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Remarks on the Theory of Scattering of Particles with a Given Total Angular Momentum

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1. WE consider the scattering of two fermions interacting by means of a boson field, but, in contrast to the usual treatment of this problem, the initial and final states of the pair of fermions are specified as states with a certain given total angular momentum. It turns out that this approach to the problem presents a number of peculiarities.

It is convenient to separate the motion of the center-of-mass from the relative motion and then to examine the process in the system of coordinates in which the center-of-mass is at rest. The matrix element which corresponds to the simplest irreducible diagram (which is the only one we shall consider for the sake of brevity) may be written in the form

$$\overline{\psi}_{\sigma_1\sigma_2}(k)\,\Gamma^{(1)}D(k-k_0)\,\Gamma^{(2)}\psi_{\sigma_1^0\sigma_2^0}(k_0), \tag{1}$$

where D is the retarded interaction function, $\Gamma^{(i)}$ is the vertex part, the matrix of which operates on the indices of the ith particle; $k(\epsilon, \mathbf{k})$ is the 4-momentum of the particle in the system of coordinates under consideration ($\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k}$); $\sigma_i = \pm ½$ define the spin states of the particles. We consider only the process of interaction which does not lead to the radiation or absorption of bosons and also the case of the absence of external fields. The diagrams which take into account the proper mass of the particles and which are not related to the interaction between the particles are also not taken into account.

The matrix elements of interest to us in the energy-angular momentum space may be most conveniently obtained from the matrix elements of type (1) by a suitable transformation which takes into account the spinor nature of the wave functions[1]

$$\mathfrak{M}^{jm\chi}_{j_0m_0\chi_0} = \sum_{\sigma_1,\sigma_2}\ \sum_{\sigma_1^0,\sigma_2^0} \iint (dk)\,(dk_0)\ A^{\chi}_{jm}A^{\chi^0}_{j_0m_0}M^{\sigma_1\sigma_2 k}_{\sigma_1^0\sigma_2^0}\,k_0. \tag{2}$$

Here j and m are the quantum numbers of the total angular momentum and of its component, and the index χ characterizes the spin state of the pair of particles (singlet and triplet). The coefficients A^{χ}_{jm}, for example, in the simplest case of the singlet state, have the following form:

$$A^{s}_{jm} = \delta(k_4 - \varepsilon)\,k^{-2}\delta \tag{3}$$

$$\times(|\mathbf{k}| - p)\,Y_{jm}(n_k)\,2\sigma_1\delta_{\sigma_1,-\sigma_2}/\sqrt{2}.$$

In the same way, one may define the coefficient for the triplet case which includes suitable spherical vectors and the required symmetric combinations of spin states (ϵ is the energy, p is the absolute magnitude of the momentum of the particles in the energy-angular momentum space).

By transforming in the manner indicated the matrix element of the zero order approximation to (1) (Möller scattering), for example for the transition singlet → singlet ($\chi = \chi_0 = s$), we obtain an expression of the form

$$e^2A(\varepsilon)\,Q_i(\Delta)\,\delta_{jj_0}\delta_{mm_0},$$

where the form of the function $A(\epsilon)$, which arises as a result of the summation in (2) [similar to (6), see below] depends on the explicit form of the interaction and on the kind of Bose particles, but does not depend on j and m. The argument of Q_j (Legendre function of the second kind) is the quantity $\Delta = 1 + \mu^2/2p^2$ where $p = |\mathbf{k}|$ is the absolute magnitude of the momentum of the particles, and μ is the mass of the boson responsible for the interaction.

In the case of electrodynamics the integration in (2) cannot be carried out formally (if one considers the particles in the initial and final states to be free) since $Q_j(1)$ diverges. For bound states $k^2 \neq m^2$ and the function Δ will contain the binding energy of the particles.

For large $j \gg l$ we have

$$Q_j(1 + \mu^2/2p^2) \approx K_0(\mu j/p).$$

Therefore, for not too large initial energies

$\mu j/p \gg 1$ and then $K_0(\mu j/p) \approx \sqrt{\pi p/2j\mu}\, e^{-j\mu/p}$.

As may be seen, the magnitude of the scattering amplitude in this case depends exponentially on the mass of the bosons responsible for the interaction, which makes scattering by means of heavy particles improbable. For energies so large that $j\mu/p \ll 1$ we have $K_0(j\mu/p) \approx \ln(p/j\mu)$, i. e., the scattering amplitude increases logarithmically.

2. Within the framework of perturbation theory the transformation (2) permits one to separate the divergences in a unique way without carrying out the evaluation of the corresponding integrals.

As an example, let us consider the correction to the matrix element which takes into account the loop corresponding to polarization of the vacuum. In the case of electrodynamics, for example, we have

$$e^4 \bar{\psi}_{\sigma_1\sigma_2}(k)\, \gamma_\mu^{(1)} D_{\mu\sigma}^{(0)}(q)\, T_{\sigma\tau}(\hat{q})\, D_{\tau\nu}^{(0)}(q)\, \gamma_\nu^{(2)} \psi_{\sigma_1^0\sigma_2^0}(k_0); \quad (4)$$

$$D_{\mu\nu}^{(0)} = (\delta_{\mu\nu} - \hat{q}^{-2} q_\mu q_\nu)\, \hat{q}^{-2}; \quad \hat{q} = \hat{k} - \hat{k}_0;$$

$$T_{\sigma\tau}(\hat{q}) = \mathrm{Sp} \int \gamma_\sigma \frac{1}{\hat{q} - \hat{\varkappa} - M} \gamma_\tau \frac{1}{\hat{\varkappa} - M} (d\varkappa).$$

Using the transformation (2) we shall obtain ($\chi = \chi_0 = s$)

$$e^4 \int \frac{Y_{jm}^*(\mathbf{n})\, Y_{j_0m_0}(\mathbf{n}_0)}{(\hat{k} - \hat{k}_0)^2} \times T(\hat{k} - \hat{k}_0)\, \Phi_{jm, j_0m_0}^{(\chi, \chi_0)}(k, k_0)\, d\mathbf{n}\, d\mathbf{n}_0, \quad (5)$$

where the integration is carried out over the directions of the momenta of the particles; $(\hat{k} - \hat{k}_0)^2 = -2p^2(\Delta - \mathbf{nn}_0)$, $T = 1/3\, T_{\mu\mu}$, and the symbol Φ denotes the sum

$$\Phi = \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{\hat{q}^2}\right) \quad (6)$$

$$\times \sum_{\sigma_1, \sigma_2} \sum_{\sigma_1^0, \sigma_2^0} 2\sigma_1 \frac{\delta_{\sigma_1, -\sigma_2}}{\sqrt{2}}\, 2\sigma_2^0 \frac{\delta_{\sigma_1^0, -\sigma_2^0}}{\sqrt{2}} (\gamma_\mu)_{k_0\sigma_1^0}^{k, \sigma_1} (\gamma_\nu)_{-k_0, \sigma_2^0}^{-k, \sigma_2}.$$

It may easily be shown that Φ does not depend on the angular variables of integration in (5) which permits one to take Φ outside the integral sign.

Further, we have[2]

$$3T(\hat{q}) = 2(\hat{q}^2 + 8M^2)\, I(\hat{q}) - 12\, I_{\mu\mu}(\hat{q}), \quad (7)$$

$$I_{(\ , \mu\mu)}(\hat{q}) = \int \frac{(1, \varkappa^2)\,(d\varkappa)}{[(\hat{\varkappa} + \hat{q}/2)^2 - M^2]\,[(\hat{\varkappa} - \hat{q}/2)^2 - M^2]}.$$

Carrying out the integration over the angles (5) under the integral sign (7) (for which we temporarily restrict the region of integration), we shall obtain, taking into account

$$\int \frac{Y_{jm}^*(\mathbf{n})\, Y_{j_0m_0}(\mathbf{n}_0)\, d\mathbf{n}\, d\mathbf{n}_0}{(a - \mathbf{nn}_0)(b - \mathbf{nn}_0)(c - \mathbf{nn}_0)} \quad (8)$$

$$= 4\pi \left[\frac{Q_j(a)}{(a-b)(a-c)} + \frac{Q_j(b)}{(b-a)(b-c)} + \frac{Q_j(c)}{(c-a)(c-b)}\right] \delta_{jj_0}\delta_{mm_0},$$

the following expression [for example, for $I(q)$]:

$$4\pi Q_j(\Delta)\, I(0) + I_{jR}(\Delta, \Delta_\varkappa, \alpha), \quad (9)$$

where $I(0)$ is a logarithmically divergent integral and the quantity

$$I_{jR} = \frac{8\pi}{p^4} \int \frac{(d\varkappa)}{\alpha} \int_{-1}^{+1} \left[\frac{Q_j(\Delta_\varkappa + \Delta)}{\Delta_\varkappa} - \frac{Q_j(\Delta_\varkappa + \Delta - \alpha t)}{\Delta_\varkappa - \alpha t}\right] \frac{dt}{t} \quad (10)$$

($\Delta_\varkappa = -2(\hat{\varkappa}^2 - m^2)/p^2$, $\alpha = 2\,|\,\varkappa\,|\,/\,p$) turns out to be finite.

3. Further, let us examine the complete scattering amplitude (1) considering the functions D and Γ to have been renormalized. If the interaction occurs by means of particles of mass μ, then the following representation holds[3]

$$D(\hat{q}^2) = \int \frac{\rho(\lambda^2)\, d\lambda^2}{\hat{q}^2 - \lambda^2 + i\varepsilon}; \quad (11)$$

$$\rho(\lambda^2) = \delta(\lambda^2 - \mu^2) + \sigma(\lambda^2);$$

$$\sigma(\lambda^2) \geqslant 0; \quad \sigma(\lambda^2) = 0 \ \text{for}\ \lambda^2 \leqslant \mu^2; \ \varepsilon \to 0^+.$$

The transformation of the matrix element which contains the function D into the space of the angular momenta is here carried out similarly to the earlier case, so that replacing the upper limit of the integral (11) by a certain quantity Λ^2, and then integrating over the angles under the integral sign we shall find that the matrix element turns out to be proportional to the quantity

$$\int_0^{\Lambda^2} \rho(\lambda^2)\, Q_j(\Delta - \lambda^2/2p^2)\, d\lambda^2.$$

The existence of the scattering amplitude as an observable imposes certain conditions on the function $\rho(\lambda^2)$ in order to guarantee the convergence of the integral. In the case of the angular momentum j the integrand for large values of λ^2 turns out to be proportional to $\rho(\lambda^2)/(\lambda^2)^{j+1}$. Thus only for $j = 0$ is the commonly used condition[4] necessary that $\int \lambda^{-2}\rho(\lambda^2)\,d\lambda^2$ should not diverge at the upper limit. For a collision with an angular momentum $j > 0$ the requirements imposed on the function $\rho(\lambda^2)$ may be relaxed.

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On the Reduction of Wave Equations for Spin 0 and 1 to the Hamiltonian Form

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RECENTLY Schrödinger[1] and Case[2] have shown that equations for particles of spin 0 and 1

$$(\beta_k \nabla_k + \varkappa)\,\psi = 0, \tag{1}$$

the matrices β_k in which satisfy the well-known conditions of Duffin

$$\beta_i\beta_k\beta_l + \beta_l\beta_k\beta_i = \delta_{ik}\beta_l + \delta_{kl}\beta_i \tag{2}$$

may be reduced to the Hamiltonian form

$$H\psi = i\dot{\psi}, \tag{3}$$

where H is the Hamiltonian operator which has the form

$$H = \gamma_\alpha p_\alpha + \varkappa\gamma_4 \quad (\alpha = 1, 2, 3), \tag{4}$$

while the matrices γ_k ($k = 1, 2, 3, 4$) also satisfy conditions (2). To Eq. (3) we must also add the initial condition of the form

$$(H\gamma_4 - \varkappa)\,\psi = 0, \tag{5}$$

which, in consequence of (3), holds at any arbitrary instant of time. These results are evidently of interest, in particular because they allow one to formulate a theory of particles with spin 0 and 1 (to a large extent by analogy with the well-developed theory of Dirac). In the work by Schrödinger[1] and Case[2] the Hamiltonian form (3) is obtained by resolving Eqs. (1) into their components for a certain specific choice of the matrices β_k. Such a noninvariant method of derivation is unnecessarily awkward and requires separate calculations for spin 0 and spin 1. Moreover, the connection between the matrices γ_k and the initial matrices β_k remains unclear in this method of procedure. The object of the present note consists of showing the method by means of which the reduction to the form (3)-(5) may be carried out simultaneously for spin 0 and 1 without introducing components, and solely on the basis of Eq. (1) and the algebra (2). In the course of the derivation the relation between the matrices γ_k and β_k is also established. The results are generalized to the case of zero rest mass.

Let us write (1) in the form $(\beta_\alpha\nabla_\alpha + \beta_4\nabla_4 + \varkappa)\psi = 0$ and multiply it by $(1 - \beta_4^2)$ and by β_4^2. Taking into account the fact that $\beta_4\beta_\alpha\beta_4 = 0$ we shall obtain

$$[((\beta_4\beta_\alpha - \beta_\alpha\beta_4)\nabla_\alpha + \varkappa\beta_4)\beta_4 - \varkappa]\,\psi = 0, \tag{6}$$

$$\beta_4[((\beta_4\beta_\alpha - \beta_\alpha\beta_4)\nabla_\alpha + \varkappa\beta_4) + \nabla_4]\,\psi = 0. \tag{7}$$

On the basis of (2) it is easily seen that the matrices

$$\gamma_\alpha = i(\beta_4\beta_\alpha - \beta_\alpha\beta_4), \quad \gamma_4 = \beta_4 \tag{8}$$

satisfy the same conditions (2) as do β_k and are likewise Hermitian. We note that (8) is easily solved with respect to β_α and gives $\beta_\alpha = -i(\gamma_4\gamma_\alpha - \gamma_\alpha\gamma_4)$. Evidently (6) can be rewritten with the aid of (8) in the form (5) where H is defined by (4) and $p_\alpha = -i\nabla_\alpha$. Thus the initial condition (5) is the result of multiplying (1) by $(1 - \beta_4^2)$. The relation (7) takes on the form

$$\gamma_4(H + \nabla_4)\,\psi = 0. \tag{9}$$

We operate on (5) by the operator $\nabla_4 = -i\,\partial/\partial t$, on (9) by the operator H and subtract the results. Taking into account that

$$H\gamma_4 H = \varkappa H, \tag{10}$$