z) / \partial(\varepsilon, \tau, p_z) = 2\pi m_0,$$

that  $v_x = -(1/2\pi m_0) \partial p_y / \partial \tau$ , and that  $C_y^{(0)}$  is independent of  $\tau$  [cf. (17)], we obtain

$$\begin{aligned} a_{xy}^{(0)} &= -\frac{e^2 t_0}{\pi m_0 k^3} \int f_0'(\varepsilon) \frac{\partial p_y}{\partial \tau} p_x dp_z d\varepsilon \\ &= -\frac{e^2 t_0}{\pi m_0 k^3} \int f_0'(\varepsilon) d\varepsilon \int dp_z \oint p_x dp_y. \end{aligned}$$

The inner integral  $\oint p_x dp_y = \pm S(\varepsilon, p_z)$ , where  $S(\varepsilon, p_z)$  is the area of the plane  $p_z = \text{constant}$  intersected by the surface  $\varepsilon(\mathbf{p}) = \varepsilon$ , and the sign is determined by the sense of rotation around the orbit, i.e., by the sign of the effective mass  $m^*$  (cf. Sec. 1). Since  $\sigma_{xy} = \gamma_0 a_{xy}$ , we have asymptotically in large fields:

$$\begin{aligned} \sigma_{xy} &= -\frac{ec}{H} \frac{2}{k^3} \left\{ \int \frac{\partial f_0}{\partial \varepsilon} d\varepsilon \right. \\ &\quad \left. \times \left[ \int S_1(\varepsilon, p_z) dp_z - \int S_2(\varepsilon, p_z) dp_z \right] \right\}. \end{aligned}$$

Here the first integral is taken over those parts of the zone where  $m^* > 0$ , and the second over those parts where  $m^* < 0$ .

If we put  $-f_0'(\varepsilon) = \delta(\varepsilon - \zeta)$ , we get

$$\begin{aligned} \sigma_{xy} &= \frac{ec}{H} \frac{2}{k^3} \left\{ \int S_1(\zeta, p_z) dp_z - \int S_2(\zeta, p_z) dp_z \right\}. \quad (25) \end{aligned}$$

Thus finally, if the closed trajectories under consideration are situated on closed surfaces, we obtain

$$\sigma_{xy} = (2ec / Hh^3) (V_1(\zeta) - V_2(\zeta)).$$

Here  $V_1(\zeta)$  is the volume enclosed by the surfaces  $\varepsilon(\mathbf{p}) = \zeta$ , within which the energy is less than  $\zeta$  (i.e.,  $m^* > 0$ ), and  $V_2(\zeta)$  the volume enclosed by the surfaces  $\varepsilon(\mathbf{p}) = \zeta$ , within which the energy is greater than  $\zeta$  (i.e.,  $m^* < 0$ ). Noting that states with energy less than  $\zeta$  are occupied by electrons, we obtain

$$\sigma_{xy} = ec(n_1 - n_2) / H. \quad (26)$$

Here  $n_1$  is the number of states occupied by electrons with positive effective mass, and  $n_2$  the

\* The first terms in the expansion of  $\sigma_{xx}$  and  $\sigma_{yy}$  in powers of  $\gamma_0$  vanish because of the particular form of  $\psi_x^{(1)}$  and  $\psi_y^{(1)}$  in (19) [cf. below, calculation of  $a_{yx}^{(0)}$ ].

number of unoccupied states with negative effective mass. It is natural to call  $n_1$  the number of "electrons", and  $n_2$  the number of "holes".

It should be emphasized that it is only possible to introduce these concepts if in the energy interval  $\delta\epsilon$  there exist only closed, nonintersecting energy surfaces. Thus, if in the interval  $\delta\epsilon$  there exist only closed surfaces, the asymptotic value of  $\sigma_{xy}$  is given by Eq. (26). In this case we note that  $\sigma_{xy}$  does not depend on the direction of the magnetic field. If, however, the closed trajectories have arisen as sections of open surfaces, then the transition from (25) to (26) cannot be made. In this case  $\sigma_{xy}$  is as before proportional to  $1/H$ , but is now strongly dependent on the direction of the magnetic field.

The case  $n_1 = n_2$  needs special consideration (here, of course, we are still considering closed energy surfaces). In this case  $a_{xy}^{(0)} = 0$ , i.e., generally speaking\*,

$$\sigma_{xy} \sim \gamma_0^2 = H_0^2 / H^2. \tag{27}$$

It should be noted that equality of the numbers of electrons and "holes" is not something exceptional; all metals with an even number of electrons possess conduction electrons only because the energy bands in the metal overlap. It is natural that the number of vacant states (number of "holes") in the lower zone should be equal to the number of occupied states (number of electrons in the upper zone [ cf. Fig. 6). We recall that it is

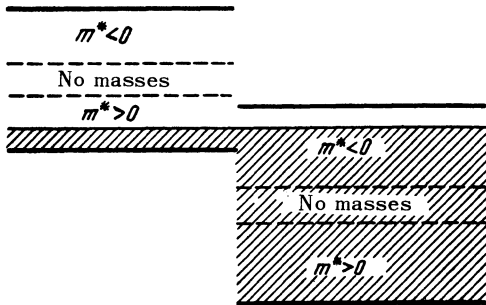


FIG. 6. Shading indicates occupation of states at  $\theta = 0$ .

only possible to speak of "holes" and "electrons" when the empty and occupied states correspond to closed energy surfaces. This will always be the case if the overlap of energy zones is not too great (Fig. 6). Strictly speaking, exact compensation ( $n_1 = n_2$ ), leading to the result (27), is possible only at the absolute zero. Thermal excitation will cause the removal of electrons from deeper states in the zone, not equivalent to closed surfaces and therefore not contributing to the "number of holes". Thus, although the number of electrons in the upper zone is equal to the number of vacant states in the lower zone, the equation  $n_1 - n_2 = 0$  is violated in this case. Evidently,

$$n_1 - n_2 \sim ne^{-\Delta\epsilon/k\theta}, \tag{28}$$

where  $n$  is the number of electrons in the zone and  $\Delta\epsilon$  is the energy gap between the Fermi surface and the nearest open surface.

Thus, if there exist no open orbits in the interval  $\delta\epsilon$ , all  $\sigma_{ik}$  (except  $\sigma_{zz}$ ) tend to zero as  $H$  tends to infinity ( $H \gg H_0$ ); but the nature of the asymptotic approach to zero is different for the different coefficients [ cf. Eq. (24)].

This statement is valid only at the absolute zero of temperature ( $\theta = 0$ ). At finite temperatures, i.e., if  $f'_0(\epsilon) \neq -\delta(\epsilon - \zeta)$ , we cannot in general put  $\bar{v}_\alpha = 0$  everywhere, since among all trajectories there may be open ones. This leads to the appearance of zero-order terms in the expansion of all the  $\sigma_{ik}$  with respect to  $\gamma_0$ . However, owing to the factor  $f'_0(\epsilon)$ , the zero-order terms  $\psi_i^{(0)}$  are of order  $\exp(-\Delta\epsilon_1/k\theta)$ , where  $\Delta\epsilon_1$  is a quantity of the order of the energy gap from the Fermi surface to the nearest surface containing open trajectories. The appearance of zero-order terms in the expansion of  $\psi_i$  leads to the appearance of zero-order terms in the expansion with respect to  $\gamma_0$  of all the matrix elements  $\sigma_{ik}$ . However, in all the elements  $\sigma_{ik}$  (except  $\sigma_{zz}$ ) these terms are negligibly small up to fields of order  $H_0 \exp(\Delta\epsilon_1/k\theta)$ , which are at present unattainable\*.

\* In the special case of isotropic dispersion and  $\hat{W} \equiv 1$ ,  $\sigma_{xy} \sim \gamma_0^3$  for  $n_1 = n_2$ .

\* Moreover, the quantization of orbits in a magnetic field<sup>6</sup> must be taken into account at a much earlier stage.



In the second case (when open trajectories do exist in the interval  $\delta\epsilon \sim k\theta$ ) all  $\psi_i^{(0)}$  are different from zero (20). In consequence of this, all the matrix elements  $\sigma_{ik}$  tend to saturation for  $H \gg H_0$ . Moreover, all the  $\sigma_{ik}$  now depend on temperature even at low temperatures, where normally (in the absence of a magnetic field) the kinetic coefficients depend only feebly on temperature.

When the energy surfaces are of the second or third type (cf. Sec. 1), an extremely strong anisotropy should be observed in the dependence of  $\sigma_{ik}$  on magnetic field. In fact, as the direction of the magnetic field approaches that in which there exist open normal sections, the size of the tensor components  $\sigma_{ik}$  changes considerably [because of the change in character of the solutions of (16)]. For instance, in the simplest case of the "undulating cylinder" (see above), as long as the magnetic field  $H$  is not perpendicular to the cylinder axis, the asymptotic dependence of  $\sigma_{ik}$  on magnetic field is given by (24), while if  $H$  is perpendicular to the axis,

$$\sigma_{ik}(\mathbf{H}) = \begin{pmatrix} \gamma_0^2 a'_{xx} & \gamma_0 a'_{xy} & \gamma_0 a'_{xz} \\ \gamma_0 a'_{yx} & a'_{yy} & a'_{yz} \\ \gamma_0 a'_{zx} & a'_{zy} & a'_{zz} \end{pmatrix} \quad (29)$$

[cf. (23) and (21)]. As before, expansion of the matrix elements  $a'_{ik}$  in powers of  $\gamma_0$  begins with the zero-order term.

Thus, an experimental determination of the asymptotic behavior of  $\sigma_{ik}$  in strong magnetic fields for various orientations in principle makes possible a clarification of the topology of the constant-energy surfaces in the region of the Fermi energy.

It should be noted that the usual experimental arrangement, in which the electric field is measured for only one current direction, does not permit a determination of the dependence of all  $\sigma_{ik}$  on magnetic field. It is essential to compare the results of measurements for three noncoplanar directions of current flow. There are at present no experimental results which can be used for this purpose.

## 5. FOURIER METHOD

In cases where only electrons moving along closed trajectories are important (no open tra-

jectories in the interval  $\delta\epsilon \sim k\theta$ ), the Fourier method can be used to analyze the solution of Eq. (10). For convenience, the position of the electron in its orbit in momentum space will be specified in this section by the phase  $\varphi$ , varying from 0 to  $2\pi$ :

$$\varphi = 2\pi \frac{t}{T} = 2\pi \int \frac{dl}{v_{\perp}} / \oint \frac{dl}{v_{\perp}} \quad (30)$$

[cf. (3) and (4)].

Expressing Eq. (10) in terms of the new variable  $\varphi$ , we obtain

$$(\partial\psi/\partial\varphi) + \gamma \hat{W} \{\psi\} = \gamma f'_0(\epsilon) \mathbf{v}. \quad (31)$$

Here  $\gamma = (T/2\pi T_0) \gamma_0 = T/2\pi t_0$  and is a function of  $\epsilon$  and  $p_z$  in distinction to the constant  $\gamma_0$  introduced in Sec. 2. We now expand  $\mathbf{v}$  and  $\psi$  in Fourier series:

$$\mathbf{v}(\epsilon, p_z, \varphi) = \sum_{k=-\infty}^{\infty} \mathbf{v}_k(\epsilon, p_z) e^{ik\varphi}, \quad (32)$$

$$\psi(\epsilon, p_z, \varphi) = \sum_{k=-\infty}^{\infty} \psi_k(\epsilon, p_z) e^{ik\varphi}.$$

Expressed in Fourier components, Eq. (31) becomes:

$$ik\psi_k + \gamma \hat{W}_{kk'} \psi_{k'} = \delta f'_0(\epsilon) \mathbf{v}_k; \quad (33)$$

here  $\hat{W}_{kk'}$  is the Fourier component of the operator  $\hat{W}$  and is an operator with respect to the variables  $\epsilon$  and  $p_z$ ; summation over doubly occurring indices is understood. Note that the equations of motion (2) (connecting the quantities  $\mathbf{p}$  and  $\mathbf{v}$  entering into the linearized equations), when applied to the Fourier components of  $\mathbf{p}$  and  $\mathbf{v}$  assume the form

$$ik\mathbf{p}_k = m^* [\mathbf{v}_k \mathbf{n}]; \quad \mathbf{H} = H\mathbf{n}. \quad (34)$$

The vector  $\psi$  can be represented as follows:

$$\psi = [\mathbf{n}\boldsymbol{\varphi}] + \theta \mathbf{n}; \quad \boldsymbol{\varphi} = [\psi \mathbf{n}]; \quad \theta = (\psi \mathbf{n}). \quad (35)$$

From (33), (34) and (35) we have

$$ik\boldsymbol{\varphi}_k + \gamma \hat{W}_{kk'} \boldsymbol{\varphi}_{k'} = ik\gamma f'_0(\epsilon) \mathbf{p}_k / m^*; \quad (36)$$

$$ik\theta_k + \gamma \hat{W}_{kk'} \theta_{k'} = \gamma f'_0(\epsilon) v_k^z.$$

In expanding  $\varphi_k$  and  $\theta_k$  in powers of  $\gamma$  it is convenient first to eliminate  $\varphi_0$  and  $\theta_0$ . Substituting  $k = 0$  in Eqs. (36), we have

$$\begin{aligned}\varphi_0 &= -\hat{W}_{00}^{-1} \hat{W}_{0k'} \varphi_{k'}; \\ \theta_0 &= \hat{W}_{00}^{-1} \{f'_0(\varepsilon) v_0^z - \hat{W}_{0k'} \theta_{k'}\} \quad (k' \neq 0),\end{aligned}\quad (37)$$

from which

$$\begin{aligned}\varphi_k - \gamma \hat{L}_{kk'} \varphi_{k'} &= \gamma f'_0(\varepsilon) \mathbf{p}_k / m^*, \\ \theta_k - \gamma \hat{L}_{kk'} \theta_{k'} &= \gamma h_k.\end{aligned}\quad (k, k' \neq 0) \quad (38)$$

Here

$$\begin{aligned}\hat{L}_{kk'} &= \frac{i}{k} \{\hat{W}_{kk'} - \hat{W}_{k0} \hat{W}_{00}^{-1} \hat{W}_{0k'}\}, \\ h_k &= -\frac{i}{k} \{f'_0(\varepsilon) v_k^z - \hat{W}_{k0} \hat{W}_{00}^{-1} f'_0(\varepsilon) v_0^z\}.\end{aligned}\quad (39)$$

The solution of Eq. (38) may be expressed as follows:

$$\begin{aligned}\varphi_k &= (1 - \gamma \hat{L})_{kk'}^{-1} \gamma f'_0(\varepsilon) \frac{\mathbf{p}_{k'}}{m^*} \\ &= \gamma f'_0 \frac{\mathbf{p}_k}{m^*} + \gamma \hat{L}_{kk'} \gamma f'_0 \frac{\mathbf{p}_{k'}}{m^*} + \dots, \\ \theta_k &= (1 - \gamma \hat{L})_{kk'}^{-1} \gamma h_{k'} = \gamma h_k + \gamma \hat{L}_{kk'} \gamma h_{k'} + \dots\end{aligned}\quad (40)$$

From this and (37) it can be seen that the expansion of  $\varphi_k$  and  $\theta_k$  in powers of  $\gamma$  begins with the first power ( $\sim 1/H$ ). An exception is  $\theta_0$ , for which the expansion begins with the term of zero order

$$\theta_0 = \hat{W}_{00}^{-1} f'_0(\varepsilon) v_0^z - \hat{W}_{0k'} \gamma h_{k'} + \dots \quad (41)$$

The same results might, of course, have been obtained by the method of successive approximations, starting from Eqs. (16) and (18).

We note that the linear term in the expression for  $\varphi_k$  does not depend on the collision integral [cf. Eq. (21)]. Using Eqs. (22), (9) and (35), we find

$$\mathbf{j} = -\frac{2e^2}{h^3} \int \{(\varphi[\mathbf{E}\mathbf{n}] + \theta(\mathbf{n}\mathbf{E}))\} \mathbf{v}(d\mathbf{p})$$

from which

$$\begin{aligned}\sigma_{\alpha\beta} &= -\frac{2e^2 t_0}{h^3} \int v_\alpha [\mathbf{n}\varphi]_\beta(d\mathbf{p}); \\ \sigma_{iz} &= -\frac{2e^2 t_0}{h^3} \int \theta v_i(d\mathbf{p}).\end{aligned}\quad (42)$$

( $\alpha, \beta = x, y$ ). We recall that  $\sigma_{ik}(\mathbf{H}) = \sigma_{ik}(-\mathbf{H})$ .

Substituting the Fourier expansions of  $\varphi$  and  $\theta$  in these formulas, we obtain

$$\begin{aligned}\sigma_{\alpha\beta} &= -\frac{2e^2 t_0}{h^3} \sum_{h=-\infty}^{\infty} \int v_{-h}^\alpha [\mathbf{n}\varphi_h]^\beta(d\mathbf{p}); \\ \sigma_{iz} &= -\frac{2e^2 t_0}{h^3} \sum_{h=-\infty}^{\infty} \int v_h^i \theta_h(d\mathbf{p}),\end{aligned}$$

or, since  $\mathbf{v}$ ,  $\varphi$  and  $\theta$  are real quantities and  $v_0^\alpha = 0$ ,

$$\begin{aligned}\sigma_{\alpha\beta} &= -\frac{4e^2 t_0}{h^3} \operatorname{Re} \sum_{k=1}^{\infty} \int v_{-k}^\alpha [\mathbf{n}\varphi_k]^\beta(d\mathbf{p}); \\ \sigma_{iz} &= -\frac{2e^2 t_0}{h^3} \left\{ \int v_0^i \theta_0(d\mathbf{p}) + 2 \operatorname{Re} \sum_{k=1}^{\infty} \int v_{-k}^i \theta_k(d\mathbf{p}) \right\}.\end{aligned}$$

Using the expressions (40) and (31), and the equations of motion (34), we obtain the components of the conductivity matrix in powers of the reciprocal of the magnetic field:

$$\begin{aligned}\sigma_{xx} &= -\frac{4c^2}{h^3 H^2} \sum_{k=1}^{\infty} \\ &\times \int p_{-k}^y (\hat{\mathcal{L}}_{kk'} - \hat{\mathcal{L}}_{k0} \hat{\mathcal{L}}_{00}^{-1} \hat{L}_{0k'}) p_{k'}^y f'_0(\varepsilon)(d\mathbf{p}) + \dots; \\ \sigma_{yy} &= -\frac{4c^2}{h^3 H^2} \sum_{k=1}^{\infty} \\ &\times \int p_{-k}^x (\hat{\mathcal{L}}_{kk'} - \hat{\mathcal{L}}_{k0} \hat{\mathcal{L}}_{00}^{-1} \hat{L}_{0k'}) p_{k'}^x f'_0(\varepsilon)(d\mathbf{p}) + \dots; \\ \sigma_{xy} &= (n_1 - n_2) ec / H + \dots; \\ \sigma_{zz} &= -\frac{2e^2}{h^3} \int v_0^z \hat{\mathcal{L}}_{00}^{-1} f'_0(\varepsilon) v_0^z(d\mathbf{p}) + \dots; \\ \sigma_{xz} &= \frac{4ec}{h^3 H} \operatorname{Re} \sum_{k=1}^{\infty} \\ &\times \int p_{-k}^y \{f'_0(\varepsilon) v_k^z - \hat{\mathcal{L}}_{k0} \hat{\mathcal{L}}_{00}^{-1} f'_0(\varepsilon) v_0^z\}(d\mathbf{p}) + \dots; \\ \sigma_{yz} &= -\frac{4ec}{h^3 H} \operatorname{Re} \sum_{k=1}^{\infty} \\ &\times \int p_{-k}^x \{f'_0(\varepsilon) v_k^z - \hat{\mathcal{L}}_{k0} \hat{\mathcal{L}}_{00}^{-1} f'_0(\varepsilon) v_0^z\}(d\mathbf{p}) + \dots.\end{aligned}$$

( $\hat{\mathcal{L}} = t_0 \hat{W}$  is the collision integral). From these expressions it is simple to derive expressions for

the matrix elements  $a_{ik}$ , but the explicit form of these will not be required in what follows.

The solutions of Eqs. (33) are particularly simple in cases where a time of relaxation exists, i.e., when\*

$$\hat{\mathcal{L}} \equiv 1/t_0 \quad (\hat{\mathcal{L}}_{kk'} \equiv (1/t_0) \delta_{kk'}),$$

$$\psi_k = \gamma f'_0(\varepsilon) \mathbf{v}_k / (\gamma + ik). \quad (43)$$

We then have

$$\psi = \gamma f'_0(\varepsilon) \sum_{k=-\infty}^{\infty} \frac{\mathbf{v}_k e^{ikh\varphi}}{ik + \gamma}, \quad (44)$$

$$\sigma^{xx'}(\mathbf{H}) = -\frac{2e^2 t_0}{h^3} \sum_{k=-\infty}^{\infty} \int \frac{\mathbf{v}_{-k}^x \mathbf{v}_k^{x'}}{ik + \gamma} \gamma f'_0(\varepsilon) (d\mathbf{p}), \quad (45)$$

or

$$\sigma^{xx'}(\mathbf{H}) = -\frac{2e^2 t_0}{h^3} \left\{ \int \overline{\mathbf{v}^x} \overline{\mathbf{v}^{x'}} f'_0(\varepsilon) (d\mathbf{p}) + 2\text{Re} \sum_{k=1}^{\infty} \int \frac{(\mathbf{v}_k^x)^x \mathbf{v}_k^{x'}}{ik + \gamma} \gamma f'_0(\varepsilon) (d\mathbf{p}) \right\}. \quad (46)$$

In the case of a quadratic isotropic dispersion law, only one term remains in the sum, and we easily obtain the well-known formulas:

$$\sigma_{xx} = \sigma_{yy} = \sigma / (1 + \Omega_0^2 t_0^2); \quad \sigma_{zz} = \sigma; \quad (47)$$

$$\sigma_{xy} = -\sigma_{yx} = \sigma \Omega_0 t_0 / (1 + \Omega_0^2 t_0^2); \quad \sigma_{xz} = \sigma_{yz} = 0$$

(where  $\sigma$  is the conductivity of the metal in the absence of a magnetic field, and  $\Omega_0 = 2\pi/T_0$  is the frequency of Larmor precession). It should be noted that such a dependence of the conductivity tensor on the magnetic field is obtained in all cases where an isotropic dispersion law is under consideration (independent of the detailed form of the collision integral). This is connected with the fact that in all such cases the action of the collision operator on a function of the form  $\chi_i(\varepsilon) v_i$  reduces to a multiplication of the latter by a quantity depending on energy but not on angle<sup>4</sup>. This leads to the identical vanishing of all  $\psi_k$  except  $\psi_0$  and  $\psi_{\pm 1}$ . This can be seen from Eq. (41), if we notice that in the isotropic case  $V_k = 0$  ( $k = \pm 2, \pm 3, \dots$ ), and we obtain Eq. (47).

\*  $t_0$  may here depend on  $\varepsilon$  and  $p_z$ .

## 6. ARBITRARY MAGNETIC FIELD ( $\gamma \sim 1$ )

The Fourier method is applicable only if the trajectories in momentum space are closed. If they are open, the expressions obtained in Sec. 5 have no meaning.

In the present section compact formulas are obtained for the conductivity in a magnetic field for those cases where the collision integral may be written in the form  $(f - f_0)/t_0$  (i.e.,  $\hat{W} \equiv 1$ ). In such a case Eq. (10) becomes a differential equation instead of an integro-differential equation\*:

$$\partial \psi_i / \partial \tau + \gamma_0 \psi_i = \gamma_0 f'_0(\varepsilon) v_i.$$

The solution of this equation, satisfying (11), has the form

$$\psi_i = \gamma_0 \int_0^{\infty} e^{-\gamma_0 \tau'} v_i (\tau - \tau') d\tau' f'_0(\varepsilon). \quad (48)$$

Substituting this expression for  $\psi_i$  into formula (23) for  $\sigma_{ik}$ , changing the order of integration and averaging, we obtain

$$\sigma_{ik} = -\frac{2e^2 t_0}{h^3} \int \varphi_{ik}(\varepsilon, p_z) f'_0(\varepsilon) (d\mathbf{p}), \quad (49)$$

$$\varphi_{ik}(\varepsilon, p_z) = \gamma_0 \int_0^{\infty} e^{-\gamma_0 \xi} \overline{v_i(\tau) v_k(\tau + \xi)} d\xi.$$

These expressions are convenient for investigating explicit laws of dispersion.

## 7. RESISTIVITY AND HALL FIELD

For the investigation of galvanomagnetic phenomena, it is usual to let a current of a given magnitude flow through the crystal in a given direction, and to measure the components of electric field in three noncoplanar directions (as far as possible, three orthogonal directions; cf, for example, Ref. 3). What is then studied is the resistivity tensor in a magnetic field.

$$\rho_{ik}(\mathbf{H}) = \sigma_{ik}^{-1}(\mathbf{H}).$$

Using the expressions obtained earlier for the asymptotic behavior of  $\sigma_{ik}(\mathbf{H})$ , it is easy to find the asymptotic behavior in high magnetic fields ( $H \gg H_0$ ) of the resistivity tensor  $\rho_{ik}(\mathbf{H})$ .

\* We revert now to the notation of Secs. 1-4.

1) When only closed trajectories are important.

a) According to (24),

$$\rho_{ik}(\mathbf{H}) = \begin{pmatrix} b_{xx} & \gamma_0^{-1} b_{xy} & b_{xz} \\ \gamma_0^{-1} b_{yx} & b_{yy} & b_{yz} \\ b_{zx} & b_{zy} & b_{zz} \end{pmatrix}. \quad (50)$$

$$\begin{aligned} b_{xx} &= (a_{yy} a_{zz} + a_{yz}^2) / a_{zz} a_{xy}^2; & b_{xy} &= -b_{yx} = 1 / a_{yx}; \\ b_{xz} &= -b_{zx} = a_{zy} / a_{zz} a_{xy}; & b_{yy} &= (a_{xx} a_{zz} + a_{xz}^2) / a_{zz} a_{xy}^2; \\ & & b_{yz} &= -b_{zy} = a_{zx} / a_{zz} a_{xy}; & b_{zz} &= 1 / a_{zz}. \end{aligned}$$

Thus if in the interval  $\delta\epsilon$  there are only closed surfaces, then  $\rho'_{xx}, \rho_{yy} \sim (n_1 - n_2)^{-2}$ , and

$$\rho_{xy} / H = 1 / (n_1 - n_2) ec + \dots \quad (51)$$

If, however, the closed trajectories are formed by sections of open surfaces (cf. Sec. 4), then

(52)

$$\rho_{xy} / H = \hbar^3 / 2 \left\{ \int S_1(\zeta, p_z) dp_z - \int S_2(\zeta, p_z) dp_z \right\}$$

and depends markedly on the direction of the magnetic fields.

Generally in experimental work the current  $\mathbf{j}$  is perpendicular to the magnetic field. Let us take the  $x$  axis parallel as to the direction of current. Then the resistivity  $\rho$  (the ratio of electric field strength parallel to the current to the current density) is  $\rho_{xx}$ \*, and  $\rho_{xy} / H$  coincides with the Hall "constant"  $R = E_y / H j$ . Thus, in this case the resistivity tends to a saturation value, and  $\rho$  depends markedly on the direction of magnetic field independently of whether the closed trajectories arise as sections of closed surfaces or not.

The asymptotic behavior of the Hall "constant" is different in these two cases. If the surfaces are closed, then from (51)  $R$  is a constant determining the difference between electrons and "holes". If, however, the surfaces are open (though the sections are closed), then  $R$  is a complicated function of angle [cf. (52)]. Thus a study of the Hall field in large magnetic fields is particularly important in studying the topology of the Fermi surface.

We should note the following fact: Generally, by

\* For arbitrary current direction ( $n_i$ ),  $\rho = \rho_{ik} n_i n_k$ .

The expansion of the matrix elements  $b_{ik}$  in powers of  $\gamma_0$  begins with the zero-order terms, which are determined by the zero-order terms of the matrix  $a_{ik}$  as follows:

Hall field we mean the electric field, perpendicular to the current, arising when a current passes through the metal in a magnetic field and changing sign when the magnetic field changes sign, i.e., with our choice of coordinates,

$$E_{\text{Hall}} = 1/2 [E_y(\mathbf{H}) - E_y(-\mathbf{H})]. \quad (53)$$

On such a definition, the even powers of  $\mathbf{H}$  naturally vanish. We should note that in the case considered the leading term in the asymptotic behavior of  $E_y(\mathbf{H})$  contains the first (odd) power of magnetic field.

b)  $n_1 = n_2$ . In this case, as has been pointed out,  $\sigma_{xy} \sim H_0^2 / H^2$ . Consequently,

$$\rho_{\alpha\beta} \sim H^2 / H_0^2; \quad \rho_{\alpha z} \sim H / H_0;$$

$$\rho_{zz} \sim \text{const} \quad (\alpha, \beta = x, y)$$

From this we see that in this case the resistivity grows quadratically with magnetic field. The significant difference of this case, moreover, lies in the presence of quadratic terms (in magnetic field) in the expression for  $E_y(\mathbf{H})$ , i.e.,

$$E_y(\mathbf{H}) = (AH^2 + BH) j,$$

where  $A$  and  $B$  are constants depending on the explicit form of the collision integral and the dispersion law. Because of this the Hall field (53) in the present case does not characterize the leading term in the asymptotic behavior of the electric field perpendicular to the current. Note that  $A$  is connected with the anisotropy of the crystal. In

the isotropic case  $A = 0^*$ .

This discussion shows that the experimentally well-known existence of a group of metals (Be, Zn, Cd, Mg, Ga, Sn, Pb, C, Bi, Sb, As, Mo, W, Ba) with anomalous group of resistance follows from the most general considerations of the modern theory of metals. Previously these results have been obtained only under extremely special assumptions<sup>8</sup>.

The linear growth of resistivity with magnetic field observed by Kapitza<sup>2</sup> in many metals corresponds to a transition from a quadratic dependence in small fields to a quadratic dependence with a different coefficient in high fields. The same conclusion was reached by Borovik<sup>3</sup>, from analysis of the data in the literature and also from experimental observation of both the quadratic dependencies and the transition region between them, which approximates well to a linear dependence.

The influence of temperature, i.e., the effect of the exponentially small terms (of order  $\exp(-\Delta\epsilon/k\theta)$ --cf. (28)] leads to a saturation of all components of the resistivity tensor. However, this saturation is reached only in extremely high fields, or order  $H \sim H_0 \exp(\Delta\epsilon/k\theta)$ .

2) When open orbits do exist in the range  $\delta\epsilon \sim k\theta$ .

Generally speaking, in this case all the matrix elements  $\rho_{ik}$  tend to saturation, which naturally

\* Note that in the presence of two overlapping zones with quadratically anisotropic dispersion laws ( $n_1 = n_2 = n$ ) and with proportional mobility tensors ( $u_{ik}^{(2)} = ku_{ik}^{(1)}$ )

$$\rho = \frac{\sigma_2 \cos^2 \alpha + \sigma_1 \sin^2 \alpha}{\sigma_1 \sigma_2} \left( 1 + \frac{H_2}{H_0^2} \right);$$

$$R = \frac{E_{\text{Hall}}}{jH} = - \frac{1}{nec} \frac{1-k}{1+k};$$

$$E_y^{(S)} = \frac{E_y(\mathbf{H}) + E_y(-\mathbf{H})}{2j} \\ = \frac{\sin \alpha \cos \alpha (\sigma_2 - \sigma_1)}{\sigma_1 \sigma_2} \left( 1 + \frac{H^2}{H_0^2} \right).$$

Here  $H_0 = c/e\sqrt{u_1^{(1)}u_2^{(2)}}$ ;  $\sigma_{1,2}$  are the principal values of the conductivity tensor  $\sigma_{ik} = ne^2u_{ik}(1+k)$ . The magnetic field is chosen to lie along one of the principal axes (the third), the current is perpendicular to the magnetic field and  $\alpha$  is the angle between the axis  $l$  and the current.

leads to saturation of the resistivity. The Hall "constant", however, decreases quadratically with increasing magnetic field. However, the symmetry of the open surface may substantially change these results. For instance, in the special case of the "undulating cylinder" (with magnetic field perpendicular to cylinder axis), according to (29) the matrix  $\rho_{ik}$  has the form

$$\rho_{ik}(\mathbf{H}) = \begin{pmatrix} \gamma_0^{-2}b'_{xx} & \gamma_0^{-1}b'_{xy} & \gamma_0^{-1}b'_{xz} \\ \gamma_0^{-1}b'_{yx} & b'_{yy} & b'_{yz} \\ \gamma_0^{-1}b'_{zx} & b'_{zy} & b'_{zz} \end{pmatrix}.$$

The elements of the matrix  $b'_{ik}$  are determined by the values of the elements  $a'_{ik}$ . It should be remembered that in the present case the  $x$  axis is along the axis of the cylinder.

Thus, if the current flows in the  $(yz)$  plane, the resistivity tends to saturation; in other cases it grows quadratically.

The Hall field always grows linearly with magnetic field. The Hall "constant" has no simple physical meaning.

At the present time, apparently, there are still no experimental data on the asymptotic behavior of the tensor  $\rho_{ik}$  in high fields which can be used to draw conclusions about the topology of the Fermi surface.

In conclusion, the authors would like to thank L. D. Landau and E. S. Borovik for helpful discussions.

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## Structure of the Intermediate State of Superconductors

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The structure of the intermediate state was studied by application of fine ferromagnetic powder to the surface of a tin specimen. Two-dimensional pictures of structures of various types were obtained for various contents of normal phase in the specimen. The influence of a number of other factors (method of transition, temperature, specimen size and others) on the character of the picture was also studied.

### INTRODUCTION

AS is well known, a superconductor in the intermediate state is broken up into regions or "domains" of superconducting and normal phases ( $s$ - and  $n$ - phases). The magnetic field is concentrated almost entirely in the  $n$ - domains. Thus the field outside the specimen and very close to its surface is also nonuniform. The dimensions of the  $s$ - and  $n$ -domains in thermodynamic equilibrium must be determined by two basic factors which act in opposing directions. The field energy close to the specimen will be reduced by reduction of the field inhomogeneity and from this point of view it is energetically favorable to increase the degree of disperseness of the state. On the other hand the presence of a positive surface tension at the phase boundaries tends to increase the dimensions of the domains.

Agreement between experimental and theoretical investigation of the geometry of the intermediate state would therefore permit the development of a method of finding the magnitude of the surface tension from observation of the equilibrium dimensions of the structure. A knowledge of this quantity and its dependence on various factors (temperature, crystallographic orientation of the phase boundary and others) would be very useful for the development of the theory of superconductivity.

This problem however is a long way from being

solved. Mathematical difficulties prevent consideration in a general form of the problem of the shape of the phase boundaries. Landau succeeded, however, in giving exact calculation of the shapes and sizes of the domains under certain simplifying conditions. In one calculation nearly plane parallel  $s$ - and  $n$ - layers were considered,<sup>1</sup> while in another<sup>2</sup> a model was considered in which the  $n$ - layers repeatedly branched as they approached the external surface so that the structure was homogeneous or "mixed" at the surface. Subsequent work by other authors<sup>3,4,5,6</sup> is based on the methods used by Landau. In particular it was shown<sup>4</sup> that a model with layers branching a limited number of times and coming out to the surface without formation of the mixed phase (i.e., in a certain sense a combination of the models of Refs. 1 and 2) is energetically more favorable than the original methods. In one way or another all the formulas proposed for connecting the surface tension with the dimensions of the domains, have been obtained only under various simplifying assumptions, still requiring experimental verification, about the shapes of the domains.

The experiments of Shal'nikov, Meshkovskii and Tumanov<sup>7-11</sup> on the pattern of the field distribution in a gap between two tin hemispheres by the methods of a bismuth micro-probe and ferromagnetic powder have shown that the real structures of the