The Application of Non-Self-Adjoint Operators to Scattering Theory

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It is proved that for any given scattering matrix there exists a non-self-adjoint operator defining the energy of the compound nucleus in a simple triangular representation. The relation between the decay of the compound nucleus and its spectrum is investigated.

IN recent years a large number of purely mathematical investigations have appeared[1-8], dealing with the spectral theory of a wide class of non-self-adjoint operators. In the general theory of scattering there appears the idea of a compound or intermediate nucleus, which is formed during the course of various nuclear reactions. The main property of the compound nucleus is to possess complex energy levels or quasi-stationary states, and in mathematical language this means that the energy operator is not self-adjoint. The spectral decomposition of a non-self-adjoint operator, discovered by the present author[5], proceeds by means of the so-called characteristic matrix-function which defines the operator to within a unitary equivalence-class. The present paper establishes the existence of a close connection between the scattering matrix[9-16] of a compound nucleus and the characteristic matrix-function of its energy-operator. This makes it possible to construct the energy operator of a compound nucleus when the scattering matrix is given. In addition, some new properties of the scattering matrix are discovered. The energy operator is put into a triangular representation in which the equation of motion and the decay of the compound nucleus can be studied and related to the character of its spectrum (discrete or continuous, etc.).

Previous knowledge of the mathematical papers quoted above is not required for understanding the present work.

1. THE ENERGY OPERATOR OF A COMPOUND NUCLEUS

To explain the essence of our method we consider the simple case of a purely elastic scattering reaction described by $a + X \to C \to X + a$. Here a is the incident particle, x the target nucleus and C the compound nucleus. There is only one channel[16], and there are no external fields.

The wave function $u(r)(0 < r < \infty)$ of this system can be represented as a vector with two components $\varphi(r)$ and $\psi(r)$ defined by

$$\varphi(r) = \begin{cases} a_0(e^{-iKr} - S(W)e^{iKr}) & r > R, \\ 0 & r < R, \end{cases} \tag{1}$$

$$\psi(r) = \begin{cases} 0 & r > R, \\ \sum_{j=0}^{n} c_j\psi_j(r) & r < R \quad (n \leqslant \infty). \end{cases} \tag{2}$$

Here K is the wave number, and $W = h^2K^2/2\mu$. The $\psi_j(r)$ are a complete orthonormal set of wave-functions for the compound nucleus in the interval $(0, R)$, the C_j are constant coefficients, and R is the channel radius[16]. The Hamiltonian* may accordingly be written as a 2×2 matrix

$$\begin{Vmatrix} T & \Gamma \\ \Gamma^* & A \end{Vmatrix}$$

where $T = -(h^2/2\mu)(d^2/dr^2)$ is the Hamiltonian in the center-of-mass system in the absence of interactions. A is an operator acting on the wave-functions $\psi(r)(r < R)$ inside the compound nucleus, and the operators Γ and Γ^* describe the probabilities for decay and formation of the compound nucleus.

The function $\varphi(r)$ may also be written

$$\varphi(r) = \text{const}\,(e^{-iK(r-R)} - S_0(W)e^{iK(r-R)}) \quad (r > R), \tag{3}$$

where $S_0(W) = e^{2iKR}S(W)$ describes the resonance scattering associated with the formation of the compound nucleus. We call the function $S_0(W)$ the reduced (one-dimensional) scattering matrix. Introducing the functions

* Here we used the method of Bethe[15] with some minor alterations.

$$\varphi_1(r) = \sin K(r-R),$$
$$\varphi_2(r) = \cos K(r-R) \quad (r > R),$$

[$\varphi_1(r) = \varphi_2(r) = 0$ for $r < R$], we may represent $\varphi(r)$ in the form

$$\varphi(r) = a\varphi_1(r) + b[\varphi_2(r) + i\varphi_1(r)]. \tag{4}$$

The function $\varphi(r)$ has a discontinuity at R of magnitude $b = \varphi(R+0) - \varphi(R-0)$.

Consider the wave-equation

$$\begin{Vmatrix} T & \Gamma \\ \Gamma^* & A \end{Vmatrix} \begin{pmatrix} \varphi(r) \\ \psi(r) \end{pmatrix} = W \begin{pmatrix} \varphi(r) \\ \psi(r) \end{pmatrix},$$

which separates into the two equations

$$T\varphi + \Gamma\psi = W\varphi, \tag{5}$$

$$\Gamma^*\varphi + A\psi = W\psi. \tag{6}$$

Equations (4) and (5) imply*

$$(\Gamma\psi, \varphi_1) = \int_0^\infty \varphi_1(r)\left(W + \frac{h^2}{2\mu}\frac{d^2}{dr^2}\right)\varphi(r)\,dr \tag{7}$$
$$= \int_{R-\varepsilon}^{R+\varepsilon} \frac{h^2}{2\mu} b\delta'(r-R)\varphi_1(r)\,dr$$
$$= (h^2/2\mu)\varphi_1'(R) = -(h^2/2\mu)bK.$$

Next, Eqs. (2) and (7) give

$$b = -\frac{2\mu}{h^2K}\sum_{j=1}^n c_j\gamma_j \quad (\gamma_j = (\Gamma\psi_j, \varphi_1)). \tag{8}$$

In the same way we can prove that $(\Gamma\psi_j, \varphi_2) = 0$. Hence we deduce from Eqs. (6) and (4)

$$a\Gamma^*\varphi_1 + b\Gamma^*\varphi_2 + ib\Gamma^*\varphi_1 + A\psi = W\psi. \tag{9}$$

Multiplying Eqs. (2) and (9) by $\psi_k(r)$ and integrating, we find

$$a(\Gamma^*\varphi_1, \psi_k) + b(\Gamma^*\varphi_2, \psi_k) \tag{10}$$
$$+ ib(\Gamma^*\varphi_1, \psi_k) + (A\psi, \psi_k) = Wc_k.$$

Since

$$(\Gamma^*\varphi_1, \psi_k) = (\varphi_1, \Gamma\psi_k) = \gamma_k^* \quad (k = 1, 2, \ldots, n),$$
$$(\Gamma^*\varphi_2, \psi_k) = (\varphi_2, \Gamma\psi_k) = 0,$$

equations (8) and (10) lead to

$$a\gamma_k^* - i\frac{2\mu}{h^2K}\sum_{j=1}^n c_j\gamma_i\gamma_k^* + \sum_{j=1}^n A_{jk}c_j = Wc_k, \tag{11}$$

with $A_{jk} = (A\psi_j, \psi_k)$.

Let f be the vector with components $c_1, c_2, \ldots, c_n$, and let the operators A, B, H, H^* be defined by

$$[Af]_k = \sum_{j=1}^n A_{jk}c_j; \tag{12}$$
$$[Bf]_k = \sum_{j=1}^n c_j\beta_j^*\beta_k \quad (\beta_j = \sqrt{2\mu/h^2K}\,\gamma_j^*),$$
$$H = A + iB, \quad H^* = A - iB,$$

where $[Af]_k$, $[Bf]_k$ $(k = 1, 2, \ldots, n)$ are the components of the vectors Af, Bf. Equation (11) can then be written in the form

$$H^*f - Wf = g_0, \tag{13}$$
$$g_0 = -a\sqrt{h^2K/2\mu}\,\{\beta_k\}_{k=1}^n.$$

If $a = 0$, there is no incident wave, and the equation $H^*f = Wf$ defines the complex-levels and quasi-stationary states of the decaying compound nucleus. The equation of motion of the compound nucleus is

$$ihdf/dt = H^*f. \tag{14}$$

2. THE SCATTERING MATRIX

We now calculate the scattering matrix. If e is the vector $e = \{\sqrt{2}\beta_k\}_{k=1}^n$, Eq. (12) gives*

$$-i(H - H^*)f = 2\sum_{j=1}^n c_i\beta_j^*\beta_k = (f, e)e. \tag{15}$$

Equations (4) and (8) imply

$$\varphi(r) = \frac{a}{2i}(e^{iK(r-R)} - e)^{-iK(r-R)} \tag{16}$$
$$- \sqrt{\frac{2\mu}{h^2K}}\frac{(f, e)}{\sqrt{2}}e^{iK(r-R)},$$

* The symbol

$$(u, v) = \int_0^\infty v^*(r)u(r)\,dr$$

represents a scalar product. The function $b\delta'(r-R)$ arises from double differentiation of the simple discontinuity.

* Here $(f, e) = \sum_{j=1}^n e_j^* c_j = \sqrt{2}\sum_{j=1}^n c_j\beta_j^*$ is the scalar product.

while Eq. (13) gives

$$f = (H^* - WI)^{-1} g_0 \tag{17}$$

$$= -\frac{a}{\sqrt{2}} \sqrt{\frac{h^2 K}{2\mu}} (H^* - WI)^{-1} e.$$

Thus the function $\varphi(r)$ becomes

$$\varphi(r) = \frac{a}{2i} (e^{iK(r-R)} - e^{-iK(r-R)}) \tag{18}$$

$$+ \frac{a}{2} ((H^* - WI)^{-1} e, e) e^{iK(r-R)}$$

$$= -\frac{a}{2i} \{e^{-iK(r-R)}$$

$$- [1 + i((H^* - WI)^{-1} e, e)] e^{iK(r-R)}\}.$$

Comparing this with Eq. (3), we obtain the following expression for the one-dimensional scattering matrix

$$S_0(W) = 1 + i((H^* - WI)^{-1} e, e), \tag{19}$$

$$S(W) = e^{-2iKR} S_0(W). \tag{20}$$

The cross section for resonance scattering[16] is

$$\sigma_{res} = \pi\lambda^2 |1 - S_0(W)|^2 \tag{21}$$

$$= \pi\lambda^2 |((H^* - WI)^{-1} e, e)|^2.$$

It is clear from Eq. (15) than an arbitrary vector f is either annihilated by the operator $-i(H - H^*)$ or is transformed into a vector parallel** to e. In addition, we have the inequality

$$(-i(H - H^*) f, f) = |(f, e)|^2 \geqslant 0. \tag{22}$$

The function $S_0(W)$, related to the operator H by Eq. (19), is called the characteristic function of H. It was studied in a number of papers[1,2,5,7] on the theory of non-self-adjoint operators. We quote without proof some of the main results of the theory.

1. The function $S_0(W)$ determines H to within a unitary equivalence-class. Thus the scattering matrix completely determines* (in the quantum mechanical sense) the motion of the intermediate nucleus C. Later we shall show how, given $S_0(W)$, H or H^* can be constructed in their simplest representation, which is one in which they become triangular matrices.

2. $S_0(W)$ is an analytic function of W.

3. The identity

$$1 - |S_0(W)|^2 \tag{23}$$

$$= \frac{W - W^*}{i} (R_W e, R_W e) \quad (\mathrm{Im}\, W \neq 0),$$

holds, implying that

$$1 - |S_0(W)|^2 > 0, \quad (\mathrm{Im}\, W > 0). \tag{24}$$

Here R_W denotes the operator $(H^* - WI)^{-1}$.

4. The eigenvalues $W_k = E_k + \frac{1}{2} i \Gamma_k$ $(k = 1, 2, \ldots)$ of H lie in the upper half-plane and are roots of the equation $S_0(W) = 0$.

5. The continuous spectrum of H (if it exists) lies on the real W-axis. A real value of W belongs to the continuous spectrum if $1 - |S_0(W + i0)|^2 > 0$. Intervals in which $1 - |S_0(W + i0)|^2 = 0$ contain no continuous spectrum. If $1 - |S_0(W + i0)|^2 > 0$ with $W > 0$, then there exist alternative reaction channels which have not been considered. Equation (14) implies

$$\frac{d|f|^2}{dt} = \frac{d(f, f)}{dt} = \left(\frac{df}{dt}, f\right) + \left(f, \frac{df}{dt}\right)$$

$$= -\frac{i}{h} (H^* f, f) + \frac{i}{h} (f, H^* f)$$

$$= -\frac{1}{h} \left(\frac{1}{i} (H - H^*) f, f\right).$$

From Eq. (22) we have

$$d|f|^2/dt = -h^{-1} |(f, e)|^2 \leqslant 0, \tag{25}$$

and therefore the total decay probability (for one particle and per unit time) is

** The departure of the operator H from Hermiticity (the rank of non-Hermiticity) is characterized in this case by a single vector e, because there exists only one reaction channel. In Ref. 5, operators with arbitrary rank of non-Hermiticity are considered. The rank of non-Hermitcity is equal to the number of channels.

* This result is important in connection with the S-matrix theory of Heisenberg (see Ref. 17). In Ref. 5, the results (1-5) are proved under the assumption that e is independent of W.

$$P = h^{-1}(e, e) = \frac{2}{h}\sum_{j=1}^{n} |\beta_j|^2. \tag{26}$$

This quantity can be calculated directly from $S_0(W)$. In fact, Eq. (19) implies*

$$S_0(W) = 1 + iW((H^*W^{-1} - I)^{-1}e, e)$$
$$= 1 - iW^{-1}[(e, e) + W^{-1}(H^*e, e) + \dots],$$

from which it follows that

$$P = \frac{1}{h}\lim_{W\to\infty} W\,|\,1 - S_0(W)\,| \tag{27}$$

$$= \frac{1}{hi}\operatorname{residue}_{W\to\infty} S_0(W).$$

Next we derive a formula for the population q of compound nuclei existing per unit volume during a sustained scattering process. Comparing the well-known expansion[18] of a plane wave e^{iKz} into spherical harmonics with Eq. (4), we obtain

$$|a| = 1/K. \tag{28}$$

The state of the compound nucleus is defined by the vector $f = R_W g_0$ according to Eq. (13). Hence, q becomes

$$q = |f|^2 = (h^2/2\mu K)(R_W e, R_W e),$$

which by Eq. (23) may be written

$$q = (h^2/2\mu K)\lim_{y\to\infty}(1 - |S_0(W + iy)|^2/2y).$$

Hence the formula for q

$$q = -\frac{W}{K^3}\left[\frac{\partial}{\partial y}|S_0(W + iy)|\right]_{y=0}. \tag{29}$$

3. A FINITE NUMBER OF COMPLEX LEVELS. TRIANGULAR MODEL.

We shall find the energy-operator H^* of the compound nucleus, in the case when this operator possesses a finite number of eigenvalues $W_k^* = E_k - \frac{1}{2}i\Gamma_k$ ($k = 1, 2, \dots$). According to the results 4 and 5 above, the reduced scattering matrix $S_0(W)$ is of the form

$$S_0(W) = \prod_{k=1}^{n}\frac{W - W_k}{W - W_k^*} \tag{30}$$
$$= \prod_{k=1}^{n}\frac{W - E_k - (i/2)\Gamma_k}{W - E_k + (i/2)\Gamma_k}.$$

* It is assumed that the spectrum of H does not extend to infinity.

By choosing a particular orthonormal basis $\psi_k(r)$ ($k = 1, 2, \dots, n$) for the wave-functions of the compound nucleus, we can reduce the operator H^* to triangular form. Thus

$$d_k = \sum_{j=1}^{n}(H^*)_{kj}c_j \quad ((H^*)_{kj} = 0,\ j > k). \tag{31}$$

The elements $(H^*)_{kk}$ on the principal diagonal* coincide with the eigenvalues $(H^*)_{kk} = W_k^*$. We now derive the remaining elements $(H^*)_{kj}$ ($j < k$). We observe that the operator $g = -i(H - H^*)f$ is defined by the equations

$$d_1 = \Gamma_1 c_1 + \frac{1}{i}H_{12}c_2 + \dots + \frac{1}{i}H_{1n}c_n, \tag{32}$$
$$d_2 = -\frac{1}{i}H_{12}^*c_1 + \Gamma_2 c_2 + \dots + \frac{1}{i}H_{2n}c_n,$$
$$\dots\dots\dots\dots\dots\dots$$
$$d_n = -\frac{1}{i}H_{1n}^*c_1 - \frac{1}{i}H_{2n}^*c_2 - \dots + \Gamma_n c_n.$$

Comparing Eqs. (32) with (15), we obtain

$$2|\beta_k|^2 = \Gamma_k, \quad H_{jk} = 2i\beta_j\beta_k^* \quad (j < k).$$

Hence

$$H_{jk} = i\sqrt{\Gamma_j\Gamma_k}\exp\{i(\varphi_j - \varphi_k)\} \quad (\varphi_j = \arg\beta_j).$$

Replacing the functions $\psi_k(r)$ by $\psi_k(r)\exp\{i\varphi_k\}$, we have

$$(H^*)_{kj} = H_{jk}^* = -i\sqrt{\Gamma_j\Gamma_k}.$$

Thus the operator $g = H^*f$ takes the form

$$d_k = \left(E_k - \frac{i}{2}\Gamma_k\right)c_k \tag{33}$$
$$- i\sum_{j=1}^{k-1}\sqrt{\Gamma_k\Gamma_j}\,c_j \quad (k = 1, 2, \dots n).$$

The operator (33) is uniquely determined by the roots of the function $S_0(W)$. In the simplest case of a single root, the system (33) reduces to one equation $d_1 = (E_1 - \frac{1}{2}i\Gamma_1)c_1$,

$$S_0(W) = \frac{W - E_1 - (i/2)\Gamma_1}{W - E_1 + (i/2)\Gamma_1},$$

and the cross section becomes

$$\sigma_{Res} = \pi\lambda^2\left|1 - \frac{W - E_1 - (i/2)\Gamma_1}{W - E_1 + (i/2)\Gamma_1}\right|^2$$
$$= \pi\lambda^2\frac{\Gamma_1}{(W - E_1)^2 + \Gamma_1^2/4}$$

* See Ref. 19 for the reduction of finite matrices to triangular form.

(Breit-Wigner). The motion of the compound nucleus is given by

$$ihdc_1/dt = (E_1 - {}^1/_2 i\Gamma_1)c_1,$$

so that

$$c_1(t) = c_1(0)e^{-itE_1/h}e^{-(\Gamma_1/2h)t}$$

and Γ_1/h is the decay probability. For an arbitrary number of levels, the decay probability becomes by Eq. (27)

$$P = \frac{1}{h}\lim_{W\to\infty} W\left|1 - \sum_{k=1}^{n}\frac{W-W_k}{W-W_k^*}\right| = \frac{1}{h}\sum_{k=1}^{n}|W_k^* - W_k| = \frac{1}{h}\sum_{k=1}^{n}\Gamma_k. \tag{34}$$

The total decay probability is equal to the sum of the probabilities for the individual levels, as one would expect.

4. CONTINUOUS SPECTRUM OF THE COMPOUND NUCLEUS. TRIANGULAR MODEL

Suppose that the number of complex levels $W_k^{(n)} = E_k^{(n)} + \frac{1}{2} i\Gamma_k^{(n)}$ ($k = 1, 2, \ldots, n$) increases indefinitely, and that in the limit as $n \to \infty$ the numbers $E_k^{(n)}$ fill a certain interval $a \le E \le b$ continuously. If the decay probability P of the compound nucleus is held fixed, then by Eq. (34) the $\Gamma_k^{(n)}$ must tend to zero as $n \to \infty$. Introducing the step-function

$$p_n(E) = \Gamma_k^{(n)}/E_{k+1}^{(n)} - E_k^{(n)} \quad (E_k^{(n)} \leqslant E < E_{k+1}^{(n)};\quad E_1^{(n)} = a,\quad E_{n+1}^{(n)} = b)$$

and using Eqs. (34) and (30), we obtain

$$P = h^{-1}\sum_{k=1}^{n} p_n(E_k^{(n)})(E_{k+1}^{(n)} - E_k^{(n)}),$$

$$S_0^{(n)}(W) = \prod_{k=1}^{n}\left[1 - i\frac{p_n(E_k^{(n)})(E_{k+1}^{(n)} - E_k^{(n)})}{W - E_k^{(n)} + \frac{i}{2}\Gamma_k^{(n)}}\right].$$

Since $E_{k+1}^{(n)} - E_k^{(n)} \to 0$ and $\Gamma_k^{(n)} \to 0$ as $n \to \infty$, we have

$$1 - i\frac{p_n(E_k^{(n)})(E_{k+1}^{(n)} - E_k^{(n)})}{W - E_k^{(n)} + \frac{i}{2}\Gamma_k^{(n)}} \approx \exp\left\{-i\frac{p_n(E_k^{(n)})(E_{k+1}^{(n)} - E_k^{(n)})}{W - E_k^{(n)}}\right\},$$

from which it follows that as $n \to \infty$ the function $S_0^{(n)}(W)$ tends to the limit

$$S_0(W) = \exp\left\{-i\int_a^b\frac{p(E)\,dE}{W-E}\right\}. \tag{35}$$

The total decay probability is

$$P = \frac{1}{h}p(E)\,dE, \tag{36}$$

and the probability for decay $P(E, E+dE)$ belonging to the interval $(E, E+dE)$ is given by

$$P(E, E+dE) = h^{-1}p(E)\,dE. \tag{37}$$

If the compound nucleus has both a continuous and a discrete (complex) spectrum, the reduced scattering matrix has the form

$$S_0(W) = \prod_{k=1}^{n}\frac{W - E_k - (i/2)\Gamma_k}{W - E_k + (i/2)\Gamma_k}\exp\left\{-i\int_a^b\frac{p(E)\,dE}{W-E}\right\}. \tag{38}$$

It is interesting that the density $h^{-1}p(E)$ of decay probability can be found directly from the function $S_0(W)$. Taking the logarithm, we find

$$\ln|S_0(W)| = \ln\left|\prod_{k=1}^{n}\frac{W-W_k}{W-W_k^*}\right| + \operatorname{Im}\int_a^b\frac{p(E)\,dE}{W-E}\quad(\operatorname{Im}W \neq 0),$$

and the Stieltjes inversion formula[20] then gives

$$h^{-1}p(E) = -(1/h\pi)\ln|S_0(E - i0)|. \tag{39}$$

Next we determine the operator H^* in the case of a pure continuous spectrum, passing to the limit in the equations

$$d_k^{(n)} = (E_k^{(n)} - \frac{i}{2}\Gamma_k^{(n)})c_k^{(n)} - i\sum_{j=1}^{k-1}c_j^{(n)}\sqrt{\Gamma_j^{(n)}\Gamma_k^{(n)}}.$$

We define the step-functions

$$f_n(E) = c_k^{(n)},\quad p_n(E) = \Gamma_k^{(n)}/E_{k+1}^{(n)} - E_k^{(n)}$$
$$\left(E_k^{(n)} \leqslant E < E_{k+1}^{(n)},\quad E_{k+1}^{(n)} - E_k^{(n)} = \frac{b-a}{n}\right)$$

and obtain in the limit as $n \to \infty$ the following representation for the operator $g = H^*f$,

$$g(E) = Ef(E) - i\int_a^E f(\xi)\sqrt{p(\xi)p(E)}\,d\xi \quad (a \leqslant E \leqslant b). \tag{40}$$

The corresponding equation of motion* is

$$ih\frac{\partial f(E,t)}{\partial t} = Ef(E,t) - i\int_a^E f(\xi,t)\sqrt{p(\xi)\,p(E)}\,d\xi. \quad (41)$$

In this case also the equation of motion is determined uniquely by $S_0(W)$.

5. DECAY OF THE COMPOUND NUCLEUS IN THE CASE OF A CONTINUOUS SPECTRUM

If $f(E,t)$ satisfies Eq. (41) and $g(E,t)$ satisfies the adjoint equation

$$ih\frac{dg}{dt} = Eg(E,t) + i\int_E^b g(\xi,t)\sqrt{p(\xi)\,p(E)}\,d\xi, \quad (42)$$

the scalar product (f, g) is constant in time. Explicitly,

$$\frac{d(f,g)}{dt} = \left(\frac{df}{dt}, g\right) + \left(f, \frac{dg}{dt}\right) = \frac{1}{ih}(H^*f, g) - \frac{1}{ih}(f, Hg) = 0.$$

It follows that

$$|f|^2_\infty \geqslant |(f_0, g_0)|^2/|g|^2_\infty \quad (43)$$

holds, where

$$|f|^2_\infty = \lim_{t\to\infty}\int_a^b |f(E,t)|^2 dE,$$

$$|g|^2_\infty = \lim_{t\to\infty}\int_a^b |g(E,t)|^2 dE,$$

$$f_0 = f(E;0), \quad g_0 = g(E,0).$$

We shall show that the initial state $f_0 = f(E,0)$ can decay incompletely** :

$$\lim_{t\to\infty}\int_a^b |f(E,t)|^2 dE > 0.$$

This means that after all decaying waves are removed there remains a certain population of nuclei which do not decay into the channel which is under consideration. Such a phenomenon obviously cannot occur when there exist only complex levels.

A solution of Eq. (42), or of

$$ih\frac{dg}{dt} = Hg \quad \left(Hg = Eg(E) + i\int_E^b g(\xi)\sqrt{p(\xi)\,p(E)}\,d\xi\right) \quad (44)$$

can be obtained from the formula

$$g(E,t) = -\frac{1}{2\pi i}\int_\gamma e^{-i\lambda t/h}(H-\lambda I)^{-1}g_0\,d\lambda, \quad (45)$$

where γ is a contour enclosing the spectrum $a \leq E \leq b$ of H. It is easy to verify Eq. (45). Putting $g_0 = e$ and using Eqs. (45) and (19), we have

$$(g, e) = -\frac{1}{2\pi i}\int_\gamma e^{-i\lambda t/h}((H-\lambda I)^{-1}e, e)\,d\lambda = \frac{1}{2\pi}\int_\gamma e^{-i\lambda t/h}(S_0^*(\lambda^*) - 1)\,d\lambda.$$

When the contour γ is contracted onto the interval $a \leq E \leq b$, this becomes

$$(g, e) = \frac{1}{2\pi}\int_a^b e^{-iEt/h}[S_0^*(E+i0) - S_0^*(E-i0)]\,dE. \quad (46)$$

Equation (25) when applied to the operator H becomes $d|g|^2/dt = h^{-1}|(g, e)|^2$, and therefore

$$|g|^2_\infty = |g_0|^2 + \frac{1}{h}\int_0^\infty |(g,e)|^2 dt = |g_0|^2 + \frac{1}{4\pi^2 h}\int_0^\infty\left|\int_a^b e^{-\frac{iEt}{h}}[S_0^*(E+i0) - S_0^*(E-i0)]\,dE\right|^2 dt \leqslant |g_0|^2 + \frac{1}{2\pi}\int_a^b |S_0(E-i0)|^2 dE. \quad (47)$$

The initial state f_0 is given by Eq. (13),

$$f_0 = -\frac{1}{K}\sqrt{\frac{h^2K}{2\mu}}(H^* - WI)^{-1}e,$$

and the scalar product is given by

* Similar equations can be derived (see Ref. 5) for any number of reaction channels.

** We give later a quantitative estimate of this limit.

$$|(f_0, g_0)|^2 = \frac{h^2}{2\mu K}|((H^* - WI)^{-1}e,e)|^2 \qquad (48)$$
$$= \frac{h^2}{2\mu K}|1 - S_0(W)|^2.$$

From Eqs. (47), (48) and (43) we derive the inequality

$$|f|^2_\infty \geqslant W|1 - S_0(W)|^2/K^3\Big[hP + \frac{1}{2\pi}\int_a^b |S_0(E+i0) - S_0(E-i0)|^2 dE\Big], \qquad (49)$$

where, by virtue of the Stieltjes inversion formula,

$$|S_0(E+i0) - S_0(E-i0)| = e^{\pi p(E)} - e^{-\pi p(E)}. \qquad (50)$$

6. A DEGENERATE REAL LEVEL

Returning to the case of a finite number of complex levels, we may take in particular $W_1^{(n)} = W_2^{(n)} = \ldots = W_n^{(n)} = E_0 + \frac{1}{2} i\Gamma_0^{(n)}$, and derive

$$S_0^{(n)}(W) = \left(\frac{W - W_1^{(n)}}{W - W_1^{*(n)}}\right)^n \qquad (51)$$
$$= \left(1 - \frac{i\Gamma_0^{(n)}}{W - E_0 + (i/2)\Gamma_0^{(n)}}\right)^n.$$

The corresponding operator $g = H^* f$ becomes

$$d_k^{(n)} = \left(E_0 + \frac{i}{2}\Gamma_0^{(n)}\right)c_k^{(n)} - i\sum_{j=1}^{k-1} c_j^{(n)}\Gamma_0^{(n)}, \qquad (52)$$

and the decay probability is $P = (n/h)\Gamma_0^{(n)}$. We fix P and let $n \to \infty$. Since $\Gamma_0^{(n)} = hP/n \to 0$, we have $W_k^{(n)} \to E_0$. Equation (51) gives the following expression for the one-dimensional scattering matrix in the case of a degenerate real level

$$S_0(W) = \exp\{- ihP/W - E_0\}. \qquad (53)$$

We define step-functions $f_n(x)$ $(0 \leq x \leq hP)$ by $f_n(x) = c_k^{(0)}$ $((k-1)hP/n \leq x \leq khP/n)$, $(k = 1, 2, \ldots, n)$, and then pass to the limit in Eq. (52). The energy operator corresponding to a single infinitely degenerate real level E_0 is thus

$$H^* f = E_0 f(x) - i\int_0^x f(\xi)\,d\xi \quad (0 \leqslant x \leqslant hP). \qquad (54)$$

The equation of motion is

$$ih\frac{\partial f(x,t)}{\partial t} = E_0 f(x,t) - i\int_0^x f(\xi, t)\,d\xi. \qquad (55)$$

Using Eq. (45), it is easy to obtain the solution $f(x, t)$ of Eq. (55) corresponding to the initial condition $f(x, 0) = f_0(x)$:

$$f(x,t) = e^{-iE_0 t/h}\Big[f_0(x) - \sqrt{\frac{t}{h}}\int_0^x f_0(\xi)\frac{J_1(2\sqrt{t(x-\xi)/h})}{\sqrt{x-\xi}}\,d\xi\Big], \qquad (56)$$

where $J_1(\lambda)$ is a Bessel function.

7. EFFECT OF AN EXTERNAL FIELD ON THE COMPOUND NUCLEUS SCATTERING MATRIX

In Secs. 1-6 we obtained the general properties of the amplitude of the elastic scattering process $a + X \to C \to X + a$ (a is the incident particle and X the target), assuming that the scattering is caused by the formation of the compound nucleus and by an impenetrable-sphere interaction. The effect of a spherically symmetrical potential can be included by the following simple argument. We suppose that for $r > R_c$ the interaction between a and X is described by a potential $V(r)$, while for $r < R_c$ the compound nucleus C is formed and can be described by a non-self-adjoint operator. Here r is the distance between a and X, and R_c is the radius of C. The radial function $U_l(r)$ $(r > R_c)$ corresponding to angular momentum l satisfies the equation

$$\frac{h^2}{2\mu}\frac{d^2U_l}{dr^2} + \Big[W - \frac{h^2}{2\mu}\frac{l(l+1)}{r^2} - V(r)\Big]U_l(r) = 0. \qquad (57)$$

Let $E_l(r)$ be the solution of Eq. (57) satisfying the initial conditions $E_l(R_c) = 1$, $E_l'(R_c) = iK$. Then

$$U_l(r) = \text{const}\,\{E_l^*(r) - S_l^{(C)}(W)E_l(r)\}. \qquad (58)$$

Since at $r = R_c$ we have

$$E_l(r) = e^{iK(r - R_C)}, \; E_l'(r) = [e^{iK(r-R_C)}]',$$

the function $S_l^{(c)}(W)$ may be regarded as the amplitude of pure resonance scattering caused by the formation of the compound nucleus C in the absence of a potential. On the other hand, as $r \to \infty$ we have the asymptotic relation

$$U_l(r) \approx \text{const}\,\{e^{-i(Kr - l\pi/2)} - S_l(W)\, e^{i(Kr - l\pi/2)}\}, \tag{59}$$

where $S_l(W)$ is the total amplitude, including the effects both of the compound nucleus and of the potential. In the case of a short-range potential $[V(r) \equiv 0 \text{ for } r > R]$ the function $U_l(r)$ can also be presented in the form

$$U_l(r) = \text{const}\,\{U_l^{(-)}(r) - S_l(W)\, U_l^{(+)}(r)\} \quad (r > R), \tag{60}$$

where $U_l^{(+)}(r)$ and $U_l^{(-)}(r)$ are the solutions of Eq. (57) which are asymptotically of the forms $U_l^{(+)}(r) \approx e^{i(Kr - l\pi/2)}$, $U_l^{(-)}(r) \approx e^{-i(Kr - l\pi/2)}$. Comparing the values of $U_l(R)$ and $dU_l/dr|_{r=R}$ obtained from Eqs. (58) and (60), we find

$$S_l(W) = a_l S_l^{(C)}(W) - b_l/(b_l^* S_l^{(C)}(W) - a_l^*),$$

with

$$a_l = \frac{dE_l}{dr} U_l^{(-)} - E_l \frac{dU_l^{(-)}}{dr}\bigg|_{r=R},$$

$$b_l = \frac{dE_l^*}{dr} U_l^{(-)} - E_l^* \frac{dU_l^{(-)}}{dr}\bigg|_{r=R}.$$

In this way the scattering amplitude with an external potential appears as a fractional linear transformation of the amplitude $S_l^{(c)}(W)$, and the coefficients a_l, a_l^*, b_l, b_l^* can be computed if the potential $V(r)$ is given. If the compound nucleus C has the complex energy-levels $W_j^{(l)}$, then by Eq. (30) the amplitude $S_l^{(c)}(W)$ is

$$S_l^{(C)}(W) = \prod_{j=1}^{n} \frac{W - E_j^{(l)} - (i/2)\,\Gamma_j^{(l)}}{W - E_j^{(l)} + (i/2)\,\Gamma_j^{(l)}}.$$

Recently, various attempts have been made to unify the idea of a compound nucleus with the theory of nuclear shells[21], so it is interesting to consider the case in which the compound nucleus C possesses in a certain energy interval a large number of almost equally spaced levels $W_j = W_0 + jD + i\Gamma/2\ (j = 1, 2, \ldots, n)$, where D is the mean level spacing and Γ the mean width. In this case

$$S_l^{(C)} = \prod_{j=1}^{n} \frac{1 - \left(W - W_0 - \frac{i}{2}\Gamma\right)/jD}{1 - \left(W - W_0 + \frac{i}{2}\Gamma\right)/jD}.$$

If $\Gamma/D \ll 1$, we have approximately

$$S_l^{(C)} = \frac{\sin \pi\left(W - W_0 - \frac{i}{2}\,\Gamma\right)/D}{\sin \pi\left(W - W_0 + \frac{i}{2}\,\Gamma\right)/D}.$$

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