Generalized Phase Analysis as a Corollary of the Unitary Property of the S-Matrix

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A method is given for the determination of the general form of the scattering amplitude, providing a generalization of the usual formulas of phase-shift analysis. An invariant operational technique is developed which simplifies the calculations connected with the expansion in terms of eigenfunctions of the total angular momentum.

INTRODUCTION

THE problem of phase-shift analysis is the determination of the general form of the scattering amplitude of particles of definite type. It is well known that in the simplest cases (spins 0 and ½) the scattering amplitude can be expressed in terms of the phases of the scattered partial waves. A phase analysis establishing only the general form of the scattering amplitude cannot determine the concrete values of the phases. The values of the phases are determined either by solution of the dynamical problem of the scattering or by means of analysis of experimental data. In spite of this, the general formula expressing the scattering amplitude or cross-section in terms of a number of independent parameters is extremely useful, since on one hand it facilitates the analysis of the experimental data, and on the other hand it reduces the dynamical problem of the scattering to the problem of determining the phases.

But in the more complicated cases (when the spin of the system exceeds ½) the simple formulas of phase analysis are no longer valid. For instance, for the scattering of a nucleon by a nucleon, instead of real phases one must introduce complex phases for the triplet states, and the amplitude for photoproduction of mesons is not expressible in terms of phases at all.

A second important point is the question of the foundation of the phase analysis. In the simplest cases the formulas of phase analysis have been obtained from a consideration of the asymptotic behavior of the solution of the Schrödinger equation. but for many important cases, in particular for the scattering of relativistic particles, the question of the existence of the corresponding Schrödinger equation cannot be regarded as settled. On the other hand it is obvious that the formulas of the phase analysis are applicable also in those cases in which a Schrödinger equation does not exist for the system under consideration.

In the present paper a method is proposed for obtaining the formulas of the phase analysis from an extremely general requirement, namely from the condition of the unitary character of the S-matrix. In the simplest cases this method leads to the well-known expressions for the scattering amplitude in terms of phases. In other cases the scattering amplitude is expressed in a more complicated fashion, but a general procedure is indicated for determining formulas expressing the scattering amplitude in terms of the minimum number of independent parameters.

An analogous program has been carried out in a number of papers devoted to the formal theory of nuclear reactions (cf., for example, Ref. 1). But in the general case the formulas there obtained have an extraordinarily cumbersome appearance because of the unfortunate choice of the eigenfunctions of the total angular momentum. Instead of the expansion in terms of eigenfunctions of the angular momentum, we develop in Sec. 2 a technique of invariant operators W. The expansion of the various operators encountered in the scattering theory in terms of the operators W generalizes to the case of spin different from zero the usual expansion in terms of Legendre polynomials for spin zero. The orthogonality conditions, Eq. (2.7), in a certain sense play the role of the addition theorem for Legendre polynomials, and this makes it possible to use the operators W for the separation of the angular variables. The use of the invariant operators W frees us from the necessity of a definite choice of a coordinate system and spares us the cumbersome calculations associated with spherical functions involving spin.

It must be remarked that the invariant operators W are determined for spin ½ in Ref. 2, and for the general case in Ref. 3, where some of their properties are investigated. Unlike the previous treatment [3] we do not use the apparatus of polarized

solid harmonics in determining the operators W .
Since the problem of the present paper did not
involve the calculation of the quantities W in
explicit form, we have not definitely specified
the forms of the eigenfunctions of the angular mo-
mentum, but have only used their orthogonality
relations. The explicit calculation of the quantities
W in closed form can be carried through by means
of the generalized spherical functions introduced
by Gel 'fand[4], which, in our opnion, should be
included in the apparatus of theoretical physics
along with the ordinary spherical functions.

The calculation of the quantities W and the con-
sideration of the formulas of the generalized phase
analysis will be the subject of another paper.

1. THE S-MATRIX AND THE SCATTERING AMPLITUDE

The scattering operator Ω , which transforms the
incident wave into the sum of the incident and
scattered waves, can be written in the form[5]

$$\Omega = I + \delta_+ (\varepsilon - \varepsilon') R. \qquad (1.1)$$

where I is the identity operator, $\delta_+ (\epsilon) = \frac{1}{2} \delta (\epsilon) + 1/2 \pi i \epsilon$, and the matrix R is a regular function
of the energy. Since energy is conserved in the
scattering, a description of the scattering requires
only the value of R on the energy surface, i. e.,
those matrix elements of R for which the energies
of the initial and final states are equal.

The operator

$$S = I + R, \qquad (1.2)$$

considered on the energy surface, is called the
S -matrix. In the theory of scattering it is shown
that the S-matrix is unitary.[5,6]

In what follows we shall consider only those
scattering processes in which two colliding par-
ticles give two scattered particles, i. e., reactions
of the type

$$a + b \to c + d \qquad (1.3)$$

We shall not, however, assume that c and d are
particles of the same kind as a and b . In parti-
cular, they can have different masses, as for
example, in photomesic reactions. When we speak
of the complete set of states of the system, we
shall have in mind noth all possible initial states
and also all possible final states that can be
formed as a result of the scattering. For example,
for the reaction

$$\gamma + p \to \pi^+ + n$$

we shall consider the initial state $\gamma + p$ and the
final state $\pi^+ + n$ as different states of one and
the same system. We note that the definition of
a complete set of states is of a somewhat con-
ventional nature and depends on the degree of
precision with which the given scattering process
is treated. For example, in studying the scattering
of π -mesons by nucleons, one may to a high degree
of accuracy neglect the radiative capture of the
π -meson. In this case the complete set of states
of the system is the manifold of all possible states
of the type " π -meson + nucleon." If, however,
we do not neglect the radiative capture, then it is
necessary to include in our consideration also
states of the type "nucleon + γ -quantum." In
this case four different processes —-the scattering
of the meson by the nucleon, the radiative capture
of the meson by the nucleon and its inverse pro-
cess, the photoproduction of a meson, and also the
scattering of light by nucleons — are united in
a single scattering process.

Accordingly, in considering any sort of scatter-
ing process, one must specify a system of states
which can, to a certain approximation, be re-
garded as closed with respect to this scattering.
Only in relation to a closed system of states can
one speak of the unitary nature of the S -matrix.
Indeed, if we confine ourselves to an unclosed
system of states, the concept of the unitary
property of the S-matrix loses its meaning. Thus,
for example, one cannot speak of the unitary
property of the S-matrix describing only the process
of photoproduction of mesons.

Let us introduce dynamical variables defining
the states of the system. It is convenient to con-
sider the reaction (1.3) in the center-of-mass
system, in which the total momentum vanishes

$$\mathbf{p} = \mathbf{p}_a + \mathbf{p}_b = \mathbf{p}_c + \mathbf{p}_d = 0. \qquad (1.4)$$

In what follows we shall assume that all quanti-
ties are referred to the center-of-mass system.

We shall describe a complete set of states by
the variables ϵ , n, σ , α, where ϵ is the total
energy of the system, n is a unit vector directed
along the momentum of one of the particles (in
virtue of Eq. (1.4), apart from sign, n determines
the direction of motion of both particles), σ is
the total spin variable of the system, and α is the
set of all other variables, which we shall not
specify explicitly. In particular, the value of α

determines the types of particles in one or another state.

With a view to the application to the scattering of relativistic elementary particles, we shall suppose that the energy ϵ is connected with the momentum p of one particle [because of (1.4) the magnitude of p is the same for both particles) in the state α by the relation

$$\varepsilon = \varepsilon_\alpha(p) = \sqrt{p^2 + M_\alpha^2} + \sqrt{p^2 + \mu_\alpha^2}, \quad (1.5)$$

where M_α and μ_α are the masses of the particles in the state α (we use a system of units in which $\hbar = 1$, $c = 1$).

The matrix elements of the operator A will be denoted by symbols $(\epsilon\, n\, \sigma\, \alpha \,|\, A \,|\, e'n\,'\sigma'\alpha\,')$. If, however, the operator is considered on the energy surface, i.e., for $\epsilon = \epsilon\,'$, then we shall omit ϵ from the labels of the matrix elements. Thus an element of the S-matrix will be denoted by the symbol $(n\,\sigma\,\alpha \,|\, S \,|\, n'\sigma'\alpha\,')$, with the understanding that this quantity is a function of ϵ. The connection between the S-matrix and the scattering amplitude is given by the relations (1.1) and (1.2). In the coordinate representation these relations have the following form. If the incident plane wave is

$$\psi_0 = e^{i\mathbf{p}_0\mathbf{x}}\delta_{\sigma\sigma_0}\delta_{\alpha\alpha_0},$$

then at great distances the sum of the incident and scattered waves is equal to

$$\Omega\psi_0 = \exp\{i\mathbf{p}_0\mathbf{x}\}\,\delta_{\sigma\sigma_0}\delta_{\alpha\alpha_0} + \frac{\exp\{ip_\alpha r\}}{r}\,F; \quad (1.6)$$

$$F = \frac{2\pi}{ip_0}\sqrt{\frac{v_0}{v_\alpha}}\,(n\sigma\alpha \,|\, R \,|\, n_0\sigma_0\alpha_0),$$

where $\mathbf{p}_0 = p_0\mathbf{n}_0$ and $\mathbf{p}_\alpha = p_\alpha\,\mathbf{n}_\alpha$ are the momenta in the initial and final states, $\mathbf{x} = r\mathbf{n}$, and $v_\alpha = \partial\epsilon_\alpha/\partial p$ is the relative velocity. According to Eq. (1.5), the absolute values of the momenta are determined in terms of the total energy ϵ_0 by the relations

$$\varepsilon_{\alpha_0}(p_0) = \varepsilon_\alpha(p_\alpha) = \varepsilon_0.$$

The asymptotic expression (1.6) for the wave function is calculated in a well-known way.[7] It is only necessary to keep in mind that the operators (1.1), (1.2) are taken to be in the ϵ-representation, and in the passage to the x-representation one must use plane waves, normalized to a δ-function of the total energy ϵ.

By means of Eq. (1.6) one can easily find the expression for the differential scattering cross-section

$$d\sigma/d\Omega = |F|^2 v_\alpha/v_0.$$

Substituting the value of F from Eq. (1.6), we obtain

$$d\sigma/d\Omega = (4\pi^2/p_0^2)\,|(n\sigma\alpha \,|\, R \,|\, n_0\sigma_0\alpha_0)|^2. \quad (1.7)$$

2. THE INVARIANT OPERATORS W AND THE EXPANSION OF THE S-MATRIX

If no external fields act on the system, then the total angular momentum M of the system (including spin) is conserved. In this case it is convenient to transform to a representation in which the quantities M^2 and M_z are diagonal. This representation will be called the angular momentum representation.

We construct a complete orthonormal system of eigenfunctions of M^2 and M_z : Y_{jm}^r (n, σ), where j, m are the values of the total angular momentum and of its projection, and for a given j the index r runs through as many values as there are distinct independent systems of states with the total angular momentum j. For example, for the system of two nucleons the total spin takes the values 0 and 1, and corresponding to this the index r runs through four values: one for the single state of the system and three for the triplet.

The orthogonality condition for the functions Y_{jm}^r has the form

$$\sum_\sigma \int Y_{jm}^{r*}(n\sigma)\, Y_{j'm'}^{r'}(n\sigma)\, d\Omega = \delta_{jj}\delta_{mm'}\delta_{rr'}; \quad (2.1)$$

$$\sum_{jmr} Y_{jm}^{r*}(n\sigma)\, Y_{jm}^r(n'\sigma') = \delta_{nn'}\delta_{\sigma\sigma'}.$$

Here $\delta_{nn'}$, the δ-function on the sphere, is defined by the equation

$$\int \delta_{nn'}f(n')\, d\Omega' = f(n).$$

The S-matrix can be expanded in terms of the system of functions Y_{jm}^r

$$(n\sigma\alpha \,|\, S \,|\, n'\sigma'\alpha') \quad (2.2)$$

$$= \sum_{jmr}\sum_{j'm'r'} (jmr\alpha \,|\, S \,|\, j'm'r'\alpha')Y_{jm}^r(n\sigma)\, Y_{j'm'}^{*r'}(n'\sigma').$$

We now make use of the conservation of angular momentum and the absence of any favored direction in space, in consequence of which the

S-matrix in the angular momentum representation will have the form

$$(jmr\alpha \mid S \mid j'm'r'\alpha') = (r\alpha \mid S_j \mid r'\alpha')\,\delta_{jj'}\delta_{mm'}. \quad (2.3)$$

Here S_j is a matrix with respect to the variables r, α, and depends on the value j of the total angular momentum, but not on the projection m. The relation (2.3) shows that in the angular momentum representation the S-matrix divides into diagonal blocks, S_j corresponding to the various values of j. In concrete cases the matrices S_j can be further broken down, by using, in addition to the conservation of angular momentum, other conservation laws as well — conservation of parity, of isotopic spin, etc. We note that the system of functions Y^r_{jm} with given jm and various values of r is determined apart from a unitary transformation. It is convenient to choose the functions Y^r_{jm} in such a way as to secure the most complete breaking up of the matrices S_j that is possible. For example, for systems consisting of two nucleons it is convenient to choose them that the singlet states are separated from the triplets, and then owing to the conservation of the total spin the matrix S_j breaks up into singlet and triplet parts.

When Eq. (2.3) is used, we can write Eq. (2.2) in the form

$$(n\sigma\alpha \mid S \mid n'\sigma'\alpha') \qquad (2.4)$$

$$= \sum_{j}\sum_{rr'}(r\alpha \mid S_j \mid r'\alpha')\,W_j^{rr'}(n\sigma;\ n'\sigma'),$$

where

$$W_j^{rr'}(n\sigma,\ n'\sigma') \qquad (2.5)$$

$$= \sum_{m=-j}^{j}Y^r_{jm}(n\sigma)\,Y^{r'}_{jm}{}^{*}(n'\sigma').$$

We indicate certain simple properties of the operators $W_j^{rr'}$ that follow directly from the definition (2.5) and the orthogonality condition (2.1):

Hermitian character

$$(W_j^{rr'}(n\sigma;\ n'\sigma'))^{*} = W_j^{r'r}(n'\sigma';\ n\sigma), \qquad (2.6)$$

orthogonality

$$\sum_{\sigma'}\int d\Omega'\,W_j^{rr_1}(n\sigma;\ n'\sigma')\,W_{j'}^{r'r_0}(n'\sigma';\ n_0 r_0) \qquad (2.7)$$

$$= \delta_{jj'}\delta_{r_1 r_2}W_j^{rr_0}(n\sigma;\ n_0\sigma_0),$$

completeness

$$\sum_{jr}W_j^{rr}(n\sigma;\ n'\sigma') = \delta_{nn'}\delta_{\sigma\sigma'}. \qquad (2.8)$$

In what follows we shall need the equation

$$\sum_{\sigma}W_j^{rr'}(n\sigma;\ n\sigma) = \frac{2j+1}{4\pi}\delta_{rr'}, \qquad (2.9)$$

which follows easily from the orthogonality conditions (2.1) if we note that as an invariant function of the single vector n the left-hand member of Eq. (2.9) must be a constant.

Taking into account the condition (2.8), we easily find for the operator $R = S - 1$ a representation analogous to Eq. (2.4)

$$(n\sigma\alpha \mid R \mid n'\sigma'\alpha'). \qquad (2.10)$$

$$= \sum_{j}\sum_{rr'}(r\alpha \mid S_j - I_j \mid r'\alpha')\,W_j^{rr'}(n\sigma;\ n'\sigma');$$

where

$$(r\alpha \mid I_j \mid r'\alpha') = \delta_{rr'}\delta_{\alpha\alpha'}.$$

In general, representations of the type (2.4) exist for any operator invariant with respect to rotations. Such operators are characterized by the fact that in the angular momentum representation their matrices are diagonal with respect to the quantities jm and do not depend on m, i. e., are of a form analogous to Eq. (2.3). Therefore all of the considerations leading to the expression (2.4) hold for such operators. In the Appendix we shall make use of this remark to accomplish the separation of the angular variables.

Let us consider the form of the operators W_j^{rr} for the simplest cases. For spin 0, the index r takes one value, which can be omitted. The functions $Y_{jm}(n)$ are identical with the ordinary spherical functions $Y_{jm}(\theta,\varphi)$, where θ and φ are the polar angles of the vector n. By means of the addition theorem for the Legendre polynomials it is easy to establish the equation

$$W_j(n;\ n') = \frac{2j+1}{4\pi}P_j(nn'), \qquad (2.11)$$

where $P_j(x)$ is the Legendre polynomial (unnormalized).

The case of spin ½, corresponding to the system π-meson + nucleon, is less trivial. As the functions Y^r_{jm} we take the spherical spinors. The index r takes two values $\pm$, corresponding to the two different states with a fixed total angular momentum j, for which the quantity j is connected with the orbital angular momentum

l by the equations $j = l \pm \frac{1}{2}$. The operators W have the following form:

$$W_j^{++} = \frac{1}{4\pi} [P'_{j+1/2}(\mathbf{nn'}) - P'_{j-1/2}(\mathbf{nn'})(\sigma\mathbf{n})(\sigma\mathbf{n'})];$$

$$W_j^{--} = \frac{1}{4\pi} [-P'_{j-1/2}(\mathbf{nn'}) + P'_{j+1/2}(\mathbf{nn'})(\sigma\mathbf{n})(\sigma\mathbf{n'})];$$

$$W_j^{+-} = \frac{1}{4\pi} [P'_{j-1/2}(\mathbf{nn'})(\sigma\mathbf{n}) - P'_{j+1/2}(\mathbf{nn'})(\sigma\mathbf{n'})];$$

$$W_j^{-+} = \frac{1}{4\pi} [-P'_{j+1/2}(\mathbf{nn'})(\sigma\mathbf{n}) + P'_{j-1/2}(\mathbf{nn'})(\sigma\mathbf{n'})].$$

(2.12)

In these equations $P_l'(x) = (d/dx)P_l(x)$, and σ are the Pauli spin matrices. The operators (2.12) are the same as the invariant polynomials defined in Ref. 2. For the system nucleon + photon the expressions for the operators W can be found in Ref. 3. We shall devote space here to the details of the determination of the explicit form of the operators W; we merely remark that in the general case the operators can be calculated in closed form by means of Gel ' fand 's generalized spherical functions.[4]

3. UNITARY PROPERTY OF THE S-MATRIX GENERALIZED PHASE ANALYSIS

We now consider the consequences that follow from the condition that the S-matrix be unitary.

From Equation (2.3) it can be seen that the diagonal blocks S_j must be unitary matrices. The condition that the matrices S_j be unitary imposes definite connections on their matrix elements, and thus reduces the number of essential parameters involved in the general expression (2.4) for the S-matrix. We assume that it is possible to carry out in a general form the further splitting up of the matrices S_j into diagonal blocks. These blocks will also be unitary matrices, which leads to a further reduction of the number of essential parameters in the S-matrix. We consider the extreme case of complete splitting-up.

In this case the condition that S_j be unitary leads to the relation

$$(r\alpha \,|\, S_j \,|\, r'\,\alpha') = \exp\{2i\delta_{jr\alpha}\}\,\delta_{rr'}\,\delta_{\alpha\alpha'}. \quad (3.1)$$

Here $\delta_{jr\alpha}$ are real numbers, and the factor 2 is placed in the exponent in order to obtain a more complete analogy with the equations of phase analysis. Substituting (3.1) into the general formula (2.4), we find that in this special case the S-matrix has the form

$$(\mathbf{n}\sigma\alpha \,|\, S \,|\, \mathbf{n'}\,\sigma'\,\alpha')$$

$$= \delta_{\alpha\alpha'} \sum_{jr} \exp\{2i\delta_{jr\alpha}\}\, W_j^{rr}(\mathbf{n}\sigma; \mathbf{n'}\,\sigma'). \quad (3.2)$$

From Eq. (3.2) the various formulas of phase analysis follow as special cases. For the particle with spin zero, taking into account Eqs. (1.6), (2.10), and (2.11), we obtain the scattering amplitude F in the form

$$F = \frac{1}{2ip_0} \sum_j (2j+1)(e^{2i\delta_j} - 1)P_j(\mathbf{nn_0}),$$

which agrees with the well-known formula of scattering theory.

For the system π-meson + nucleon a complete splitting-up also takes place owing to the laws of conservation of parity and isotopic spin. The corresponding formulas of phase analysis can be found in Ref. 2.

It must be pointed out that a complete breaking-up of the matrices S_j on the basis of the general conservation laws alone is possible only in exceptional cases. More generally speaking, only a partial breaking-up can be achieved. In this case the simple formula (3.2) must be replaced by the general formula (2.4). If in Eq. (2.4) the maximum possible splitting-up of the matrices S_j has been carried out, then, using the unitary property, we can express the scattering amplitude of the partial waves in terms of a certain number of parameters. In the general case these parameters play precisely the same role as the scattering phases do in the simplest cases. Ordinarily the number of these parameters is not large, and we arrive in this way at a generalized phase analysis, the merit of which is that without the solution of the dynamical problem one obtains the general form of the scattering amplitude, containing the minimum number of arbitrary parameters.

The actual values of these parameters can be obtained from the solution of the dynamical problem or by means of the analysis of experimental data. But even without fixing the values of the parameters it often turns out to be possible to draw certain general conclusions about the character of the scattering, and in particular about the shape of the angular distribution.

In the general case the differential scattering cross-section is expressed by the formula (1.7).

We state the expression for the total cross-section for an unpolarized beam, which is obtained from Eq. (1.7) by integration over all final states and averaging over the spin of the initial state. Using Eqs. (2.6) – (2.10) and the unitary property of the matrices S_j , we find

$$\sigma = \frac{2\pi}{N_0 p_0^2} \sum_{jr} (2j+1) \left[-\mathrm{Re}\,(r\alpha_0 \,|\, R_j \,|\, r\alpha_0) \right]. \quad (3.3)$$

Here N_0 is the number of orientations of the spin in the initial state. For a particle with spin zero, $R_j = e^{2i\delta_j} - 1$, and Eq. (3.3) goes over into the well-known expression of scattering theory

$$\sigma = \frac{4\pi}{p_0^2} \sum_{j} (2j+1) \sin^2 \delta_j.$$

APPENDIX. SEPARATION OF ANGULAR VARIABLES

Using the equation of the stationary scattering problem* as an example, we shall show how the separation of the angular variables can be carried out in the general case by means of the operators W .

The scattering amplitude R (not on the energy surface) defined in Eq. (1.1), satisfies the equation

$$(1)$$

$$- (1/2\pi i)(\varepsilon\mathbf{n}\sigma \,|\, R \,|\, \varepsilon_0\,\mathbf{n}_0\,\sigma_0) = (\varepsilon\mathbf{n}\sigma \,|\, V \,|\, \varepsilon_0\,\mathbf{n}_0\,\sigma_0)$$

$$+ \sum_{\sigma'} \int d\varepsilon'\, d\Omega' \,(\varepsilon\mathbf{n}\sigma \,|\, V \,|\, \varepsilon'\,\mathbf{n}'\,\sigma')$$

$$\times\; \delta_+(\varepsilon' - \varepsilon_0)\,(\varepsilon'\,\mathbf{n}'\,\sigma' \,|\, R \,|\, \varepsilon_0\,\mathbf{n}_0\,\sigma_0).$$

We assume that the interaction Hamiltonian V is invariant with respect to rotations and consequently permits an expansion analogous to Eq. (2.4). It is obvious that in this case R also has such an expansion:

$$(\varepsilon\mathbf{n}\sigma \,|\, V \,|\, \varepsilon'\,\mathbf{n}'\,\sigma') \quad (2)$$

$$= \sum_{j} \sum_{rr'} (\varepsilon r \,|\, V_j \,|\, \varepsilon'\, r')\, W_j^{rr'}(\mathbf{n}\sigma;\, \mathbf{n}'\,\sigma'),$$

$$(\varepsilon\mathbf{n}\sigma \,|\, R \,|\, \varepsilon'\,\mathbf{n}'\,\sigma')$$

$$= \sum_{j} \sum_{rr'} (\varepsilon r \,|\, R_j \,|\, \varepsilon'\, \acute{r}')\, W_j^{rr'}(\mathbf{n}\sigma;\, \mathbf{n}'\,\sigma').$$

Substituting Eq. (2) into Eq. (1) and taking into account the orthogonality conditions (2.7), we obtain a system of integral equations in which the angular variables are eliminated.

$$-\frac{1}{2\pi i}\,(\varepsilon r \,|\, R_j \,|\, \varepsilon_0\, r_0) = (\varepsilon r \,|\, V_j \,|\, \varepsilon_0\, r_0) \quad (3)$$

$$+ \sum_{r'} \int d\varepsilon' \,(\varepsilon r \,|\, V_j \,|\, \varepsilon'\, r')\, \delta_+(\varepsilon' - \varepsilon_0)\,(\varepsilon' r' \,|\, R_j \,|\, \varepsilon_0\, r_0).$$

The system (3) takes the simplest form in the case of complete resolution of the matrices V_j , i.e., when we have the equation

$$(\varepsilon r \,|\, V_j \,|\, \varepsilon'\, r') = (\varepsilon \,|\, V_j \,|\, \varepsilon')\, \delta_{rr'}.$$

In this case the system of equations (3) breaks up into individual equations involving the quantities $(\epsilon r_0 \,|\, R_j \,|\, \epsilon_0\, r_0)$.

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