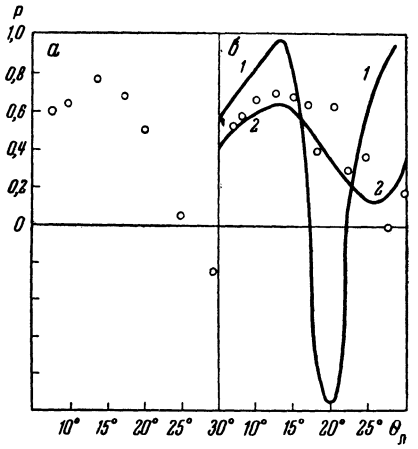

a – polarization of protons on He (E = 315 mev), *b* – the same on C (E = 290 mev). *1* – results of calculation with a rectangular well, *2* – the same, taking into account the volume distribution.

$$A_l^{\pm} = (\exp\{2i\delta_l^{\pm}\} - 1)/2i, \qquad (11)$$

$\delta_l^{\pm}$ is the phase shift for the partial wave with $J = l \pm \frac{1}{2}$, which we calculate in the WKB approximation; P_l^0, P_l^1 are the Legendre polynomials. The results of the calculation of $P(\vartheta)$ are given by the graph, and show that, because of the more accurate description of the behavior of the potential at the edge of the nucleus by means of introducing a certain nucleon distribution inside it, we have succeeded in obtaining better agreement with experiment. This is explained by the inclusion of higher phases. The polarization remains approximately constant for g varying from $3e$ to $5e$ which can be understood from the nature of the phenomenon. A decrease in ϵ makes the agreement with experiment worse.

In conclusion, I express my thanks to Prof. D. Ivanenko for his continued interest in the work and for the discussion of results.

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Calculation of Elastic Scattering of Slow Electrons in Hydrogen by the Integral Equation Method

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THE problem of the elastic scattering of slow electrons by the hydrogen atom, taking exchange into account, was first solved by Morse and Allis[1] by means of a numerical integration of the corresponding integro-differential equation and later by Massey and Moiseiwitsch[2] by means of a variational principle. The object of the present calculation consisted of checking the effectiveness of the method of integral equations proposed by Drukarev[3].

We shall start with the same approximate representation of the wave function of the system "electron + hydrogen atom" which was used by Morse and Allis:

$$\Psi(r_1 r_2) = \psi(r_1) F^{\pm}(r_2) \pm \psi(r_2) F^{\pm}(r_1), \qquad (1)$$

where $\psi(r)$ is the function of the ground state of the hydrogen atom. $F^{+}(r)$ and $F^{-}(r)$ are considered to be spherically symmetric, which means that only s-scattering is taken into account.

With the choice of $\Psi(r_1 r_2)$ made above the functions $f^{\pm}(r) = rF^{\pm}(r)$ must satisfy the equation

$$\left(\frac{d^2}{dr^2} + k^2\right) f^{\pm}(r) = V(r) f^{\pm}(r) \pm 2u(r) \times \int_0^{\infty} u(r') f^{\pm}(r') \left\{\gamma(rr') + \varepsilon - \frac{k^2}{2}\right\} dr' \qquad (2)$$

and the boundary conditions

$$f^{\pm}(0) = 0, \qquad (3)$$

$$f^{\pm}(r) \sim \frac{1}{k}\sin kr + a^{\pm} e^{ikr}, \qquad (4)$$

where

$$V(r) = 2\left\{\int_0^{\infty} u^2(r')\gamma(rr')\,dr' - \frac{1}{r}\right\}, \qquad u(r) = r\psi(r),$$

$$\gamma(rr') = \begin{cases} 1/r & r \geqslant r' \\ 1/r' & r \leqslant r', \end{cases}$$

ϵ is the energy of the atom, k is the wave number of the incident electron.

We note that usually[2,4] in order to derive this equation the wave function (1) is substituted into Schrödinger's equation and after multiplying it by $\psi(r_1)$, and integrating over r_1, Eq. (2) is obtained. In so doing it is generally overlooked that the Schrodinger equation cannot be exactly satisfied by any function of the form (1). However, Eq. (2) may be justified quite rigorously if one uses the variational principle formulated for collision problems.

Using the method indicated by Drukarev[3], Eq. (2) may, by taking (3) into account, be brought to the form of the following two integral equations:

$$X(r) = \varphi_1 + \int_0^r K(rs)\, X(s)\, ds, \tag{5}$$

$$Y(r) = \varphi_2 + \int_0^r K(rs)\, Y(s)\, ds, \tag{6}$$

with

$$f^{\pm}(r) = \alpha_1 X(r) + \alpha_2 Y(r). \tag{7}$$

Here the following notation is used:

$$\alpha_1 = f'(0), \quad \varphi_1 = \frac{1}{k}\sin kr; \tag{8}$$

$$\varphi_2 = \frac{1}{k}\int_0^r \sin k(r-s)\, u(s)\, ds;$$

$$\alpha_2 = \int_0^\infty u(s)\, f(s)\left(\frac{1}{s} + \varepsilon - \frac{k^2}{2}\right) ds;$$

$$K(rs) = \frac{1}{k}\sin k(r-s)\, V(s) \pm \frac{2}{k}\, u(s)\int_s^r \sin k(r-t)\, u(t)\left(\frac{1}{t} - \frac{1}{s}\right) dt.$$

Substituting (7) into (8) we shall obtain the equation relating α_1 to α_2,

$$\alpha_2 = \alpha_1 \int_0^\infty u(r')\, X(r')\left(\frac{1}{r} + \varepsilon - \frac{k^2}{2}\right) dr' + \alpha_2 \int_0^\infty u(r')\, Y(r')\left(\frac{1}{r'} + \varepsilon - \frac{k^2}{2}\right) dr'. \tag{9}$$

This equation and the condition (4) allow us to find both constants and by this, together with (5) and (6), completely determine the solution.

In the antisymmetric case, one may simplify the calculations considerably if one takes into account

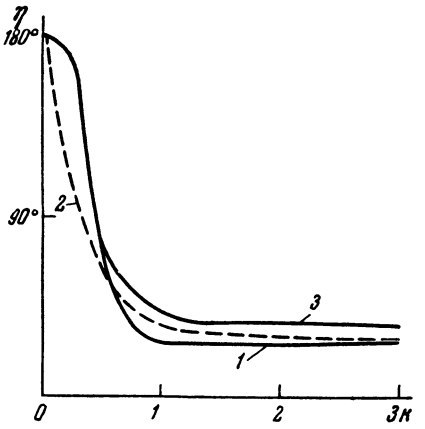

FIG. 1. The scattering phase for the symmetric case: *1* – numerical integration, *2* – variational method, *3* – method of integral equations.

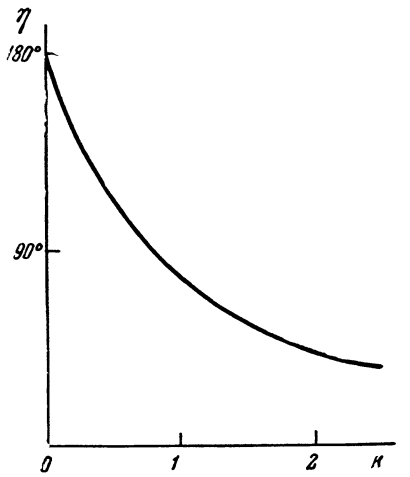

FIG. 2. The scattering phase in the antisymmetric case. The results of all three methods coincide within the scale of the drawing.

that the complete wave function is invariant with respect to replacing $f(r) \to f(r) + Cu(r)$, where C is an arbitrary constant. On this basis one may consider either α_1 or α_2 to be arbitrary. In particular, setting $\alpha_1 = 0$ we shall obtain

$$f(r) = \alpha_2 Y(r). \tag{10}$$

Equation (9) then reduces to

$$\int_0^\infty u(r')\, Y(r')\left(\frac{1}{r'} + \varepsilon - \frac{k^2}{2}\right) dr' = 1.$$

Using (6) and the fact that

$$(1/r + \varepsilon)\, u = d^2u / dr^2,$$

it is not hard to show that this equality is satisfied identically, and that, therefore, it does not amount to some supplementary condition imposed on the function $Y(r)$.

An analogous simplification may be obtained by setting $\alpha_2 = 0$. But one must keep in mind that the iteration process may not converge equally rapidly in the two cases.

The functions $f^+(r)$ and $f^-(r)$ were calculated in the first approximation from the formulas (5)-(7), (9) and (10) and the scattering phases were found from them. The results are given in Figs. 1 and 2. For comparison, the results of a numerical integration[1] of Eq. (2) and the results a variational calculation[2] are also given there. In the antisymmetric case all three curves coincide within the scale of the drawing. In the symmetric case one obtains a somewhat higher value of the phase for $k > 0.5$; at lower energies in this case also a complete agreement is obtained in the first approximation with the result of the numerical integration.

In conclusion, we express our gratitude to G. F. Drukarev for his interest in this work and for a number of valuable suggestions.

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On the Relation between the Distribution of a Quasi-Monochromatic Stationary Process and the Distribution of its Envelope

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NOT long ago Blanc-Lapierre and others[1] showed that if the probability density $w_A(A)$ of the envelope of a quasi-monochromatic stationany stochastic process ξ is known, the characteristic function $f_\xi(u)$ of the process is

$$f_\xi(u) = \int_0^\infty w_A(A)\, J_0(Au)\, dA. \tag{1}$$

Rytov[2] continued the calculation and obtained the following formula:

$$w_\xi(\xi) = \frac{1}{\pi} \int_{|\xi|}^\infty \frac{w_A(A)}{\sqrt{A^2 - \xi^2}}\, dA, \tag{2}$$

connecting the probability density $w_\xi(\xi)$ of a stationary stochastic process with $w_A(A)$. In the present note I wish to show another way of deriving Eqs. (1) and (2), whereby they are obtained as the zero-order approximation in neglecting the narrow passband width of an amplifier. The use in this derivation of the time average, correct in the case of a stationary stochastic process, gives the possibility of obtaining in a natural manner the correction terms accounting for the finite width of the passband.

A quasi-monochromatic stationary process can be written in the form $\xi(t) = A(t) \cos[\omega_0 t + \varphi(t)]$, where $A(t)$ and $\varphi(t)$ are functions of time varying slowly in comparison with $\cos \omega_0 t$. Then the characteristic function $f_\xi(u)$ is

$$f_\xi(u) = \lim_{T\to\infty} \frac{1}{T} \times \int_0^T \exp\{iuA(t) \cos[\omega_0 t + \varphi(t)]\}\, dt. \tag{3}$$

Let us break up the interval $0, T$ into N small intervals, each of length $\tau = 2\pi/\omega_0$. Then

$$f_\xi(u) = \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \frac{1}{\tau} \times \int_{(m-1)\tau}^{m\tau} \exp\{iuA(t) \cos[\omega_0 t + \varphi(t)]\}\, dt. \tag{4}$$

Taking into account the smallness of the change of $A(t)$ and $\varphi(t)$ in the time τ, we expand the expression under the integral sign in a series in A and φ and limit ourselves to terms of the first order of smallness.

Then after simple transformations we obtain in place of Eq. (4),