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Cascade Showers Produced by Electrons and Photons in Light and Heavy Elements

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A regular method is given for plotting cascade curves from known moments. The cascade curves in light elements for primary electrons and photons with an energy E_0 are computed (E_0 is of the order of the critical energy for the given substance). A recursion formula is deduced which permits the calculation of all cascade curve moments with account of scattering, ionization losses and the dependence of the total photon absorption coefficient $\sigma(E)$ on energy for an arbitrary primary particle spectrum. Numerical calculations of the first two momenta of the electron distribution curve in lead were carried out for a primary photon spectrum of the form $1/E$. The cascade curves in lead were computed from the known moments $\overline{t}$ and $\overline{t^2}$. The results of the calculations are compared with experiment.

THE cascade theory of showers, developed in Ref. 1, describes with sufficient completeness the propagation of electron-photon high energy showers [when $\ln(E_0/\beta) > 1$] in light elements. The basic equations of the theory are usually solved by the method of functional transforms (the Laplace-Mellin transform in the energy E and the Laplace transform in the depth t)*. We employ this method because the asymptotic function (used in the theory) for the cross section of the processes of pair formation by photons and bremsstrahlung are homogeneous functions of the energy of the particles.

If the energy E_0 of the initial particle which starts the shower is not very great in comparison with the critical energy β for the given element, then we cannot obtain a solution of the basic equations by the method of functional transforms. The solution of the equations of cascade theory in analytic form in this energy region is of considerable mathematical difficulty. In the passage of electrons and photons through heavy elements, the formation of showers becomes very intense. We recall the difficulties which are encountered in the theory of showers for heavy elements: (1) the total photon absorption coefficient $\sigma(E)$ depends very strongly on the energy, σ changing from the value 0.773 for high energies to 0.24 at its minimum; (2) the scattering of the shower particles in heavy elements is very great.

A method was developed in Refs. 2 and 3 for the solution of the basic equations of cascade theory (integrated over the depth) in the variable σ, which permits us to obtain an expression for the "equilibrium" spectrum of electrons and photons and, using this spectrum to calculate successively, in principle, all the moments of the distribution function of the shower particles. The effect of scattering on the magnitude of the moments and the form of the curve for heavy elements was considered in Refs. 3 and 4. The treatment in Ref. 3 was highly approximate, since it was carried out without consideration of the ionization loss. A recursion formula was obtained in Ref. 4 which permits successive calculation of all the moments $\overline{t^n}$ of the distribution function, with consideration of scattering and ionization losses for an original spectrum of δ-shape. A recursion formula was obtained in Ref. 5 for the determination of the moments $\overline{t^n}$ without regard to scattering in the case of an arbitrary spectrum of the primary particles. The first two moments were calculated in lead for a spectrum of primary photons of the form $1/E$. The computed moments agree with experiment. However, inasmuch as scattering was not considered in this work, accuracy is lacking. For the solution of the fundamental equation of cascade theory, it is possible to approach the problem differently, i.e., to determine the function $N(E_0,0,t)$ (the number of particles with energy greater than zero at the depth t^n, created by an initial particle with energy E_0) along with its moments $\overline{t^n}(E_0, 0)$.

The purpose of the present work is the calculation, by the method of moments, of the cascade curves from primary electrons and photons of not very great energy E_0 in light and heavy elements.

* The depth t is measured in so-called "cascade" units.

In the first part of the work, explicit analytical expressions are derived for the first and second moments of cascade curves in light elements; the first three moments of cascade curves are calculated for light elements; these cascade curves are calculated by the use of available values of the moments. In the second section, a recursion formula is derived which permits successive calculation of all moments of the cascade curve with consideration of scattering and ionization losses for an arbitrary spectrum of the primary particles. This conclusion is a generalization of the results of Ref. 5, where a similar formula was obtained without taking scattering into account, and of Ref. 4, where a similar formula was obtained for a δ-type spectrum with consideration of scattering. Numerical calculations are carried out for the first two moments of the electron distribution function in lead for a primary spectrum of the form $1/E$. According to the method developed in the first part of the work, the cascade curves are then computed for this same spectrum.

1. CASCADE SHOWERS IN LIGHT ELEMENTS

1. The basic equations of cascade theory have the form

$$L_1[P(E_0, E, t), \Gamma(E_0, E, t)] = \frac{\partial P(E_0, E, t)}{\partial t}; \quad (1)$$

$$L_2[P(E_0, E, t), \Gamma(E_0, E, t)] = \frac{\partial \Gamma(E_0, E, t)}{\partial t}.$$

Here $P(E_0, E, t)$ and $\Gamma(E_0, E, t)$ are distribution functions for electrons and photons at the depth t and energy E, L_1, L_2 are integro-differential operators (linear in P and Γ) which take into account pair formation by photons, bremsstrahlung and ionization losses for electrons. For small energies, the operators must also take the Compton effect into account[1]. The cascade curve--the total number of particles as a function of the depth--is determined by the relation

$$N(E_0, 0, t) = \int_0^{E_0} P(E_0, E, t)\, dE.$$

The moments of the cascade curves are determined by the expression

$$\overline{t^n}(E_0, 0) \quad (1a)$$

$$= \int_0^\infty N(E_0, 0, t)\, t^n\, dt \Big/ \int_0^\infty N(E_0, 0, t)\, dt.$$

A recursion formula was obtained in Ref. 3 which permits successive calculation of all moments of the cascade curve

$$\{\overline{t_P^n}(E_0, 0)\}^P \quad (2)$$

$$= \frac{n}{E_0} \int_0^{E_0} [P_P(E_0, E)\{\overline{t_P^{n-1}}(E, 0)\}^P$$

$$+ \Gamma_P(E_0, E)\{\overline{t_P^{n-1}}(E, 0)\}^\Gamma]\, E dE,$$

where the superscripts on P and Γ distinguish the moments of the electron distribution functions in a shower produced by an electron or a photon, respectively. $P_P(E_0, E)$ and $\Gamma_P(E_0, E)$ are the "equilibrium" spectra (integrated over the depth) of electrons and photons from the primary electron. Thus, in principle, the problem of the solution of the basic equations of the theory reduce to the finding of the equilibrium spectrum when one has recursion formulas of the type (2). An analytical expression for the equilibrium spectrum was first found by Tamm and Belen'kii[2] who solved approximately the basic equations of the theory, integrated over the depth. One can estimate the mathematical approximations used in this solution by the method of successive approximations developed in Ref. 6. The correction to the solution of Tamm and Belen'kii does not exceed 4.5%. The computation was carried out for $E_0/\beta = 2.29$; with increase in E_0, the correction diminishes.

The error introduced in the spectrum of Tamm and Belen'kii by the inaccuracy of the initial assumptions (in the equations, use was made of asymptotic relations for the cross section of processes of pair formation and bremsstrahlung which are strictly correct only for very high energies; the Compton effect was taken approximately into account; the yield of high energy electrons which is obtained in collisions of electrons of the shower with electrons of the medium was neglected) can be estimated only by comparison with the results of computations in which these simplifying assumptions were not made.

Rossi and Klapman[7], by integrating Eqs. (1), averaged over t, obtained $N_P(E_0, E)$ for a fixed value of E as a function of the primary energy E_0. The computation was carried out for air: $E = 10^7$ ev, $E_0 \geq 10^7$ ev; in the calculations, all processes were considered that take place in the substance as a result of radiation; exact formulas were obtained for the cross section. Pair creation, radiation in the field of the electrons and the effect of the density on the ionization loss were not considered. The spectrum of Tamm and Belen'kii did not differ by more than 4% from the spectrum computed by Rossi and Klapman.

Thus it can be said that for $E \geq 10^7$ ev, in light elements, the approximate "equilibrium" spectrum differs from the exact by not more than 4%. We note that a mathematically exact solution of the equation (1), integrated over the depth, differs by 8.5% from the calculation of Rossi and Klapman. Consequently, the equilibrium spectrum of Tamm and Belen'kii is a better approximation to the actual case.

Richards and Nordheim[8], with the help of difficult computational methods, solved Eqs. (1) approximately (integrated over t). For $E/\beta < 1$, and also for $E/\beta > 1$, where the equation is applicable, the spectra of Tamm and Belen'kii and Richards and Nordheim coincide.

Carrying out calculations of the moments of the electron distribution function for the showers produced by photons of energy E_0, we get the following expressions

$$\{\bar{t}_P(E_0,\ 0)\}^\Gamma \tag{3}$$
$$= \frac{1}{f(0)}\int_0^{\varepsilon_0}\left(1+\frac{1}{\sigma(\varepsilon)}\right)\chi_2(\varepsilon_0,\varepsilon)\,d\varepsilon + \frac{1}{\sigma(\varepsilon_0)};$$

$$\{\bar{t}_P^2(E_0,\ 0)\}^\Gamma = \tau_{\Gamma_1}+\tau_{\Gamma_2}+\tau_{\Gamma_3}+\tau_{\Gamma_4}+\tau_{\Gamma_5}; \tag{4}$$

$$\tau_{\Gamma_1} = \frac{2}{[f(0)]^2}\int_0^{\varepsilon_0}\int_{\varepsilon_1}^{\varepsilon_0}\left(1+\frac{1}{\sigma(\varepsilon_1)}\right)$$
$$\times\left(1+\frac{1}{\sigma(\varepsilon_{11})}\right)\chi_2(\varepsilon_0,\varepsilon_1)\,\chi_2(\varepsilon_0,\varepsilon_{11})\,d\varepsilon_{11}\,d\varepsilon_1;$$

$$\tau_{\Gamma_2} = \frac{2}{f(0)}\int_0^{\varepsilon_0}\left(\frac{1}{\sigma(\varepsilon_0)}+\frac{1}{\sigma^2(\varepsilon_1)}+\frac{1}{\sigma(\varepsilon_0)\,\sigma(\varepsilon_1)}\right)\chi_2$$
$$\times(\varepsilon_0,\varepsilon_1)\,d\varepsilon_1 + \frac{2}{\sigma^2(\varepsilon_0)};$$

$$\tau_{\Gamma_3} = \frac{2}{[f(0)]^2}\int_0^{\varepsilon_0}\int_{\varepsilon_1}^{\varepsilon_0}\left(1+\frac{1}{\sigma(\varepsilon_1)}\right)\left(1+\frac{1}{\sigma(\varepsilon_{11})}\right)\chi_2(\varepsilon_0,\ \varepsilon_{11})$$
$$\times[\chi_1(\varepsilon_{11},\ \varepsilon_1)-\chi_2(\varepsilon_0,\ \varepsilon_1)]\,d\varepsilon_{11}\,d\varepsilon_1;$$

$$\tau_{\Gamma_4} = \frac{2}{[f(0)]^2}\int_0^{\varepsilon_0}\int_{\varepsilon_1}^{\varepsilon_0}\left(1+\frac{1}{\sigma(\varepsilon_1)}\right)\frac{1}{\sigma(\varepsilon_{11})}\chi_2$$
$$\times(\varepsilon_0,\ \varepsilon_{11})\ [\chi_2(\varepsilon_{11},\ \varepsilon_1)-\chi_1(\varepsilon_0,\varepsilon_1)]\,d\varepsilon_{11}\,d\varepsilon_1;$$

$$\tau_{\Gamma_5} = \frac{2}{[f(0)]^2}\int_0^{\varepsilon_0}\int_{\varepsilon_1}^{\varepsilon_0}\left(1+\frac{1}{\sigma(\varepsilon_1)}\right)$$
$$\times\frac{\partial\chi_1(\varepsilon_{11},\ \varepsilon_1)}{\partial\varepsilon_{11}}\varepsilon_{11}\,\chi_2(\varepsilon_0,\ \varepsilon_{11})\,d\varepsilon_{11}\,d\varepsilon_1,$$

where

$$\varepsilon_0 = f(0)E_0/\beta;\quad \varepsilon = f(0)E/\beta;$$
$$f(0) = 2.29;\quad \sigma_0 = 0{,}773;$$
$$\chi_1(\varepsilon_0,\ \varepsilon) = \varepsilon e^{\varepsilon}\int_{\varepsilon}^{\varepsilon_0}\frac{e^{-x}}{x^2}\,dx;\quad \chi_2(\varepsilon_0,\ \varepsilon)$$
$$= \chi_1(\varepsilon_0,\ \varepsilon) - \frac{\varepsilon}{\varepsilon_0^2}\left(1-e^{-(\varepsilon_0-\varepsilon)}\right).$$

Similar expressions for the moments of the electron distribution function in a shower produced by an electron of energy E_0 were obtained in Ref. 1. The integrals which enter into both these expressions are expressed by elementary functions, exponential integrals and rapidly converging series. Carrying out the appropriate steps, we get, in the case of the primary electron,

$$\{\bar{t}_P(E_0,0)\}^P = \frac{1+1/\sigma_0}{f(0)}\left\{\frac{1}{\varepsilon_0}-1-\frac{e^{-\varepsilon_0}}{\varepsilon_0}\right. \tag{5}$$
$$\left.+C+\ln\varepsilon_0-\mathrm{Ei}(-\varepsilon_0)\right\};$$

$$\tau_{P_4} = \frac{(2/\sigma_0)(1+1/\sigma_0)}{[f(0)]^2}\left\{-1{,}5\left(\frac{1}{\varepsilon_0}-\frac{e^{-\varepsilon_0}}{\varepsilon_0}-1\right)\right.$$
$$-0.5\,(\ln\varepsilon_0+C-\mathrm{Ei}(-\varepsilon_0))$$
$$\left.-\int_0^{\varepsilon_0}e^{-x}\left(\frac{1}{2!2}+\frac{x}{3!3}+\ldots\right)dx\right\};$$

$$\tau_{P_2} = \frac{2}{f(0)\,\sigma_0^2}\left\{\frac{1}{\varepsilon_0}-1-\frac{e^{-\varepsilon_0}}{\varepsilon_0}\right.$$
$$\left.+C+\ln\varepsilon_0-\mathrm{Ei}(-\varepsilon_0)\right\},$$

$$\tau_{P_5} = \frac{2(1+1/\sigma_0)}{[f(0)]^2}\left\{2\left(\frac{1}{\varepsilon_0}-\frac{e^{-\varepsilon_0}}{\varepsilon_0}-1\right)\right. \tag{6}$$
$$\left.+\ln\varepsilon_0+C-\mathrm{Ei}(-\varepsilon_0)\right\},$$

$$\tau_{P_1}+\tau_{P_3} = \frac{2(1+1/\sigma_0)^2}{[f(0)]^2}(1+1/\sigma_0)^2$$
$$\left\{C\left(\frac{1}{\varepsilon_0}-\frac{e^{-\varepsilon_0}}{\varepsilon_0}-1+\ln\varepsilon_0+C-\mathrm{Ei}(-\varepsilon_0)\right)\right.$$
$$+(1-e^{-\varepsilon_0})\left(\frac{\ln^2\varepsilon_0}{2}+\frac{\ln\varepsilon_0}{\varepsilon_0}\right)$$
$$-\ln\varepsilon_0-0.5\int_0^{\varepsilon_0}\ln^2 x e^{-x}\,dx$$
$$+2\int_0^{\varepsilon_0}e^{-x}\left(\frac{1}{2!2}+\frac{x^2}{4!4}+\frac{x^4}{6!6}+\ldots\right)dx$$
$$+\int_0^{\varepsilon_0}e^{-x}\left[\frac{1}{2!}-\frac{x}{3!2}+\frac{x^2}{4!3}+\ldots\right]dx$$
$$\left.-\mathrm{Ei}(-\varepsilon_0)\int_0^{\varepsilon_0}\left(\frac{e^{-x}}{x^2}-\frac{1}{x^2}+\frac{1}{x}\right)dx\right\},$$

where $C = 0.5772$ is Euler's constant, $\mathrm{Ei}(-\epsilon) = -\int_{\epsilon}^{\infty} (e^{-x}/x)\,dx$.

In the case of a primary photon,

$$\{\bar{t}_P(E_0, 0)\}^{\Gamma} = \{\bar{t}_P(E_0, 0)\}^{P} + \frac{1+1/\sigma_0}{f(0)}\left(-0.5 + \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_0^2} + \frac{e^{-\varepsilon_0}}{\varepsilon_0^2}\right) + \frac{1}{\sigma_0}. \tag{7}$$

$$\begin{aligned}\{t_P^2(E_0,0)\}^{\Gamma} &= \{\overline{t_P^2}(E_0,0)\}^{P} \\ &+ \frac{2}{[f(0)]^2}\Bigg[\left(1+\frac{1}{\sigma_0}\right)\Bigg\{\left(1-\frac{1}{\sigma_0}\right)\Bigg[0.5\Big(C+\ln\varepsilon_0-\mathrm{Ei}(-\varepsilon_0) \\ &+ \frac{1}{\varepsilon_0} - \frac{e^{-\varepsilon_0}}{\varepsilon_0} - 1\Big) - 0.5\left(\frac{e^{-\varepsilon_0}}{\varepsilon_0^2} + \frac{1}{\varepsilon_0^2} - \frac{1}{\varepsilon_0} + 0,5\right)\Bigg] \\ &+ \left(\frac{1}{\sigma_0}-1\right)\Bigg[\frac{\varepsilon_0-1}{\varepsilon_0^2}\Big(C+\ln\varepsilon_0-\mathrm{Ei}(-\varepsilon_0) \\ &+ \frac{1}{\varepsilon_0} - \frac{e^{-\varepsilon_0}}{\varepsilon_0} - 1\Big) \\ &- \frac{e^{-\varepsilon_0}}{\varepsilon_0^2}\left(\frac{1}{\varepsilon_0} + e^{\varepsilon_0} - \frac{e^{\varepsilon_0}}{\varepsilon_0} - \mathrm{Ei}(\varepsilon_0) + \ln\varepsilon_0 + C\right)\Bigg] \\ &- \Bigg[\frac{\varepsilon_0^2 - 2\varepsilon_0 + 2}{\varepsilon_0^2}\Big(C+\ln\varepsilon_0-\mathrm{Ei}(-\varepsilon_0) \\ &+ \frac{1}{\varepsilon_0} - \frac{e^{-\varepsilon_0}}{\varepsilon_0} - 1\Big) - \frac{e^{-\varepsilon_0}}{\varepsilon_0^2}\Big(\varepsilon_0 - \frac{2}{\varepsilon_0} + \varepsilon_0 e^{\varepsilon_0} - 4e^{\varepsilon_0} \\ &+ 2(\mathrm{Ei}(\varepsilon_0) - \ln\varepsilon_0 - C) + \frac{2e^{\varepsilon_0}}{\varepsilon_0} + 2\Big)\Bigg] \\ &+ \frac{1}{\sigma_0}\Bigg[0.25 - 0.5\frac{\varepsilon_0-1}{\varepsilon_0^2} \\ &- 0.5\frac{e^{-\varepsilon_0}}{\varepsilon_0^2} - \frac{1}{\varepsilon_0^2}(\mathrm{Ei}(-\varepsilon_0) - C - \ln\varepsilon_0 + \varepsilon_0) \\ &+ \frac{e^{-\varepsilon_0}}{\varepsilon_0^2}(e^{\varepsilon_0} - \mathrm{Ei}(\varepsilon_0) + \ln\varepsilon_0 + C - 1)\Bigg]\Bigg\} \\ &+ f(0)\left(1+\frac{2}{\sigma_0}\right)\frac{1}{\sigma_0}\left(-0.5 + \frac{\varepsilon_0-1}{\varepsilon_0^2} + \frac{e^{-\varepsilon_0}}{\varepsilon_0^2}\right) \\ &+ \frac{[f(0)]^2}{2}(\sigma_0+1)\tau_{P_2} + \frac{[f(0)]^2}{\sigma_0^2}\Bigg],\end{aligned} \tag{8}$$

where $\mathrm{Ei}(\epsilon) = \int_{-\infty}^{\epsilon} (e^{x}/x)\,dx$.

The values of these moments are listed in Table I for values of ϵ_0 from 0.2 to 10. The third moment $\overline{t^3}$, which is in the form of unwieldy explicit formulas, was calculated by means of numerical integration.

For $E_0/\beta \gg 1$, Eqs. (5), (6) and (7) are written in the following form (with accuracy up to terms of order $1/\epsilon_0$):

$$\{\bar{t}_P(E_0, 0)\}^{P} = \ln(E_0/\beta) + 0.41; \tag{9}$$

$$\{\bar{t}_P(E_0, 0)\}^{\Gamma} = \ln(E_0/\beta) + 1{,}2;$$

$$\{\overline{t_P^2}(E_0, 0)\}^{P} - \{\bar{t}_P^2(E_0, 0)\}^{P} = 1.76\ln(E_0/\beta) - 0{,}21;$$

$$\{\overline{t_P^2}(E_0, 0)\}^{\Gamma} - \{\bar{t}_P^2(E_0, 0)\}^{\Gamma} = 1.76\ln(E_0/\beta) + 2.32.$$

Values are given in Table II for these moments, for ϵ_0 from 15 to 2290. The values were computed from Eqs. (9) and from the formulas of Rossi and Greisen[10]:

$$\{\bar{t}_P(E_0, 0)\}^{P} = 1{,}01\ln(E_0/\beta) + 0.4; \tag{10}$$

$$\{\bar{t}_P(E_0, 0)\}^{\Gamma} = 1.01\ln(E_0/\beta) + 1{,}2;$$

$$\{\overline{t_P^2}(E_0, 0)\}^{P} - \{\bar{t}_P^2(E_0, 0)\}^{P} = 1.61\ln(E_0/\beta) - 0.2;$$

$$\{\overline{t_P^2}(E_0, 0)\}^{\Gamma} - \{\bar{t}_P^2(E_0, 0)\}^{\Gamma} = 1.61\ln(E_0/\beta) + 0{,}9.$$

It is seen from Table II that the first and second moments, computed by Eqs. (9) and (10), differ from one another by not more than 1% over a wide energy range. Thus, the expression for the moments, according to Tamm and Belen'kii, does not differ markedly from the solution of the basic equation of cascade theory obtained by the method of Snyder. As shown in Ref. 1, Snyder's solution is the first term in the expansion of the total energy in powers of β/E_0. However, the method of Snyder permits us to find only the total number of particles as a function of the depth and does not give any possibility of calculating the energy spectrum of the particles.

2. For high energy showers, when $\ln(E_0/\beta) > 1$, we can determine the position of the maximum and the number of particles in the maximum of the cascade curve if the quantities E_0/β, $\bar{t}$ and $\overline{t^2}$ are known. For low energy showers, where $E_0/\beta \sim 1$, similar relations cannot be obtained, since the procedure employed in deriving the relations

$$t_{\max} \text{ and } N_{\max} \sim f(E_0/\beta, \bar{t}, \overline{t^2}),$$

TABLE I

ε_0	$\{\bar{t}_P\}^P$	$\{\bar{t}_P\}^\Gamma$	$\{\overline{t_P^2}\}^P$	$\{\overline{t_P^2}\}^\Gamma$	$\{\overline{t_P^3}\}^P$	$\{\overline{t_P^3}\}^\Gamma$
0.2	0.096	1.357	0.148	3.694		
0.4	0.188	1.421	0.301	3.899		
0.6	0.273	1.480	0.459	4.119		
0.8	0.353	1.536	0.619	4.375		
1.0	0.429	1.590	0.780	4.609	2.940	19.79
1.2	0.500	1.641	0.943	4.850		
1.4	0.568	1.691	1.107	5.083		
1.6	0.633	1.740	1.271	5.313		
1.8	0.694	1.785	1.434	5.540		
2.0	0.752	1.829	1.599	5.762	6.364	26.51
3.0	1.006	2.027	2.392	6.803	9.817	32.78
4.0	1.213	2.195	3.132	7.746	13.37	38.79
5.0	1.386	2.340	3.845	8.625	16.84	44.52
6.0	1.536	2.468	4.503	9.429	20.29	50.02
7.0	1.666	2.582	5.192	10.26	29.53	61.29
8.0	1.782	2.685	5.691	10.87	32.73	65.86
9.0	1.886	2.778	6.233	11.53	42.83	77.59
10.0	1.979	2.862	7.140	12.53	48.16	86.07

TABLE II

ε_0	$\{\bar{t}_P\}^P$		$\{\bar{t}_P\}^\Gamma$		$\{\overline{t_P^2}\}^P$		$\{\overline{t_P^2}\}^\Gamma$	
	according to Belen'kii	according to Rossi and Greisen	according to Belen'kii	according to Rossi and Greisen	according to Belen'kii	according to Rossi and Greisen	according to Belen'kii	according to Rossi and Greisen
15	2.290	2.298	3.080	3.098	8.343	8.108	13.40	13.52
20	2.577	2.589	3.367	3.389	10.24	9.992	15.74	15.87
25	2.800	2.814	3.590	3.614	11.84	11.56	17.66	17.81
30	2.983	2.999	3.773	3.799	13.22	12.94	19.32	19.47
40	3.270	3.289	4.060	4.089	15.52	15.22	22.06	22.22
63.6	3.734	3.757	4.524	4.557	19.58	19.27	26.84	27.02
229	5.015	5.051	5.805	5.851	33.04	32.73	42.64	42.55
2290	7.318	7.377	8.108	8.177	65.5	65.34	78.78	78.88

is equivalent to the method of steepest descents, usually employed in cascade theory. One can put the problem of the calculation of the cascade curve in terms of known moments $\overline{t^n}$; knowing all moments, we can, in principle, find the function itself. However, in practice, it is possible to find only the first 2-3 moments; therefore, one must find approximation formulas which describe the cascade curves with sufficient accuracy by means of the first 2-3 moments. In Ref. 1, the cascade curves are approximated for very high values of E_0 by a curve of the form

$$\exp(\alpha t^{1/2} - \gamma t). \tag{11}$$

The coefficients α and γ are so chosen that (11) gives the correct value of the area under the cascade curve (law of conservation of energy) and the first moment $\bar{t}$. For smaller E_0, we can look for the cascade curve in the form

$$\exp(\alpha t^{1/2} - \gamma t) + \chi e^{-\gamma t}. \tag{11'}$$

The coefficients α, γ, and χ are so chosen that (11′) gives correct results for the area under the curves $\bar{t}$ and $\overline{t^2}$. The cascade curves for lead were obtained in the work of Zatsepin[11] with the help of approximations similar to (11′).

The approximation used in Ref. 11 describes the cascade process satisfactorily for the case $\ln(E_0/\beta) > 1$ and $t > t_{max}$. To determine the coefficients which enter into the approximating formula (11) we must solve a system of 3 transcendental equations, which greatly complicates the computation. We also note that in the method of Ref. 11 for constructing the curve from three moments, it is necessary to choose another form of the approximating dependence, which cannot be determined without detailed investigation. For the determination of the coefficients, we must solve a system of 4 transcendental equations.

We can suggest a regular method of constructing cascade curves, using a set of polynomials that are orthogonal in the interval $(0, \infty)$. We approximate the function $\varphi(x)$ by the sum of polynomials

$$\varphi(x) = A_0 + A_1 L_1^i(x) + A_2 L_2^i(x) + A_3 L_3^i(x) + \ldots \tag{12}$$

Here $L_n^i(x)$ are the Laguerre polynomials which are determined for real values of $i > -1$ by the equation[12]

$$L_n^i(x) = e^x \frac{x^{-i}}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+i}) \qquad (i = 0, 1, 2, \ldots).$$

The general expression for the polynomial of nth order is given by

$$L_n^i(x) = \sum_{k=0}^{n} \frac{\Gamma(n+i+1)}{\Gamma(k+i+1)} \frac{(-x)^k}{k!\,(n-k)!}.$$

The Laguerre polynomials are orthogonal in the interval $(0, \infty)$ with weight $x^i e^{-x}$:

$$\int_0^\infty e^{-x} x^i L_m^i(x) L_n^i(x)\, dx = \begin{cases} 0 & \text{for} \quad m \neq n,\ i > -1 \\ \dfrac{\Gamma(n+i+1)}{n!} & \text{for} \quad m = n,\ i > -1 \end{cases}$$

$$n = 0, 1, 2, 3, \ldots$$

The approximate improves as the mean square error

$$M = \int_0^\infty e^{-x} x^i (A_0 + A_1 L_1^i(x) + A_2 L_2^i(x) + A_3 L_3^i(x) + \ldots - \varphi(x))^2\, dx$$

grows smaller. The coefficients A_n, determined from the condition of minimum mean square error, are equal to

$$A_n = \frac{n!}{\Gamma(n+i+1)} \int_0^\infty e^{-x} x^i \varphi(x) L_n^i(x)\, dx.$$

We replace x in Eq. (12) by γt, where γ is an arbitrary positive number, and multiply both parts by $(\gamma t)^i e^{-\gamma t}$; introducing the notation

$$(\gamma t)^i e^{-\gamma t} \varphi(\gamma t) = N(t), \tag{13}$$

we get

$$N(t) = (\gamma t)^i e^{-\gamma t} \sum_{n=0}^{\infty} A_n L_n^i(\gamma t). \tag{14}$$

The coefficients A_n in our case are equal to*

$$A_n = \frac{\gamma n!}{\Gamma(n+i+1)} \int_0^\infty N(t) L_n^i(\gamma t)\, dt. \tag{15}$$

3. We approximate the cascade curves from the primary photons with the help of the Laguerre polynomials $L_n^1(x)$

$$N(t) = \gamma t e^{-\gamma t} \sum_{n=0}^{k} A_n L_n^1(\gamma t). \tag{16}$$

Here

$$L_0^1(x) = 1; \quad L_1^1(x) = 2 - x; \quad L_2^1(x) = 3 - 3x + \frac{x^2}{2};$$

$$L_3^1(x) = 4 - 6x + 2x^2 - \frac{x^3}{6}.$$

In accordance with Eq. (15),

$$A_n = \frac{\gamma}{n+1} \int_0^\infty N(t) L_n^1(\gamma t)\, dt;$$

$$A_0 = \gamma \frac{E_0}{\beta}; \quad A_1 = \frac{\gamma}{2} \frac{E_0}{\beta} (2 - \gamma \{\overline{t_P}(E_0, 0)\}^\Gamma);$$

$$A_2 = \frac{\gamma}{3} \frac{E_0}{\beta} (3 - 3\gamma \{\overline{t_P}(E_0, 0)\}^\Gamma + \frac{\gamma^2}{2} \{\overline{t_P^2}(E_0, 0)\}^\Gamma), \text{ etc.}$$

The coefficient γ is taken to be equal to the overall absorption coefficient of the most penetrating part of the radiation--the photons. We recall that the cascade theory (see, for example, Ref. 1) makes it possible to calculate the dependence of the total number of particles in the shower on the depth t for particles of large energy $\ln(E_0/\beta) > 1$

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* The coefficients A_n can be determined from the orthogonality condition of the Laguerre poylnomials. Multiplying Eq. (14) by $L_m^i(x)$ and integrating over x from 0 to ∞, we obtain expressions for the A_n that coincide with Eq. (15).

(in the work, formulas are obtained for the cascade curve from a primary electron). On the other hand, according to the approximation Eq. (16), we can compute the cascade curves from primary particles of high energy by making use of the first two moments ($k = 2$).

For a primary photon of high energy, it is easy to obtain the following exact formulas for the number of electrons:

$$\{N_P(E_0, 0, t)\}^\Gamma \tag{17}$$
$$= \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} H_1(s)\, D_1(s) \exp(\lambda_1(s)\, t + ys)\, \frac{ds}{s};$$

where

$$y = \ln\frac{E_0}{\beta};\quad D_1(s) = \frac{2\sigma_0 [f(\lambda_1(s))]^s}{(\lambda_1'(s)+\sigma_0)\,\Gamma(s-2)};$$
$$H_1(s) = \frac{\sigma_0 + \lambda_1(s)}{\lambda_1(s) - \lambda_2(s)}$$

(for the determination of the remaining terms, see Ref. 1). We compute the integral in Eq. (17) by the method of steepest descents with accuracy up to terms of $1/t^2$:

$$\{N_P(E_0, 0, t)\}^\Gamma \tag{18}$$
$$= \frac{H_1(s)\, D_1(s)}{s}\, e^{ys+\lambda_1(s)\, t}\, (2\pi\lambda_1''(s)\, t)^{-1/2}\left[1 + \frac{\alpha(s)}{t}\right],$$

where

$$\alpha(s) = -\frac{1}{2\lambda_1''(s)}\frac{d^2}{ds^2}\ln\frac{H_1(s)\, D_1(s)}{s}$$
$$+ \frac{1}{8}\frac{\lambda_1^{IV}(s)}{[\lambda_1''(s)]^2} - \frac{5}{24}\frac{[\lambda_1'''(s)]^2}{[\lambda_1''(s)]^3}.$$

The parameter s is determined from the condition

$$t = -\frac{1}{\lambda_1'(s)}\left[y + \frac{d}{ds}\ln\frac{H_1(s)\, D_1(s)}{s}\right].$$

The values of the converging functions here are tabulated in Table III.

The cascade curves for $\epsilon_0 = 63.62$ are plotted in Fig. 1. Curve *1* is based on the exact Eq. (18), curve *2* on the approximate formula (16). The curves differ after the maximum by no more than 5%, before the maximum by no more than 15%*, but in this latter region, Eq. (18) has already ceased to be applicable [condition of the applicability of Eq. (18): $t > 1$].

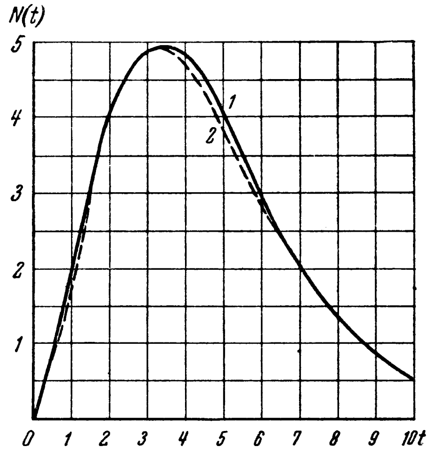

FIG. 1. Cascade curves from a primary photon of energy $\epsilon_0 = 63.62$. Curve *1* is drawn according to Eq. (18), curve *2*, according to Eq. (16).

The cascade curves from photons were also approximated with the help of a sum of Laguerre polynomials:

$$N(t) = e^{-\gamma t}\sum_{n=0}^{k} A_n L_n^0(\gamma t);$$

here

$$L_0^0(x) = 1;\quad L_1^0(x) = 1 - x;$$
$$L_2^0 = 1 - 2x + {}^1\!/_2 x^2,\ \text{etc.}$$

According to Eq. (15),

$$A_n = \gamma\int_0^\infty N(t)\, L_n^0(\gamma t)\, dt; \tag{18'}$$

$$A_0 = \gamma\frac{E_0}{\beta};\quad A_1 = \gamma\frac{E_0}{\beta}(1 - \gamma\{\overline{t_P}(E_0, 0)\}^\Gamma);$$

$$A_2 = \gamma\frac{E_0}{\beta}\left(1 - 2\gamma\{\overline{t_P}(E_0, 0)\}^\Gamma + \frac{\gamma^2}{2}\{\overline{t_P^2}(E_0, 0)\}^\Gamma\right).$$

For approximation by means of the polynomials, we require $N(t)|_{t=0} = 0$. For this, we must now insert the additional term

$$N(t) = e^{-\gamma t}\left(\sum_{n=0}^{k} A_n L_n^0(\gamma t) + C L_{k+1}^0(\gamma t)\right) \tag{19}$$

in the approximating formula. The constant C is determined by the condition

* We note that the curves differ by more than 15% at very small depths; there the curves change so rapidly that it is difficult to represent the difference in Fig. 1.

TABLE III

s	$\lambda_1^{III}(s)$	$\lambda_1^{IV}(s)$	$D_1(s)$	$\frac{d}{ds}\ln\frac{H_1(s)D_1(s)}{s}$	$\alpha(s)$
0.5	—38.07		1.173	—0.589	
0.7	—11.58	62.9	1.573	—0.475	—0.0924
0.8	— 7.31	33.45	1.79	—0.421	—0.1057
1.0	— 3.43	11.45	2.291	—0.390	—0.0791
1.1	— 2.50	7.62	2.58	—0.384	—0.0503
1.2	— 1.905	5.04	2.89	—0.381	—0.095
1.4	— 1.2	2.52	3.63	—0.391	—0.084
1.6	— 0.829	1.42	4.53	—0.403	—0.193
1.8	— 0.605		5.71	—0.408	
2.0			6.93	—0.417	

$$\{N_P(E_0, 0, t)\}_{t=0}^{\Gamma} = 0. \tag{19'}$$

The cascade curves calculated by Eqs. (16) and (19) for certain values of the primary energy ϵ_0 are drawn in Fig. 2. The curves *1* are drawn according to Eq. (16) for the first two moments ($k = 2$). The triangles denotes points of the curves established by Eq. (19) for $k = 2$ and for the same value of ϵ_0. The curves coincide within 10 percent. Such coincidence (in percent) exists over a wide range of energies ϵ_0, from 0.2 to $\sim$ 100.

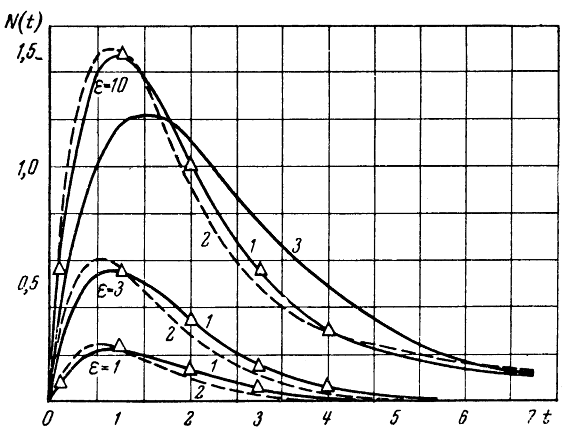

FIG. 2. Cascade curves from a primary photon of energy $\epsilon_0 = 1$, 3, 10. The curves are drawn: *1* – according to Eq. (16) for $k = 2$ [triangular points computed by Eq. (19)]; *2* – according to Eq. (16) for $k=3$; *3* – according to Eq. (16) for $k = 1$.

Thus the approximation by means of the polynomials $L_n^0(x)$ with the additional condition (19′) is completely equivalent to the approximation with the help of the polynomials $L_n^1(x)$ in the energy range from 0.2 to 100 and beyond.

Since we wish to obtain cascade curves for energies E_n^0 of the order of the critical energy β (for which the number of particles at the maximum is of the order of unity (in "cascade" units), and the maximum lies at the order of 1 t-unit in depth), it is particularly important that the approximating curve satisfy the limiting condition exactly, i.e., to obtain the greatest precision in the first t-units of the absorber. In this same Figure, the curves *2* are drawn: these are constructed from Eq. (16) with the use of the three first moments ($k = 2$). Curve *3* was constructed for $\epsilon_0 = 10$ by Eq. (16), using only the single first moment ($k = 1$). It differed from curves *1* and *2* by 30% and more. This demonstrates that the series (16) converges sufficiently rapidly. We recall that the curves corresponding to primary particles of high energy, constructed by Eq. (16) for $k = 2$ differ from the exact by no more than 5%. Taking these two circumstances into account, we can specify that only the three first terms of the series approximate the exact solution with error not exceeding 10%.

Figure 3 shows cascade curves which correspond to a primary photon for from 0.2 to 10, computed by Eq. (16) for $k = 2$. An experimental curve was obtained in carbon in Ref. 13 for a spectrum of primary photons of the form

$$\Phi_\gamma(E) = \begin{cases} 1/E \text{ where } E < 330 \text{ Mev}, \\ 0 \text{ where } E > 330 \text{ Mev}. \end{cases} \tag{20}$$

Averaging the curves (16) over the primary spectrum (20) we can compute the corresponding theoretical curve. The results of the calculation are given

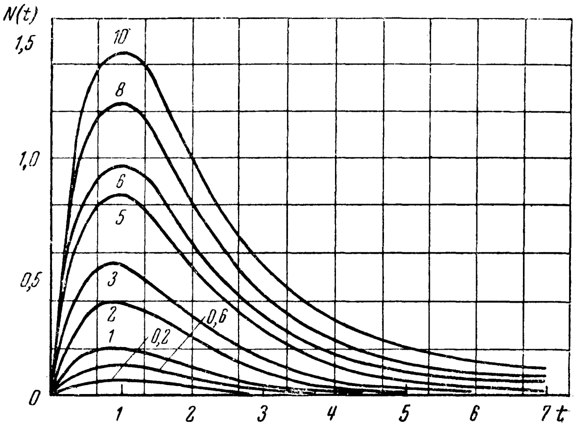

FIG. 3. Cascade curves from a primary photon for ϵ_0 from 0.2 to 10 (number on curve). Curves are constructed from Eq. (16) for $k = 2$.

in Fig. 4: *1* – experimental curve[13], *2* – theoretical curve (normalized for $t = 0.1$). Up to $t = 2$, the

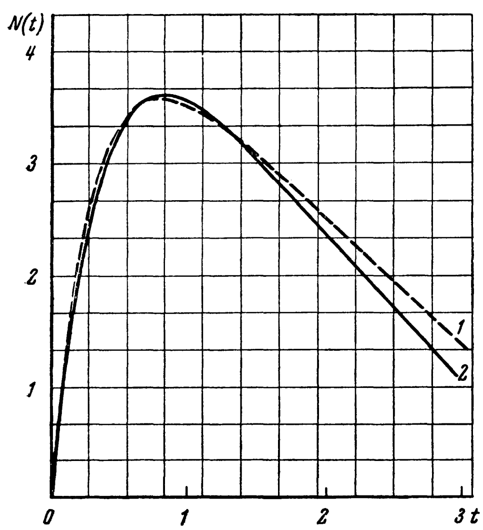

FIG. 4. *1* – experimental curve[13] obtained in graphite for a photon spectrum of the form $1/E$; *2* – obtained by averaging Eq. (16) over a photon spectrum $1/E$.

curves differ from one another by less than 5%; on the tail the difference reaches 20%, the latter because the coefficient γ was assumed to be equal to the asymptotic value of the absorption coefficient of the photons, $\gamma = 0.773$, since the σ of the photons in the energy interval considered is approximately 15% smaller, which leads to large increases in the capacity of the cascade. Making use of the values of the moments $\{\overline{t}_p\}^P$ and $\{\overline{t_p^2}\}^P$, we can compute the cascade curves from the primary electrons. We approximate the cascade curve from the primary electrons with the help of the sum of Laguerre polynomials

$$\{N_P(E_0, 0, t)\}^P = e^{-\gamma t}\left(\sum_{n=0}^{k} A_n L_n^0(\gamma t) + C L_{k+1}^0(\gamma t)\right). \tag{21}$$

We find the constant C from the condition $\{N_P(E_0, 0, t)\}^P_{t=0} = 1$, the constant γ we take equal to the asymptotic value of the photon absorption coefficient σ_0. The cascade curves constructed from Eq. (21) for $k = 2$ and for ϵ_0 running from 1 to 10 are shown in Fig. 5.

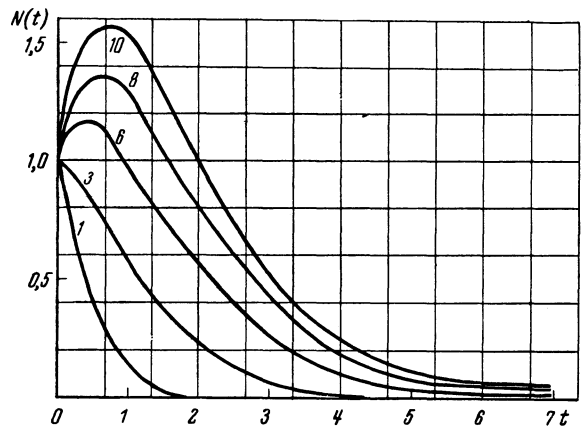

FIG. 5. Cascade curves from a primary electron for ϵ_0 from 1 to 10 (number on curve). The curves are drawn according to Eq. (21) for $k = 2$.

To verify the correctness of the cascade curves for small energies, the energy lost by the electron in radiation for the length of its mean free path was calculated by use of the data of Ref. 14. The electron of energy $\epsilon_0 = 1$ loses $\epsilon_{\text{rad}} \approx 0.3\epsilon_0$ in radiation, the electron with energy $\epsilon_0 = 3$ loses $\epsilon_{\text{rad}} \approx 0.54\epsilon_0$. The ratio of the area under the cascade curve from $t = 0$ to $t \approx E_0/\beta$ and $t \approx E_0/\beta$ to $t = \infty$ must be approximately equal to $(\epsilon_0 - \epsilon_{\text{rad}})/\epsilon_{\text{rad}}$. This ratio for curves computed from Eq. (21) for $\epsilon_0 = 1$ is 2 and for $\epsilon_0 = 3$, is equal to 1.

To estimate the accuracy of the formula, we computed the cascade curve from the primary electron with $\epsilon_0 = 229$ according to exact formulas of the theory (up to terms in $1/t^2$) and according to the approximation (21). The curves after the maximum differ from one another by no more than 5%. Thus, with the help of the method of moments developed above on the basis of "equilibrium" spectrum of Tamm and Belen'kii, one can compute the cascade curves from primary electrons and photons in light elements with errors not exceeding 10% in a broad range of depths and energies ϵ_0 from 0.2 to $\sim$ 100.

2. CASCADE SHOWERS IN HEAVY ELEMENTS

1. We rewrite the basic equations of the cascade theory with account of scattering

$$\cos\vartheta\partial P/\partial t = L_1[P(t,E,\vartheta), \qquad (22)$$

$$\Gamma(t, E, \vartheta)] + (E_k^2/4E^2)\,\Delta_\vartheta P(t, E, \vartheta),$$

$$\cos\vartheta\,\partial\Gamma/\partial t = L_2[P(t, E, \vartheta), \Gamma(t, E, \vartheta)]$$

Here $P(t, E, \vartheta)$ and $\Gamma(t, E, \vartheta)$ are the electron and photon distribution functions at depth t (everywhere measured in cascade units of length), energy E and angle ϑ, L_1, L_2 are the same operators as in Eq. (1). The last term in Eq. (22) takes into account Rutherford scattering of the electrons

$$E_k = 21\text{ Mev}; \quad \Delta_\vartheta = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right).$$

The boundary conditions we put in the form*

$$P(0, E, \vartheta) = 0; \qquad (23)$$

$$\Gamma(0, E, \vartheta) = \Phi(E_0, E)\,\delta(\vartheta),$$

where E_0 is the upper limit of the spectrum of primary photons. We neglect here the "inverse flow" of particles at $t = 0$. In Ref. 13, it was shown that the number of electrons going through the boundary of the material in the opposite direction is not large (of the order of 5% of the incident beam); therefore, neglect of the inverse flow is reasonable.

The moments of the distribution functions for the shower particles are determined by Eq. (1′). We multiply Eq. (22) by t^n and integrate over t from 0 to ∞, and over all solid angles

$$L_1[P_{n,0}(E_0, E), \Gamma_{n,0}(E_0, E)] \qquad (24)$$

$$= -nP_{n-1,1}(E_0, E),\ L_2[P_{n,0}(E_0, E), \Gamma_{n,0}(E_0, E)] = -n\Gamma_{n-1,1}(E_0, E).$$

For $n = 0$, keeping in mind the boundary conditions, we have

$$L_1[P_{0,0}(E_0, E), \Gamma_{0,0}(E_0, E)] = 0, \qquad (25)$$

$$L_2[P_{0,0}(E_0, E), \Gamma_{0,0}(E_0, E)] = -\Phi(E_0, E),$$

where

$$P_{n-i,k}(E_0, E) = \int_0^\infty\int_{(\omega)} P(t, E, \vartheta)\, t^{n-i}\cos^k\vartheta\, d\omega\, dt;$$

$$\Gamma_{n-i,k}(E_0, E) = \int_0^\infty\int_{(\omega)} \Gamma(t, E, \vartheta)\, t^{n-i}\cos^k\vartheta\, d\omega\, dt.$$

If an electron or a photon with energy E_0 is incident on the boundary of the material, then for the quantity

$$P_{(P,\Gamma),0,0}(E_0, E) = \int_0^\infty\int_{(\omega)} P_{(P,\Gamma)}(t, E, \vartheta)\, d\omega\, dt \text{ and } \Gamma_{(P,\Gamma),0,0}(E_0, E) = \int_0^\infty\int_{(\omega)} \Gamma_{(P,\Gamma)}(t, E, \vartheta)\, d\omega\, dt$$

we get equations similar to Eq. (25).

Let the quantities $P_{P,0,0}$, $\Gamma_{P,0,0}$ and $P_{\Gamma,0,0}$, $\Gamma_{\Gamma,0,0}$ be known; then the solution of Eq. (24) can be written in the form

$$P_{n,0}(E_0, E) = n\int_E^{E_0}\{P_{n-1,1}(E_0, E_1)\,P_{P,0,0}(E_1, E) + \Gamma_{n-1,1}(E_0, E_1)\,P_{\Gamma,0,0}(E_1, E)\}\,dE_1, \qquad (26)$$

$$\Gamma_{n,0}(E_0, E) = n\int_E^{E_0}\{P_{n-1,1}(E_0, E_1)\,\Gamma_{P,0,0}(E_1, E) + \Gamma_{n-1,1}(E_0, E_1)\,\Gamma_{\Gamma,0,0}(E_1, E)\}\,dE_1.$$

Integrating Eq. (26) over E from 0 to E_0, and replacing $P_{n-1,1}$ and $\Gamma_{n-1,1}$ by $P_{n-2,2}$ and $\Gamma_{n-2,2}$ we get

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* Boundary conditions of the type (23) are not essentially different from the case of an arbitrary spectrum of primary particles.

$$\int_0^{E_0} P_{n,0}(E_0, E)\,dE \tag{27}$$
$$= n(n-1)\int_0^{E_0}[P_{n-2,2}(E_0, E')\,F^P_{1,\vartheta}(E')$$
$$+ \Gamma_{n-2,2}(E_0, E')\,F^\Gamma_{1,\vartheta}(E')]\,dE'.$$

Similarly,

$$\int_0^{E_0} \Gamma_{n,0}(E_0, E)\,dE \tag{27'}$$
$$= n(n-1)\int_0^{E_0}[P_{n-2,2}(E_0, E')\,\tilde F^P_{1,\vartheta}(E')$$
$$+ \Gamma_{n-2,2}(E_0, E')\,\tilde F^\Gamma_{1,\vartheta}(E')]\,dE'.$$

Here we introduce the notation

$$F^P_{1,\vartheta}(E)$$
$$= \int_0^E dE'\int_0^{E'} dE''\,[P_{P,0,1}(E, E')\,P_{P,0,0}(E', E'')$$
$$+ \Gamma_{P,0,1}(E, E')\,P_{\Gamma,0,0}(E', E'')],$$

$$F^\Gamma_{1,\vartheta}(E)$$
$$= \int_0^E dE'\int_0^{E'} dE''\,[P_{\Gamma,0,1}(E, E')\,P_{P,0,0}(E', E'')$$
$$+ \Gamma_{\Gamma,0,1}(E, E')\,P_{\Gamma,0,0}(E', E'')];$$

$$\tilde F^P_{1,\vartheta}(E)$$
$$= \int_0^E dE''\int_0^{E'} dE''\,[P_{P,0,1}(E, E')\,\Gamma_{P,0,0}(E', E'')$$
$$+ \Gamma_{P,0,0}(E, E')\,\Gamma_{\Gamma,0,0}(E', E'')],$$

$$\tilde F^\Gamma_{1,\vartheta}(E)$$
$$= \int_0^E dE'\int_0^{E'} dE''\,[P_{\Gamma,0,1}(E, E')\,\Gamma_{P,0,0}(E', E'')$$
$$+ \Gamma_{\Gamma,0,1}(E, E')\,\Gamma_{\Gamma,0,0}(E', E'')]$$

and by definition,

$$F^P_{0,\vartheta}(E) = \int_0^E P_{P,0,0}(E, E')\,dE';$$
$$F^\Gamma_{0,\vartheta}(E) = \int_0^E P_{\Gamma,0,0}(E, E')\,dE';$$
$$\tilde F^P_{0,\vartheta}(E) = \int_0^E \Gamma_{P,0,0}(E, E')\,dE';$$
$$\tilde F^\Gamma_{0,\vartheta}(E) = \int_0^E \Gamma_{\Gamma,0,0}(E, E')\,dE'.$$

Consequently, substituting in Eq. (27) $P_{n-3,3}$, $\Gamma_{n-3,3}$ for $P_{n-2,2}$, $\Gamma_{n-2,2}$, etc., we get

$$\int_0^{E_0} P_{n,0}(E_0, E)\,dE \tag{28}$$
$$= n!\int_0^{E_0}[P_{0,n}(E_0, E)\,F^P_{n-1,\vartheta}(E)$$
$$+ \Gamma_{0,n}(E_0, E)\,F^\Gamma_{n-1,\vartheta}(E)]\,dE,$$

$$\int_0^{E_0} \Gamma_{n,0}(E_0, E)\,dE$$
$$= n!\int_0^{E_0}[P_{0,n}(E_0, E)\,\tilde F^P_{n-1,\vartheta}(E)$$
$$+ \Gamma_{0,n}(E_0, E)\,\tilde F^\Gamma_{n-1,\vartheta}(E)]\,dE.$$

The functions $F^P_{n,\vartheta}$ and $F^\Gamma_{n,\vartheta}$ are subject to the following recursion relations:

$$F^P_{n,\vartheta}(E) = \int_0^E [P_{P,0,n}(E, E_1)\,F^P_{n-1,\vartheta}(E_1) \tag{29}$$
$$+ \Gamma_{P,0,n}(E, E_1)\,F^\Gamma_{n-1,\vartheta}(E_1)]\,dE_1,$$
$$F^\Gamma_{n,\vartheta}(E) = \int_0^{E_0}[P_{\Gamma,0,n}(E, E_1)\,F^P_{n-1,\vartheta}(E_1)$$
$$+ \Gamma_{\Gamma,0,n}(E, E_1)\,F^\Gamma_{n-1,\vartheta}(E_1)]\,dE_1$$

and similarly for $\tilde F^P_{n,\vartheta}$ and $\tilde F^\Gamma_{n,\vartheta}$.

We multiply Eq. (22) by $\cos^n\vartheta$ and integrate over t from 0 to ∞ and over all solid angles for boundary conditions of the type (23). As a result, we get the following equation

$$L_1[P_{0,n}(E_0, E), \Gamma_{0,n}(E_0, E)] + \frac{E_k^2}{4E^2}\int_{(\omega)} \Delta_\vartheta P_0(E_0, E)\cos^n\vartheta\, d\omega = 0, \tag{30}$$

$$L_2[P_{0,n}(E_0, E), \Gamma_{0,n}(E_0, E)] = -\Phi(E_0, E).$$

In the case of a primary electron with energy E_0, we get equations for the functions $P_{P,0,n}$ and $\Gamma_{P,0,n}$ that are similar to (30), except that on the right side of the equation, there is the term $-\delta(E_0 - E)$, while the right side of the second equation is equal to zero. In the case of a primary photon with energy E_0, the right side of the first equation is equal to zero, while the right side of the second equation is $-\delta(E_0 - E)$ (the equation pertaining to the functions $P_{\Gamma,0,n}$ and $\Gamma_{\Gamma,0,n}$.

Let the functions $P_{(P,\Gamma),0,0}$ and $\Gamma_{(P,\Gamma),0,0}$ be known; then we can write the solution to Eq. (30) in the following form:

$$P_{0,n}(E_0, E) = \int_E^{E_0} \Phi(E_0, E')\, P_{\Gamma,0,n}(E', E)\, dE', \tag{31}$$

$$\Gamma_{0,n}(E_0, E) = \int_E^{E_0} \Phi(E_0, E')\, \Gamma_{\Gamma,0,n}(E', E)\, dE'.$$

Substituting Eq. (31) in (38), changing the order of integration in E and E' and making use of Eq. (29), we get

$$\int_0^{E_0} P_{n,0}(E_0, E)\, dE = n!\int_0^{E_0} \Phi(E_0, E)\, F^{\Gamma}_{n,\vartheta}(E)\, dE \tag{32}$$

and similarly,

$$\int_0^{E_0} \Gamma_{n,0}(E_0, E)\, dE = n!\int_0^{E_0} \Phi(E_0, E)\, \tilde{F}^{\Gamma}_{n,\vartheta}(E)\, dE.$$

The functions $\tilde{F}^{P}_{n,\vartheta}$ and $\tilde{F}^{\Gamma}_{n,\vartheta}$ coincide with accuracy to a constant multiplier, with similar functions introduced in Ref. 4. With their help, the moments of cascade curves from primary electrons or from photons of energy E_0 can be determined. Consequently, the moments of the cascade curves for an arbitrary spectrum of primary particles can be expressed by the corresponding moment of the cascade curves from a primary electron or photon of energy E_0:

$$\{\overline{t_P^n}(E_0, 0)\}^{\Phi} = \int_0^{E_0} \Phi_\gamma(E_0, E)\, \{\overline{t_P^n}(E, 0)\}^{\Gamma} \times \int_0^{E} P_{\Gamma,0,0}(E, E')\, dE' dE \Big/ \int_0^{E_0} P_{\Phi,0,0}(E_0, E)\, dE, \tag{33}$$

$$\{\overline{t_\Gamma^n}(E_0, 0)\}^{\Phi} = \int_0^{E_0} \Phi_\gamma(E_0, E)\, \{\overline{t_\Gamma^n}(E, 0)\}^{\Gamma} \times \int_0^{E} \Gamma_{\Gamma,0,0}(E, E')\, dE' dE \Big/ \int_0^{E_0} \Gamma_{\Phi,0,0}(E_0, E)\, dE.$$

Equation (33) can be considerably simplified by use of the conservation law for electrons and protons. For electrons,

$$\int_0^{E_0} P_{P,0,0}(E_0, E)\, dE = \int_0^{E_0} P_{\Gamma,0,0}(E_0, E)\, dE = \frac{E_0}{\beta}. \tag{34}$$

These relations have a simple physical meaning: all the energy of the primary particle in the final total will be dissipated in ionization. It is shown that for photons also, if we consider $\sigma(E)$ as the variable quantity, similar relations are satisfied[5]:

$$\int_{E'}^{E_0} \Gamma_{P,0,0}(E_0, E)\, dE = \int_{E'}^{E_0} \Gamma_{\Gamma,0,0}(E_0, E)\, dE \approx \frac{E_0}{\beta_\Gamma}, \tag{34'}$$

where β_Γ are functions only of the lower limit. The equality (34′) is satisfied only for the condition $E_0 \gg E'$. Numerical calculations carried out in Ref. 5 for various values of E_0 in lead show that (34′) is satisfied to within 5% for $\epsilon_0 > 10(\epsilon_0 \gg \epsilon')$. Making use of Eqs. (34) and (34′), we can rewrite Eq. (33) in the form

$$\{\overline{t_P^n}(E_0, 0)\}^{\Phi} = \int_0^{E_0} \Phi_\gamma(E_0, E)\, E\, \{\overline{t_P^n}(E, 0)\}^{\Gamma} dE \Big/ \int_0^{E_0} \Phi_\gamma(E_0, E)\, E\, dE, \tag{35}$$

$$\{\overline{t_\Gamma^n}(E_0, E_1)\}^{\Phi} = \int_{E_1}^{E_0} \Phi_\gamma(E_0, E)\, E\, \{\overline{t_\Gamma^n}(E, 0)\}^{\Gamma} dE \Big/ \int_{E_1}^{E_0} \Phi_\gamma(E_0, E)\, E\, dE.$$

Thus, if the functions $P_{(P,\Gamma),0,n}$; $\Gamma_{(P,\Gamma),0,n}$ are known, we can, by making use of Eq. (2), compute successively all the moments of the cascade curves from the primary electron or photon with energy E_0, and then, making use of Eq. (35) we can successively calculate all the moments of the cascade curve for an arbitrary spectrum of primary particles.

2. The zero moments for our problem are the functions $P_{(P,\Gamma),0,n}$ and $\Gamma_{(P,\Gamma),0,n}$. In Ref. 4, explicit expressions were found for the functions $P_{P,0,n}$ and $P_{P,0,n}$:

$$\{N_P(E,\vartheta)\}^P = \sum_{n=0}^{\infty} f_n^P(\varepsilon)\, P_n(\cos\vartheta); \tag{36}$$

$$f_n^P(\varepsilon) = \frac{2n+1}{4\pi q}\varepsilon_0 \frac{1}{\sqrt{\varepsilon^2 + a_n^2}}\left[1 - \frac{1}{1+a_n}\varphi(y)\right],$$

where $\varphi(y) = e^{-y}(y+1)\sqrt{y^2+1}$ [the remaining quantities are defined below in Eq. (39)].

Let a photon of energy E_0 be incident on a layer of the material. In the first interaction in the material, a photon can create an electron-positron pair with energy E' for the positron, with probability $W_P(E_0, E')$, or can create a Compton electron and remain a photon of high energy E' with probability $W_{\text{comp}}(E_0, E')$.

For calculation of the differential or integral "equilibrium spectrum", the first photon is equivalent to a secondary particle created by it in the primary act[6]. Consequently,

$$P_\Gamma(E_0, E, \vartheta) = \frac{1}{\sigma(E_0)}\left[2\int_E^{E_0} W_P(E_0, E')\, P_P(E', E, \vartheta)\, dE' + \int_E^{E_0} W_{\text{comp}}(E_0, E')\, P_\Gamma(E', E, \vartheta)\, dE'\right], \tag{37}$$

$$\{N_P(E_0, E, \vartheta)\}^\Gamma = \frac{1}{\sigma(E_0)}\left[2\int_E^{E_0} W_P(E_0, E')\{N_P(E', E, \vartheta)\}^P dE' + \int_E^{E_0} W_{\text{comp}}(E_0, E')\{N_P(E', E, \vartheta)\}^\Gamma dE'\right].$$

For higher energies (greater than 10^7 ev in lead) the probability W_{comp} is very small. Therefore, the second term in (37) can be neglected. Thus, Eq. (37) permits the calculation of P_P and $\{N_P\}^P$ from the known P_Γ and $\{N_P\}^\Gamma$. As shown in Ref. 7, the probability of pair formation, $W_P(E, E')$, can be represented in the form

$$W_P(E, E')\, dE' = \frac{\sigma(E)}{\sigma_0}\frac{dE'}{E} U\left(\frac{E'}{E}\right); \tag{38}$$

$$U\left(\frac{E'}{E}\right) = \left(\frac{E'}{E}\right)^2 + \left(1 - \frac{E'}{E}\right)^2 + \left(\frac{E'}{E}\right)\left(1 - \frac{E'}{E}\right)\left(\frac{2}{3} - 2b\right);$$

$$1/b = 18\ln(191\, Z^{-1/3})$$

with a high degree of accuracy.

The function U is close to unity, and for small energies can be even more accurately replaced by one. Substituting Eqs. (38) and (36) in (37) and carrying out the integration, we get the following for $\{N_P(E,\vartheta)\}^\Gamma$:

$$\{N_P(\varepsilon,\vartheta)\}^\Gamma = \sum_{n=0}^{\infty} f_n^\Gamma(\varepsilon)\, P_n(\cos\vartheta); \tag{39}$$

$$f_n^\Gamma(\varepsilon) = \frac{2n+1}{4\pi q \sigma_0 \varepsilon_0}(\varepsilon_0^2 - \varepsilon^2)\left[1 - \frac{1}{1+a_n}\varphi(y)\right](\varepsilon^2 + a_n^2)^{-1/2},$$

where

$$y = \varepsilon / a_n; \quad \varepsilon = qE/\beta; \quad q = 2{,}29;$$

$$a_n = {}^1/_2\,\varepsilon_k\sqrt{n(n+1)/q}.$$

The greatest practical interest is presented by a cascade process which is created in lead by initial photon or electron distributions of the form $1/E$. Numerical calculations of $\{\overline{\tau}_P\}^\Phi$ and $\{\overline{t_P^2}\}$ are carried out exactly for the spectrum $1/E$. For this purpose, it is necessary to know the moments $\{\overline{\tau}_P\}^P$, $\{\overline{\tau}_P\}^\Gamma$ and $\{\overline{t_P^2}\}^\Gamma$ as functions of the upper limit. In accord with Eq. (2),

$$\{\bar{t}_P(E_0, 0)\}^P \tag{40}$$

$$= \frac{1}{q}\int_0^{\varepsilon_0}\left(1+\frac{1}{\sigma(\varepsilon)}\right)\frac{1}{\sqrt{\varepsilon^2+a_1^2}}\left[1-\frac{1}{1+a_1}\varphi\left(\frac{\varepsilon}{a_1}\right)\right]d\varepsilon;$$

$$\{\bar{t}_P(E_0, 0)\}^\Gamma = \frac{1}{q\sigma_0}\int_0^{\varepsilon_0}\left(1+\frac{1}{\sigma(\varepsilon)}\right)\frac{1}{\sqrt{\varepsilon^2+a_1^2}}$$

$$\times\left(1-\frac{\varepsilon^2}{\varepsilon_0^2}\right)\left[1-\frac{1}{1+a_1}\varphi\left(\frac{\varepsilon}{a_1}\right)\right]d\varepsilon+\frac{1}{\sigma(\varepsilon_0)};$$

$$\{\overline{t_P^2}(E_0, 0)\}^P$$

$$= \frac{8\pi}{3\varepsilon_0}\int_0^{\varepsilon_0}\left\{\left[-\frac{\partial f_0^P}{\partial\varepsilon}-\frac{2}{5}\frac{\partial f_2^P}{\partial\varepsilon}\right]\{\bar{t}_P(\varepsilon, 0)\}^P\varepsilon\right.$$

$$\left.+\left[f_0^P(\varepsilon)+\frac{2}{5}f_2^P(\varepsilon)\right]\frac{\{\bar{t}_P(\varepsilon, 0)\}^\Gamma}{\sigma(\varepsilon)}\right\}d\varepsilon;$$

$$\{\overline{t_P^2}(E_0, 0)\}^\Gamma$$

$$= \frac{8\pi}{3\varepsilon_0}\int_0^{\varepsilon_0}\left\{\left[-\frac{\partial f_0^\Gamma}{\partial\varepsilon}-\frac{2}{5}\frac{\partial f_2^\Gamma}{\partial\varepsilon}\right]\{\bar{t}_P(\varepsilon, 0)\}^P\varepsilon\right.$$

$$\left.+\left[f_0^\Gamma(\varepsilon)+\frac{2}{5}f_2^\Gamma(\varepsilon)\right]\frac{\{\bar{t}_P(\varepsilon, 0)\}^\Gamma}{\sigma(\varepsilon)}\right\}d\varepsilon$$

$$+\frac{2}{\sigma(\varepsilon_0)}\{\bar{t}_P(\varepsilon_0, 0)\}^\Gamma.$$

Inasmuch as the dependence of σ on ϵ for lead cannot be given analytically, $\{\bar{t}_P\}^{P,\Gamma}$ and $\{\overline{t_P^2}\}^{P,\Gamma}$ are found by means of numerical integration of Eq. (40). The quantity $\sigma(\epsilon)$ for different ϵ is obtained from Ref. 14. Further, with the help of Eq. (35), which, for a spectrum of the type (20), takes the form

$$\{\overline{t_P^n}(E_0, 0)\}^\Phi = \frac{1}{E_0}\int_0^{E_0}\{\overline{t_P^n}(E, 0)\}^\Gamma dE,$$

the quantities $\{\bar{t}_P(E_0, 0)\}^\Phi$ and $\{\overline{t_P^2}(E_0, 0)\}^\Phi$ are found by numerical integration. These same quantities are computed on the basis of experimental data[13]. Particles are measured experimentally with energy greater than a certain amount. In Ref. 13, the lower limit lay below 1 mev; therefore, the quantities $\{\bar{t}_P(E_0, E^0)\}^\Phi$ and $\{\overline{t_P^2}(E_0, E^0)\}^\Phi$ were computed, E^0 being taken = 0.5 mev. The results of the calculations for the spectrum (20) are

$\bar{t}_{\text{theoret}}$	$\bar{t}_{\text{exp}}$	$\overline{t^2}_{\text{theoret}}$	$\overline{t^2}_{\text{exp}}$
5.89 ± 0.12	5.9 ± 0.2	53.4 ± 2	55 ± 4

The mean value of $\bar{t}$ varies by 1.0%, of $\overline{t^2}$ by 4.0%; however, the difference lies with the errors of the experiment and the calculation. While the theoretical value of $\bar{t}$ for $\sigma = \sigma_0$, without consideration of scattering, is less than the experimental by 33%, for $\sigma = \sigma(\epsilon)$, it is larger than the experimental by about 10%. Making use of the method of construction of cascade curves developed in the first part of this work, we can obtain (for lead) not only the moments but also the cascade curves themselves. We approximate the cascade curves from the primary photon with energy E_0 with the aid of the Laguerre polynomials:

$$\{N_p(E_0, 0, t)\}^\Gamma = \gamma t e^{-\gamma t}\sum_{n=0}^{k} A_n L_n^1(\gamma t). \tag{41}$$

The coefficients A_n are determined by Eq. (15). The coefficient γ is taken to be equal to the minimum value of the absorption coefficient of the most penetrating portion of the radiation--for photons, $\gamma = 0.24$.

The cascade curves computed with Eq. (41) for $k = 2$ are averaged over the spectrum of primary photons of type (20). The results of the calculations are shown in Fig. 6: curve *1* is the experimental transmission curve for lead[13], curve *2* is the calculated curve. The values of curves *1* and *2* differ by less than 5%. Thus the cascade theory satisfactorily describes the development of showers in heavy elements.

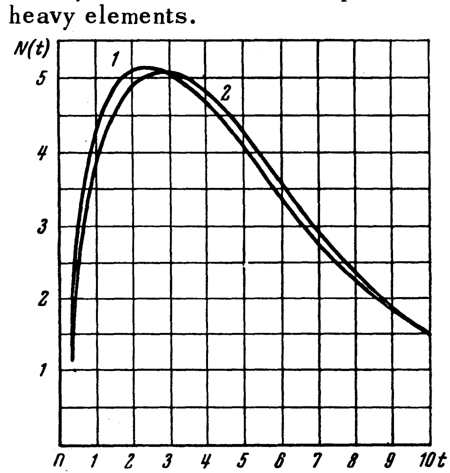

FIG. 6. *1* – experimental curve obtained in lead for a photon spectrum of the form $1/E$; *2* – obtained by averaging Eq. (41) over the photon spectrum of the form (20).

A series of papers by several authors have illuminated the problem under discussion. In Ref. 15, the fundamental equations of cascade theory are solved by a perturbation method. As a first approximation, the solution was used that had been obtained by Snyder[16], who used asymptotic expressions for the cross section of elementary processes. The correction to them was calculated by means of a more accurate approximation of the cross section due to Bethe and Heitler[17]. However, the results obtained by numerical methods are valid only for light elements (where the accurate cross sections of Bethe and Heitler differed only slightly from the asymptotic values) and for initial energies $\ln(E_0/\beta) > 1$. The method developed in Ref. 15 was not applied to heavy elements, where the cross sections of elementary processes change strongly with change of element. In Ref. 18, the basic equations of the theory were solved with approximate cross section of the absorption coefficients of photons and electrons. However, the resultant curves are not satisfactory, since the law of conservation of energy is not satisfied for them--the areas under the curves are approximately $0.6\, E_0/\beta$. Further, the energy spectra of electrons in the maximum shower in air and in lead differ distinctively from each other and from the equilibrium spectrum which was established incorrectly*.

In Ref. 19, an approximate formula was obtained for $N(E_0, E, t)$--the number of particles with energy greater than E at the depth t, created by primary particles with energy E_0:

$$N(E_0, E, t) = \frac{e^{K_0}}{\sqrt{t+K_1}} \exp\left(-t + 2\sqrt{(t+K_1)(y+K_2)}\right); \tag{42}$$

$$K_n = K_{n0} + \frac{\varepsilon K_{n1}(Z)}{E+\beta}\ ; \quad y = \ln \frac{E_0}{E+\bar{\varepsilon}}\ ; \quad \bar{\varepsilon} = \frac{\beta}{2.3},$$

$K_{n,t}$ are constants that depend only on the atomic number Z of the element. The curve computed from Eq. (42) differs (for air) by about 30% from the accurate curve of Snyder. The energy spectra at the maximum of the shower for air and lead differ markedly from the experimental values. At an arbitrary point of the shower they differ significantly for air from the spectra given by the usual cascade theory. All this does not permit us to assume Eq. (42) to be satisfactory for the description of the cascade process.

For individual cases, cascade curves were computed for lead in Ref. 20 by a method of random sampling on a special machine. Here electrons from photons with energies less than 10 mev and Rutherford scattering of charged particles were considered approximately.

In Ref. 5, the values of $\overline{t}$ and $\overline{t^2}$ were computed from an arbitrary spectrum of primary particles, but scattering of shower particles was not considered; the role of the latter is important at low energies. Moreover, there are errors in Ref. 5 for the formulas for

$$\{\overline{t}_P(E_0, E)\}^{P,\Gamma};\ \{\overline{t_P^2}(E_0, E)\}^{P,\Gamma} \text{ and } \{\overline{t_P^n}(E_0, E^0)\}^{\Phi}.$$

In conclusion, I express my deep appreciation to S. Z. Belen'kii for his advice and help with the work.

* In cascade theory, it is shown that for arbitrary dependence of the absorption coefficient of photons on the energy spectrum of electrons at maximum must be close to the "equilibrium" spectrum[1].

[1] S. Z. Belen'kii, *Cascade processes in cosmic rays*, Gostekhizdat, 1948.

[2] I. Tamm and S. Belen'kii, J. Phys. USSR **1**, 177 (1939).

[3] S. Z. Belen'kii, J. Exptl. Theoret. Phys. (U.S.S.R.) **19**, 940 (1949).

[4] S. Z. Belen'kii and B. I. Maksimov, J. Exptl. Theoret. Phys. (U.S.S.R.) **22**, 102 (1952).

[5] V. Ia. Fainberg, J. Exptl. Theoret. Phys. (U.S.S.R.) **22**, 112 (1952).

[6] B. Rossi, *High energy particles*, Gostekhizdat, Moscow, 1955 (Russian translation).

[7] B. Rossi and S. Klapman, Phys. Rev. **61**, 414 (1942).

[8] J. A. Richards and L. W. Nordheim, Phys. Rev. **74**, 1106 (1948).

[9] L. Eyges, Phys. Rev. **74**, 1801 (1948).

[10] B. Rossi and K. Greisen, *The interaction of cosmic rays with matter*, IIL, 1948 (Russian translation).

[11] G. T. Zatsepin, Dokl. Akad. Nauk SSSR **63**, 243 (1948).

[12] N. Lebedev, *Special functions and their application*, 1953.

[13] Blocker, Kenney and Panofsky, Phys. Rev. **79**, 419 (1950).

[14] W. Heitler, *The quantum theory of radiation*, Oxford, 1953.

[15] I. B. Bernstein, Phys. Rev. **80**, 995 (1950).

[16] H. Snyder, Phys. Rev. **53**, 960 (1938).

[17] H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **159A** (1937).

[18] N. Dellaporta and E. Clementel, Nuovo Cimento **3**, 235 (1946); **4**, 1 (1947); **5**, 78 (1948).

[19] W. Heisenberg, *Kosmische Strahlung*, Berlin, Springer, 1953.

[20] R. R. Wilson, Phys. Rev. **86**, 261 (1952).

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