Variational Principles in Hydrodynamics

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New variational principles are formulated, characterized by a special variation subject to auxiliary constants. This makes possible the solution of boundary problems of hydrodynamics by direct methods.

SEVERAL attempts have been made recently to formulate a general variational principle for hydrodynamics.

Bateman[1] established a variational principle for isentropic gas flows. Lin and Rubinov[2] generalized Bateman's principle to plane isoenergetic flow. In the work of Ito[3], Bateman's principle was further generalized and applied to the construction of a quantum hydrodynamics and the study of the flow of liquid He II, which had been treated earlier by Landau[4] and London[5].

In these papers, the variational functional contained several field parameters which were varied independently (density, velocity, pressure and other parameters in Refs. 1 and 3; density and stream function in Ref. 2). However, such a set of parameters is not minimal, and as we shall show later, in the general case of three-dimensional flow, two quantities determining the flow are sufficient.

1. THE PRINCIPLE OF LEAST STREAMING POTENTIAL[6]

Let ψ, ϑ be parameters defining a particular stream line of the hydrodynamic field. ψ and ϑ remain constant along this line. The surfaces ψ = const and ϑ = const are surfaces of flow. At each point of the stream line ψ, ϑ we set up an orthogonal axis system $\bar{x}_i$ so that one axis, say $\bar{x}_1$, is along (ψ, ϑ). To each such axis system we associate an energy-momentum tensor T_{ik} in the particular coordinate system determined by the axes. We denote the component T_{11} of the energy-momentum tensor with respect to such a system by T_{ss}, so that $T_{11} = T_{ss}$. We define the field in terms of the quantities $\psi(x_i)$, $\vartheta(x_i)$, satisfying the relations:

$$\rho V_1 = \frac{D(\psi,\vartheta)}{D(x_2, x_3)}; \quad \rho V_2 = \frac{D(\psi,\vartheta)}{D(x_3, x_1)}; \quad (1)$$

$$\rho V_3 = \frac{D(\psi,\vartheta)}{D(x_1, x_2)}.$$

With this choice of the defining quantities, the equation of continuity is satisfied identically both for actual as well as the varied field. For a plane flow, the defining quantity is ψ, where

$$\rho V_1 = \partial\psi/\partial x_2; \quad \rho V_2 = -\partial\psi/\partial x_1. \quad (2)$$

Relation (2) follows from (1) if we set $\vartheta = x_3$, which characterizes plane-parallel fields. The function ψ in (2) is the stream function, and physically speaking determines the flux of matter through a stream tube. The difference $\psi_1 - \psi_2$ gives the flux of liquid through the region bounded by the stream lines ψ_1 and ψ_2. The pair of functions (ψ, ϑ) should be regarded as the generalized stream functions for three-dimensional flow. In this case the product $(\psi_1 - \psi_2)(\vartheta_1 - \vartheta_2)$ gives the value of the flux of material through the region bounded by the flow surfaces ψ_1, ψ_2 and ϑ_1, ϑ_2.

Let σ be the surface which bounds a certain closed volume Ω of the field which contains no strong or weak discontinuities, and on which the distribution of ψ and ϑ is known, i.e., we know the distribution of the matter flux through σ. We shall also assume that the distribution of the total mechanical energy per unit mass of liquid, $E(\psi, \vartheta)$, is known on the surface of σ. For the actual field along the stream line $(\psi, \quad)$ within Ω, we can write the equation of change of mechanical energy in the form:

$$\int \frac{\partial T_{sk}}{\partial x_k} dS = \frac{V^2}{2} + \int \frac{dP}{\rho} + R(\rho, \psi, \vartheta) = E(\psi, \vartheta), \tag{3}$$

where R is the work of the frictional forces per unit mass of liquid. The quantity R is that part of the mechanical energy which is converted into heat. We demand that Eq. (3) be satisfied for the varied as well as the actual field.

In addition, the law of conservation of energy (first law of thermodynamics) should be satisfied for both the real and the varied field:

$$dQ = dU + Pd(1/\rho), \tag{4}$$

where Q is the heat and U the internal energy per unit mass of material, expressed in mechanical units. Under the conditions (3) and (4), the Lagrangian is $L = T_{ss}$, and the integral

$$I = \int_\Omega T_{ss} d\omega \tag{5}$$

is an extremum for the actually established field. The integral (5) represents the work of the total flux of momentum of the directed motion of the gas molecules and is called the stream potential. The condition for an extremal of I gives equations of the form:

$$\frac{\partial T_{ss}}{\partial \psi} - \frac{\partial}{\partial x_i \partial (\partial \psi / \partial x_i)} \frac{\partial T_{ss}}{} = 0; \qquad \frac{\partial T_{ss}}{\partial \vartheta} - \frac{\partial}{\partial x_i \partial (\partial \vartheta / \partial x_i)} \frac{\partial T_{ss}}{} = 0. \tag{6}$$

The equations (6) give the components of the vorticity which lie along the normals to the flow surfaces ψ = const and ϑ = const. Without loss of generality, we shall give the proof of the principle of minimum streaming potential formulated above, for the case of plane flow of an ideal gas which is barotropic with respect to a given particle. In this case

$$L = T_{ss} = P + \rho V^2; \quad R = 0;$$

$$\rho V_x = \partial \psi / \partial y; \quad \rho V_y = -\partial \psi / \partial x.$$

We introduce the geometrical characteristic of the flux

$$\Theta = (\partial \psi / \partial x)^2 + (\partial \psi / \partial y)^2 = \rho^2 V^2.$$

Then P, defined by the first law of thermodynamics, can be regarded as a function of ρ and ψ, and ρ as a function of Θ and ψ. We have

$$I = \int_\Omega \left[P(\rho, \psi) + \frac{\Theta}{\rho} \right] d\omega.$$

In general we can find ρ from Eq. (3) as a function of Θ and ψ. Regarding $\rho = \rho(\Theta, \psi)$ as the result of elimination of Eq. (3), we write the equation determining ψ from the condition for minimum I:

$$\frac{\partial}{\partial x}\left\{\left[\left(\frac{\partial P}{\partial \rho} - \frac{\Theta}{\rho^2}\right)\frac{\partial \rho}{\partial \Theta} + \frac{1}{\rho}\right] 2 \frac{\partial \psi}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial P}{\partial \rho} - \frac{\Theta}{\rho^2}\right)\frac{\partial \rho}{\partial \Theta} + \frac{1}{\rho}\right] 2 \frac{\partial \psi}{\partial y}\right\} = 0. \tag{7}$$

Differentiating (3) with respect to Θ for ψ = const, i.e., along a stream line:

$$-1/2\rho = (\partial P / \partial \rho - \Theta / \rho^2)\, \partial \rho / \partial \Theta. \tag{8}$$

Again differentiating (3) with respect to ψ for Θ = const:

$$\rho E'(\psi) = -\frac{\Theta}{\rho^2}\frac{\partial \rho}{\partial \psi} + \rho \frac{\partial}{\partial \psi} \int \frac{dP(\rho, \psi)}{\rho} + \frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial \psi}. \tag{9}$$

Substituting (8) and (9) in (7) we get

$$\text{curl}\,\mathbf{V} = \frac{\partial P(\rho, \psi)}{\partial \psi} - \rho \frac{\partial}{\partial \psi} \int \frac{\partial P(\rho, \psi)}{\rho} + \rho E'(\psi). \tag{10}$$

Equation (10) gives the expression for the vorticity in a plane flow, and is equivalent to the field equation $\partial T_{2k} \partial x_k = 0$.

If the variation of ψ upon a certain surface σ is arbitrary, then from the equation $\delta I = 0$ we get (10) and the natural boundary condition $\mathbf{V} = V \cdot \mathbf{n}$, where $\mathbf{n}$ is the normal to the surface σ.

For adiabatic flow, for which $P = C^k(\psi)\rho^k$, Eq. (10) becomes

$$\text{curl}\,\mathbf{V} = -\rho\left(\frac{dE}{d\psi} - \frac{k}{k-1} P^{(k-1)/k} \frac{dC}{d\psi}\right). \tag{11}$$

If E and C do not depend on ψ and ϑ, then $L = L \times(\Omega)$, where

$$\Theta = \left(\frac{D(\psi, \vartheta)}{D(x_2 x_3)}\right)^2 + \left(\frac{D(\psi, \vartheta)}{D(x_3 x_1)}\right)^2 + \left(\frac{D(\psi, \vartheta)}{D(x_1 x_2)}\right)^2$$

is the geometrical characteristic of the flux for the case of three-dimensional flow. The equation $\delta \int L d\omega$ leads to the condition of irrotational flow, curl $\mathbf{V} = 0$.

2. RELATIVISTIC GENERALIZATION OF THE PRINCIPLE OF LEAST STREAMING POTENTIAL FOR A STATIONARY FIELD

In relativistic hydrodynamics[7], the energy-momentum tensor of an ideal gas is

$$T_{ik} = (P + \rho c^2)\, u_i u_k + P\delta_{ik}, \tag{12}$$

where u_i is the four-velocity of the flow of the gas. In this case

$$L = T_{ss} = \beta^2 (P + \rho c^2) V^2 + P. \tag{13}$$

The equation of continuity has the form:

$$\operatorname{div} n \frac{\mathbf{V}}{c} \beta = 0; \quad \beta = \frac{1}{1 - V^2/c^2}, \tag{14}$$

where n is the density of the gas in the reference frame in which the particular element of the gas is at rest. As before, we set

$$nu_x = n \frac{V_x}{c} \beta = \frac{D(\psi, \vartheta)}{D(y, z)}; \tag{15}$$

$$nu_y = n \frac{V_y}{c} \beta = \frac{D(\psi, \vartheta)}{D(z, x)};$$

$$nu_z = n \frac{V_z}{c} \beta = \frac{D(\psi, \vartheta)}{D(x, y)}.$$

Under the conditions (15), the equation (14) is satisfied identically. The first law of thermodynamics has the form[7]:

$$Pd(1/n) + d(\rho c^2/n) = 0. \tag{16}$$

For an ideal gas, the energy integral along a stream line takes the form:

$$\int \frac{\partial T_{sk}}{\partial x_k} ds = \beta^2 \frac{V^2}{2} + \int \beta^2 \frac{dP}{P + \rho c^2} = E(\psi, \vartheta). \tag{17}$$

In relativistic hydrodynamics, the adiabatic flow of an ideal gas possesses vorticity, except in the case of one-dimensional flow. We shall assume that E = const, $P = P(n)$, $\rho = \rho(n)$, and limit ourselves to the case of plane flow. Then

$$L = [P/c^2 + \rho]\, \Theta n^{-2} + P.$$

In classical hydrodynamics the condition for minimum I would lead in this case to vanishing vorticity: curl V = 0. In relativistic hydrodynamics, because of the relativistic change of mass, the minimum I occurs for a vorticity different from zero. To simplify the computations, we shall actually carry out the calculation for the limiting case when V is close to c. Then $\rho \to \alpha n^k$, where k is the relativistic component of an adiabat and is equal to 4/3, $P = 1/3\, \alpha k^2 n^k$. Substituting these values in L, we get

$$I = \int \left(\alpha k n^{k-2} \Theta + \frac{\alpha}{3} c^2 n^k\right) dx\, dy. \tag{18}$$

The Euler equation for (18) is:

$$\frac{\partial}{\partial x}\left\{\left[\alpha k n^{k-2} + \left(\alpha k (k-2) n^{k-3}\Theta + \frac{k\alpha c^2}{3} n^{k-1}\right)\frac{dn}{d\Theta}\right] 2 \frac{\partial \psi}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{\left[\alpha k n^{k-2} + \left(\alpha k (k-2) n^{k-3}\Theta + \frac{k\alpha c^2}{3} n^{k-1}\right)\frac{dn}{d\Theta}\right] 2 \frac{\partial \psi}{\partial y}\right\} = 0. \tag{19}$$

We differentiate (17) with respect to Θ and determine $dn/d\Theta$:

$$dn/d\Theta = n/2\,(\Theta - {}^1/_3\, \beta^2 n^2).$$

Substituting this value of $dn/d\Theta$ in (19), we find after simple transformations,

$$\alpha k n^{k-1} \beta \operatorname{curl} \mathbf{V} + k(k-1)\, \alpha n^{k-3} \nabla\psi \nabla n = 0, \tag{20}$$

from which, if we assume that curl V = 0, it follows that $\nabla\psi \cdot \nabla n = 0$, but this is possible only for the case of one-dimensional flow. From (20) we obtain, in particular, the expression for the vorticity in the limiting case:

$$\operatorname{curl} \mathbf{V} = (1/3nc)\,(\mathbf{V}\nabla n). \tag{21}$$

3. NONSTATIONARY FIELD. THE PRINCIPLE OF LEAST ACTION

We introduce three "stream functions" ψ, ϑ, σ in four-dimensional space-time, and define the four-velocity u_i by the following formulas:

$$nu_1 = D(\psi, \vartheta, \sigma)/D(x_2 x_3 x_4); \tag{22}$$

$$nu_3 = D(\psi, \vartheta, \sigma)/D(x_4 x_1 x_2);$$

$$nu_2 = D(\psi, \vartheta, \sigma)/D(x_3 x_4 x_1);$$

$$nu_4 = D(\psi, \vartheta, \sigma)/D(x_1 x_2 x_3).$$

Under these conditions the continuity equation

$$(\partial/\partial x_i)\,(nu_i) = 0 \tag{23}$$

is satisfied identically. As before, the geometrical characteristic of the flux is

$$\Theta = \sum \left(\frac{D(\psi, \vartheta, \sigma)}{D(x_k x_r x_l)} \right)^2,$$

where the density n is related to Θ by the equation

$$\Theta = -n^2. \quad (24)$$

In the four-dimensional space, the "streamworld line" is defined by a triple of numbers $(\psi, \vartheta, \sigma)$ constant along the whole "streamworld line". The physical meaning of the stream function is that the product $d\psi d\vartheta d\sigma$ determines the "flux" of liquid through the region bounded by the stream hypersurfaces ψ, $\psi + d\psi$, ϑ, $\vartheta + d\vartheta$; σ, $\sigma + d\sigma$, or the flux through the region ψ, $\psi + d\psi$; ϑ, $\vartheta + d\vartheta$ of three-dimensional space during the time interval dt. For fixed t, the line $(\psi, \vartheta) = \text{const}$ is the field line $\bar{s}$ tangent to the velocity vector **V** in three-dimensional space.

To construct the Lagrangian L we construct at any given time, at each point of the actual stream line $\bar{s}$, an orthogonal set of four axes $\bar{x}_i$ so that one of the real axes, say $\bar{x}_1$, is always directed along the stream line $\bar{s}$. To each such axis system we associate an energy-momentum tensor T_{ik}, defined in the system of coordinates determined by the particular axes. We denote the tensor component T_{11} for such an axis system by $T_{\bar{s}\bar{s}}$, so that $T_{11} \equiv T_{\bar{s}\bar{s}}$. Then $L \equiv T_{\bar{s}\bar{s}}$. For an ideal gas,

$$L = T_{\bar{s}\bar{s}} = P + \left(\frac{P}{c^2} + \rho\right) u_s^2. \quad (25)$$

The integral of the energy along the stream line $\bar{s}$ has the form:

$$\int \frac{\partial T_{\bar{s}k}}{\partial \bar{x}_k} d\bar{s} = E(\bar{s}, t). \quad (26)$$

The relation (26) can be regarded as the generalized Hamilton-Jacobi equation for the field. In particular, for $c \to \infty$, Eq. (26) goes over into the classical equation:

$$\int \frac{\partial V}{\partial t} d\bar{s} + \frac{V^2}{2} + \int \frac{dP}{\rho} = E(\bar{s}, t). \quad (27)$$

Again we write the first law of thermodynamics:

$$Pd(1/n) + dU = TdH, \quad (28)$$

where U is the internal energy of the gas, and H is the entropy of the gas per molecule. If $U = v\rho c^2$, where $v = 1/n$ is the volume per molecule, then as is shown in relativistic hydrodynamics[7], $H = \text{const}$ and (28) takes the form $Pd(1/n) + dU = 0$. The variational principle for the hydrodynamic field when the conditions (24), (26) and (28) are satisfied consists in the statement that the integral

$$S = \int T_{\bar{s}\bar{s}}\left(\psi, \vartheta, \sigma, \frac{\partial\psi}{\partial x_i}, \frac{\partial\vartheta}{\partial x_i}, \frac{\partial\sigma}{\partial x_i}\right) d\Omega \quad (29)$$

has a stationary value for the actual field. The quantities defining the field are ψ, ϑ, and σ, which are related by (24).

For $c \to \infty$,

$$L = P + \rho V^2, \quad \rho V_x = D(\psi, \vartheta, \sigma)/D(y, z, t);$$

$$\rho V_y = D(\psi, \vartheta, \sigma)/D(z, t, x);$$

$$\rho V_z = D(\psi, \vartheta, \sigma)/D(t, x, y);$$

$$n = \rho = D(\psi, \vartheta, \sigma)/D(x, y, z);$$

$$dQ = TdH = dU + Pd(1/\rho).$$

In this case the integral $S = \int_{t_1}^{t_2} dt \int_\omega (P + \rho V^2) d\omega$ takes on a stationary value for the actual field. The quantity S has, for a nonstationary field, the dimensions of action, so that the principle thus established for a non-stationary field in relativistic hydrodynamics may be regarded as a principle of least action.

4. THE EXISTENCE OF A STRONG MINIMUM FOR THE STREAMING POTENTIAL

We shall prove the existence of a strong minimum for the actual subsonic, adiabatic flow of a gas. To simplify the computations we shall, without loss of generality, limit ourselves to considering plane flow of the gas. Let ψ be the solution, subject to the given boundary conditions, of the equation

$$\delta \int L dx dy = \delta \int F(\Theta) dx dy = 0, \quad (30)$$

where $L = F(\Theta)$ is the Lagrangian for the hydrodynamic field of an ideal gas, and Θ has the same meaning as earlier.

The stream function ψ gives a strong minimun for the functional $I = \int L d\omega$. In proving this, we shall make use of the theory of the Weierstrass function in the calculus of variations[8]. For the construction of the Weierstrass function, it is sufficient to satisfy the following two conditions:

1) Every solution of (30) satisfies the self-adjoint Butler equation

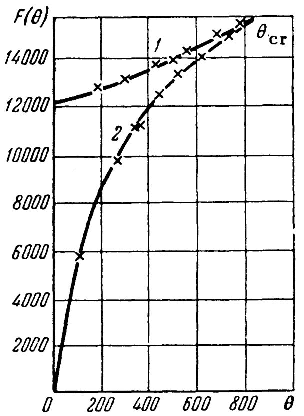
$$\frac{\partial L}{\partial \psi} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial\psi/\partial x)} - \frac{\partial}{\partial y}\frac{\partial L}{\partial(\partial\psi/\partial y)} = 0; \quad (31)$$

2) If ψ is a solution of (31), then there exists at least one system φ of Lagrange surfaces which covers the extremal surface ψ, which depends on a single parameter ξ, and for which the Jacobi condition

$$P = \partial\varphi/\partial x = \partial\psi/\partial x = p; \quad Q = \partial\varphi/\partial y = \partial\psi/\partial y = q.$$

is satisfied on the extremal surface.

Conditions 1) and 2) are sufficient for the construction of a field consisting of Lagrange surfaces covering the surface-extremals ψ, which guarantees the possibility of constructing the Weierstrass function of the variational problem. Conditions 1) and 2) are satisfied for the continuous hydrodynamic field of an ideal gas. In particular, we can take $\psi + \xi$ for φ. The Figure shows the dependence of $L = F(\Theta)$*. The branch *1* which is concave upwards corresponds to subsonic flow, the curve *2* which is concave downwards, to supersonic flow of the gas. At the turning point Θ_{cr} the critical condition is reached where the flow velocity is equal to the local sound velocity.



Corresponding to the Lagrangian L, we construct the phase surface

$$Z = L = F(\Theta) = F(p^2 + q^2) = Z(p, q) \quad (32)$$

in the space $p = \partial\psi/\partial x$, $q = \partial\psi/\partial y$. Let the aggregate of three numbers $\bar{p}$, $\bar{q}$, $\bar{L} = z(\bar{p}, \bar{q})$ represent a point of phase surface. The equation of the tangent plane at the point $(\bar{p}, \bar{q}, \bar{L})$ of the surface Z has the form:

$$Z = \bar{L} + (\partial\bar{L}/\partial\bar{p})(p - \bar{p}) + (\partial\bar{L}/\partial\bar{q})(q - \bar{q}), \quad (33)$$

where Z, P, q are running coordinates in the plane. We denote by $G(p, q, \bar{p}, \bar{q})$ the difference in the values of Z on the surface (32) and on the tangent plane (33) to the point $\bar{L}$, $\bar{p}$, $\bar{q}$, taken for the same point (p, q), and call it the deviation of the surface (32) from the plane (33) at that point. If this difference is positive, the deviation is positive, Thus

$$G(p, q, \bar{p}, \bar{q}) = L(p, q) - L(\bar{p}, \bar{q}) - \frac{\partial\bar{L}}{\partial\bar{p}}(\bar{p} - p) - \frac{\partial\bar{L}}{\partial\bar{q}}(q - \bar{q}). \quad (34)$$

Expression (34) coincides exactly with the Weierstrass function[8] of the variational problem which provides a sufficient condition for the existence of a strong minimum. If $G \geq 0$ everywhere within the domain of definition of L, the solution of Eq. (30) guarantees a strong minimum for the functional I. From the Figure we see that the phase surface $Z = L(p, q)$ for subsonic flow is a convex surface, and $G \geq 0$. For supersonic flow the phase surface has a saddle at every point. In this case the actual field ψ_0 produces a minimax of I. If $I(\psi_0) = I_0$ then it is easily shown that ψ_0 gives a strong minimum, equal to I_0, for the functional $I^* = \frac{1}{2} I_0(I_0/I + I/I_0)$. Therefore, the actual field ψ_0, determined from the equation $\delta I = 0$, satisfies the equation $\delta I^* = 0$. In this way the minimax property of the streaming potential for supersonic flow is eliminated, and one can apply the direct method of Ritz to it just as for the streaming potential of subsonic flow. For subsonic, adiabatic gas flows, the Weierstrass function is essentially positive and goes to zero if, and only if, $p \equiv \bar{p}$, $q \equiv \bar{q}$. From this we get the uniqueness theorem: there cannot exist two solutions satisfying the same conditions for ψ on the boundary surfaces of the region of the flow.

* The curve was drawn for the initial conditions $P_0 = 1$ atm, $\rho_0 = 0.125$ kgm-sec^2/m^4, $k = c_p/c_v = 1.41$, $V_0 = a_0/2$, where a_0 is the sound velocity.

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The Dynamical Magnetic Moment of the Deuteron

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The dynamical magnetic moment of the deuteron is considered on the basis of the pseudoscalar meson theory with the pseudoscalar type of coupling , in the fifth order of perturbation theory. Exchange currents in the deuteron make an essential contribution to the dynamical magnetic moment.

INTRODUCTION

THE well-known experimental result that the constant magnetic moment of the deuteron differs from the sum of the magnetic moments of the neutron and proton is commonly explained phenomenonologically by the existence of a tensor interaction between nucleons. Because of this the ground state of the deuteron consists of an *S*-state with an admixture of a *D*-wave, which on one hand leads to the existence of the quadrupole moment of the deuteron, and on the other to the nonadditivity of the magnetic moments in the deuteron[1].

Although such an interpretation is not in qualitative contradiction with experiment, it still does not correspond exactly to the effect, since it does not take into account the existence of meson exchange currents in the deuteron, which are shown by experiment to have an essential effect on the electromagnetic properties of the deuteron, particularly on radiative effects in the neighborhood of the energy threshold for production of π-mesons.

Unfortunately there is at present no consistent phenomenological theory of exchange currents. Of the attempts in this direction most deserving of attention, mention must be made of the work of Sachs[2], and a paper of Villars[3] is devoted to the meson-field treatment of this effect.

An essential difficulty in the treatment of the magnetic moment of the deuteron on the basis of the meson theory of nuclear forces arises from the circumstance that within the framework of this theory the relativistic problem of two nucleons is at the present time unsolved, and the magnetic moments of the separate nucleons are explained only qualitatively by the theory with weak interaction between the nucleon and meson fields, so that any theoretical investigations in this subject are as yet only of a qualitative nature. Nevertheless, it can turn out that the perturbation theory to a certain extent gives a correct indication of the general tendencies in the behavior of the two-nucleon system in interaction with high-energy γ-ray quanta.

The present paper is devoted to a consideration of the dynamical magnetic moment of the deuteron on the basis of the pseudoscalar meson theory with the pseudoscalar type of coupling. In interaction with the meson field of the vacuum the nucleons making up the deuteron can emit and then absorb virtual mesons, so that the real nucleon can be thought of as surrounded by a certain stationary cloud. If the charged meson clouds of the neutron and proton overlap, exchange meson currents arise, flowing from one nucleon to the other. The interaction of the meson field surrounding the deuteron with the electromagnetic field can be interpreted as a supplementary direct electromagnetic interaction of the deuteron itself.

In the expression obtained for this supplementary interaction one can single out the terms that represent the energy of a certain additional magnetic moment in the electromagnetic field. The size of this magnetic moment will depend on the frequency of the electromagnetic field. Thus the energy of