A Quasi-Unidimensional Interpretation of the Hydrodynamic Theory of Multiple Particle Production

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(Submitted to JETP editor May 12, 1955)
J. Exptl. Theoret. Phys. (U.S.S.R.) 31, 278-287 (August, 1956)

The hydrodynamic theory of multiple particle production developed by L. D. Landau is based on introducing two stages of liquid breakup——unidimensional motion and conical breakup. The validity range of the second stage is exceedingly difficult to estimate. This work investigates a variant of hydrodynamic theory, in which only the unidimensional stage is considered. This variant is shown to give very good approximation for final temperatures $T_k \sim 1.5\,\mu - 2\mu$. At $T_k \sim \mu$ the unidimensional approximation (particularly for slow secondary particles) gives a result that is merely of the right order of magnitude. The dependence of the fastest particle on T_k is also investigated. It turns out that the condition $T_k \lesssim \mu$ must be satisfied for the calculated velocity to agree with the experimentally-observed one. This leads to the preliminary deduction that when $T_k \sim \mu$ the interaction cross section of the secondary particles (apparently pi-mesons) is of the same order of magnitude as the geometric cross section.

1. Landau,[1] on the basis of an idea expressed by Fermi,[2] developed a hydrodynamic theory for the interaction of high-energy particles. The characteristic parameter of the theory is the value of the final temperature T_k , determined by the cross section σ for the interaction of the secondary particles. Landau assumes $T_k \sim \mu$ * (μ is the meson mass). However, this choice is by no means unique; if the cross section σ diminishes with diminishing energy, T_k may become much greater than μ.[3] On the other hand, if the cross section is $\sigma \sim 1/\mu$, then $T_k \sim \mu$.[4] The value of σ is thus closely linked to the value of the final temperature. To date there are no direct experimental data concerning the character of the meson—meson interaction. One must therefore choose a different path, namely: first obtain the value of T_k by comparing the results of the hydrodynamic theory with the experimental data, and then deduce the magnitude of σ.

We shall consider in this investigation the variation of the energy characteristics of the elementary act with T_k. We encounter here the very difficult problem of accounting for the lateral breakup of the liquid. The conical breakup introduced by Landau for this purpose is a rather rough approximation, the validity of which is moreover difficult to estimate. We shall therefore consider another approximation that is more advantageous for our purposes; in the calculation of the energy characteristics (more accurately, the four-velocity u), we neglect the lateral breakup entirely (quasi-unidimensional approximation).

2. For our purpose it is quite important to investigate the portion of nuclear matter carrying the principal energy fraction (the so-called leading edge). However, it is precisely to this section that the Landau results are inapplicable, for the condition $\xi >> \Delta$* on which his solution is based is not satisfied here. We therefore first obtain a solution for the forward front. The remaining computations are based on the concept that if a very high energy is concentrated in the volume in which the process occurs, the transverse components of the four-velocity u are much smaller than the longitudinal one, and the energy of the particles is therefore determined almost completely by the unidimensional motion. Inasmuch as Khalatnikov[5] (cf. also Ref. 6) obtained an accurate solution for the leading edge, our results are approximate only in that the transverse components of u have been neglected.

It is thus necessary to determine first of all the applicability limits of the unidimensional solution. Landau's estimate of this limit is expressed in the form of the inequality $t\,\xi << a^2$. As was already noted, he employed the unidimensional solution; we shall refine this estimate, relying on Khalatnikov's solution of the unidimensional problem. The equations of motion of an ultra-relativistic ideal liquid are given by the following equations:

*We employ here a system of units in which $\hbar = c = M = 1$ (M is the nucleon mass). The temperature is expressed in energy units.

*Hereinafter we shall employ Landau's notation.

$$\partial T_{ik}/\partial x_k = 0, \quad (1)$$

$$T_{ik} = (\varepsilon/3)(4u_iu_k + g_{ik}), \quad (2)$$

where ϵ is the energy density in the intrinsic coordinate system, u_i the four-velocity components, $g_{11} = g_{22} = g_{33} = 1$, $g_{00} = -1$, and $g_{ik} = 0$ if $i = k$; i and k run through the values 0, 1,2 and 3.* The index 0 denotes the temporal components, and the index 1 denotes the components along the motion of the primary particles (axis of motion). In the case of unidimensional motions, the indices i and k are either 0 or 1.

Let us write the solution given by Khalatnikov for this case in its parametric form

$$x_0 = t = e^{-y}\left(\frac{\partial\chi}{\partial y}\cosh\rho - \frac{\partial\chi}{\partial\rho}\sinh\rho\right), \quad (3)$$

$$t - x_1 = \xi = e^{-y}\left(\frac{\partial\chi}{\partial y} + \frac{\partial\chi}{\partial\rho}\right)(\cosh\rho - \sinh\rho), \quad (4)$$

$$\chi = \sqrt{3}\,\Delta e^{y}\int_{-\rho/\sqrt{3}}^{y} e^{-2z} I_0\left(\sqrt{z^2-\rho^2/3}\right)dz, \quad (5)$$

where $y = ln\,(T/T_0)$, $u_0 = \cosh\rho$, T_0 the initial temperature, T the running temperature of the liquid element, and I_0 Bessel's function for imaginary argument. It will be more convenient for what follows to write the derivatives $\partial\chi/\partial y$ and $\partial\chi/\partial\rho$ in explicit form:

$$\frac{\partial\chi}{\partial y} = \sqrt{3}\,\Delta e^{-y} I_0\left(\sqrt{y^2-\rho^2/3}\right) - e^{y}\int_{\rho/\sqrt{3}}^{-y} e^{2z} I_0\left(\sqrt{z^2-\rho^2/3}\right)dz, \quad (6)$$

$$\frac{\partial\chi}{\partial\rho} = e^{y}\Delta\left[e^{2\rho/\sqrt{3}} + \frac{\rho}{\sqrt{3}}\int_{\rho/\sqrt{3}}^{-y} e^{2z} I_1\left(\sqrt{z^2-\rho^2/3}\right)\frac{dz}{\sqrt{z^2-\rho^2/3}}\right].$$

Consider next, two cases: a) $y^2 - \rho^2/3 \lesssim 1$ (leading edge) and b) $y^2 - \rho^2/3 >> 1$, corresponding to the slowest particles. As will be seen below, both cases overlap in practice. Expanding the I functions and restricting ourselves to the first three terms of this series, we obtain for case a):

$$t = u\Delta\Big\{e^{-2y}\Big[\sqrt{3} + \frac{\sqrt{3}}{4}\left(y^2 - \frac{\rho^2}{3}\right) - \frac{6+\rho}{4\sqrt{3}} + \frac{\sqrt{3}}{64}\left(y^2-\frac{\rho^2}{3}\right)^2 - \frac{12+\rho}{16\sqrt{3}}\left(\frac{y^2}{2} + \frac{y}{2} + \frac{1}{4} - \frac{\rho^2}{6}\right)\Big] - e^{2\rho/\sqrt{3}}\Big[1 - \frac{6+\rho}{4\sqrt{3}} + \frac{12+\rho}{16\sqrt{3}}\left(\frac{\rho}{2\sqrt{3}} - \frac{1}{4}\right)\Big]\Big\}, \quad (7)$$

$$\xi = \frac{\Delta}{2u}\Big\{e^{-2y}\Big[\sqrt{3} + \frac{\sqrt{3}}{4}\left(y^2-\frac{\rho^2}{3}\right) - \frac{6-\rho}{4\sqrt{3}} + \frac{\sqrt{3}}{64}\left(y^2-\frac{\rho^2}{3}\right) - \frac{12-\rho}{16\sqrt{3}}\left(\frac{y^2}{2}+\frac{y}{2}+\frac{1}{4}-\frac{\rho^2}{6}\right)\Big] + e^{2\rho/\sqrt{3}}\Big[1 + \frac{6-\rho}{4\sqrt{3}} - \frac{12-\rho}{16\sqrt{3}}\left(\frac{\rho}{2\sqrt{3}} - \frac{1}{4}\right)\Big]\Big\}. \quad (8)$$

The relationship $\sinh\rho \sim \cosh\rho \sim e^{\rho}/2$ was used in the derivation of (7) and (8). If condition a) is satisfied, (7) and (8) converge very rapidly. Furthermore, it can be shown that the denominators of the series terms increase so rapidly that the contribution of the discarded terms of power $k \geq r$ is only on the order of the rth term.

Particularly simple are the equations obtained for the limits of the self-modeling and nontrivial motion ($\alpha = -y - \rho/\sqrt{3} = 0$). In this case

$$t = \frac{\sqrt{3}-1}{2}\Delta\exp\left\{\frac{\rho}{\sqrt{3}}(2+\sqrt{3})\right\}; \quad (9)$$

$$\xi = (\sqrt{3}+1)\,\Delta\exp\left\{\frac{\rho}{\sqrt{3}}(2-\sqrt{3})\right\},$$

$$u^2 = [(\sqrt{3}+1)/2(\sqrt{3}-1)]\,t/\xi; \quad (10)$$

$$\varepsilon = \varepsilon_0\Delta^2/t\xi;$$

$$t = \left(\frac{\sqrt{3}-1}{2}\right)(\sqrt{3}+1)^{-(2+\sqrt{3})/(2-\sqrt{3})} \times \xi^{(\sqrt{3}+2)(2-\sqrt{3})}\Delta^{-2\sqrt{3}/(2-\sqrt{3})}.$$

If $\alpha << 1$, then, restricting ourselves to the first terms of the series (7) and (8) we can obtain a more general expression, valid not merely at $\alpha = 0$:

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*The equation of state $\epsilon/3 = p$ is used in the derivation of (2).

$$u^2 = {}^1/_4 (t\xi)^{\sqrt{3}/2} \Delta^{-\sqrt{3}}, \quad (11)$$

$$\varepsilon \doteq \left(\frac{2}{\sqrt{3}-1}\right)^{2(2-\sqrt{3})/\sqrt{3}}$$

$$\times (\sqrt{3}+1)^{2(2+\sqrt{3})/\sqrt{3}} \varepsilon_0 \Delta^6 \xi^{-(3\sqrt{3}+4)/\sqrt{3}}$$

$$\times \; t^{-(3\sqrt{3}-4)/\sqrt{3}}.$$

Let us consider further the case b):

$$y^2 - \rho^2/3. \quad (12)$$

For this purpose we employ the asymptotic expansion of the Bessel function:

$$I_p(\omega) \sim \frac{e^\omega}{\sqrt{2\pi\omega}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\omega)^k} \frac{\Gamma(p+k+{}^1/_2)}{k!\Gamma(p-k+{}^1/_2)}. \quad (13)$$

We neglected the term e^{ω} in equation (13); this is quite appropriate at $\omega > 1$.

Before inserting the series (13) into (6), let us evaluate the integral

$$J_p = \int_{\rho/\sqrt{3}}^{-y} e^{2z} \frac{I_p(\sqrt{z^2-\rho^2/3})}{(\sqrt{z^2-\rho^2/3})^m} dz.$$

Let us break this integral up into two. In the first (J_{p1}), for which $z^2 - \rho^2/3 > 1$ (corresponding to $\alpha \gtrsim 1/\rho$), we shall employ the asymptotic expansion; in the second (J_{p2}; $z^2 - \rho^2/3 < 1$) we shall employ the series (6). Since the contribution of the second integral is slight, it can be adequately represented by the first terms of series (6);

$$J_{11} \sim 2J_{12} \sim \frac{1}{2} e^{2\rho/\sqrt{3}} (e^\alpha - 1) \sim \frac{1}{\rho} e^{2\rho/\sqrt{3}} \quad (14)$$

The integral J_{12} reduces to the sum of integrals

$$R_k \sim \int \frac{\exp\{2z + \sqrt{z^2-\rho^2/3}}{(z^2-\rho^2/3)^{1/2(k+m+1/2)}} dz.$$

The important factor in this integral is the exponential term. Let us therefore first evaluate the integral

$$\Phi(z) = \int \exp\{2z + \sqrt{z^2-\rho^2/3}\} dz,$$

and then make allowances for the effect of the denominator. We seek $\Phi(z)$ in the form

$$\Phi(z) = \left[\frac{1}{2 + z/\sqrt{z^2-\rho^2/3}} + \varphi(z)\right] \quad (15)$$

$$\times \; \exp\{2z + \sqrt{z^2-\rho^2/3}\},$$

where $\varphi(z)$ satisfies the equation

$$[2 + z/\sqrt{z^2-\rho^2/3}]\,\varphi(z) + \varphi'(z) \quad (16)$$

$$+ \rho^2/3\sqrt{z^2-\rho^2/3}\,(2\sqrt{z^2-\rho^2/3}+z)^2 = 0.$$

Assuming $z^2 - \rho^2/3 \gg 1$, we can approximate (16) by

$$3\varphi(z) + \varphi'(z) + \rho^2/27z^3 = 0, \quad (17)$$

hence

$$\varphi(z) = \frac{\rho^2}{27} e^{-3z} \int \frac{e^{3z}}{z^3} dz. \quad (18)$$

Using the asymptotic expansion

$$\int \frac{e^{rz}}{z^k} dz \sim \frac{e^{rz}}{rz^k}\left[1 + \frac{k}{2z} + \frac{k(k+1)}{(rz)^2} + \ldots + \frac{k(k+1)\ldots(k+i)}{(rz)^{i+1}} + \ldots\right] \quad (19)$$

and restricting ourselves to the first term only, which is quite appropriate as long as we are interested in relatively small k and large z, we can finally write

$$\varphi(z) \sim \rho^2/81z^3. \quad (20)$$

The function $\varphi(z) \ll \rho^2/54z^2$ and can therefore be neglected in (15); in our approximation

$$\Phi(z) \sim {}^1/_3(1 - \rho^2/18z^2) \quad (21)$$

$$\times \exp\{2z + \sqrt{z^2-\rho^2/3}\}.$$

If $z^2 - \rho^2/3 \sim 1$, we cannot represent (16) by (17); analysis of (16) shows that in this case $\varphi(z) \sim 1/(2+z)$ and neglecting this function results therefore in an error by a factor of approximately 2. In the second (logarithmic) approximation, the factor in front of the exponent in (15) must be set equal to unity.

Let us evaluate the role of the denominator in the principal integral. Using an asymptotic expansion analogous to (19) and restricting ourselves to the first term, we can obtain

$$R_k \sim {}^1/_3(1 - \rho^2/18z^2)(z^2-\rho^2/3)^{-1/2(k+m+1/2)} \quad (22)$$

$$\times \exp\{3z - \rho^2/6z^2\};$$

at $z^2 - \rho^2/3 \gg 1$

$$R_k \sim {}^1/_3\left(1 + \frac{m+k+{}^1/_6}{6z^2}\right) z^{-(k+m)-1/_6} \quad (23)$$

$$\times \exp\{3z - \rho^2/6z^2\};$$

if $z^2 - \rho^2/3 \sim 1$

$$R_k \sim (2+z)^{-1}(z^2 - \rho/3)^{-1/2(k+m+1/2)} \tag{24}$$
$$\times \exp\{2z + \sqrt{z^2 - \rho^2/3}\}.$$

We shall assume hereinafter that (23) represents the upper limit of the integral, and (24) the lower limit. This is equivalent to the condition $y^2 - \rho^2/3 >> 1$. Let us next evaluate

$$J_{11} - R_0(\sqrt{(\rho^2/3)+1}) \sim \frac{e^{2\rho/\sqrt{3}}}{\rho(2+z)}(2+z-\rho).$$

Since $z \sim \rho/\sqrt{3}$ in this case, and we are interested in $z \sim 2$ — 4 then

$$J_{11} - R_0\left(\sqrt{\frac{\rho^2}{3}+1}\right) \sim \sqrt{3}e^{2\rho/\sqrt{3}}/\rho(2\sqrt{3}+\rho).$$

It follows from this that

$$J_{11} - R_0(\sqrt{(\rho^2/3)+1})/R_0(-y) \sim e^{-z}.$$

Thus, restricting ourselves to an approximate accuracy of 10 – 15 %, we can neglect R_0 and the integrals J_{11} and J_{12}.

Examination of (13) shows that to obtain the same accuracy, approximately 10%, we need use only the first term of the series.

Let us now derive several specific equations. Neglecting the factors in front of the exponents in (23) and (24) and the quantities J_1, $R[\sqrt{(\rho^2/3)+1}]$, and $\partial X/\partial\rho$, we get

$$\tau \sim \rho - 2y + \sqrt{y^2 - \rho^2/3}, \tag{25}$$
$$\eta \sim -\rho - 2y + \sqrt{y^2 - \rho^2/3},$$

where

$$\tau = \ln(t/\Delta), \quad \eta = \ln(\xi/\Delta). \tag{26}$$

As can be seen from the preceding estimates, the errors due to the items neglected are quite large if $y^2 \sim \rho^2/3$; the relationships (25) and (26) are merely of the right order of magnitude

Solving both (25) and (26) for ρ and y and then using $u = e^\rho/2$ and $\epsilon = \epsilon_0 e^{4y}$, we obtain

$$u^2 = t/\xi, \tag{27}$$
$$\varepsilon = \varepsilon_0 \exp\{-4/3(n + \tau - \sqrt{\eta\tau})\}.$$

Equations (27) were first introduced by Landau[1].* Somewhat simpler expressions can be obtained if $y^2 >> \rho^2/3$. In this case, neglecting the factor $e^{\rho\sqrt{3}y}$, we can write

$$e^\tau = \sqrt{3/8\pi e^{-3y}e^{\rho}y^{-1/4}}, \tag{28}$$
$$e^n = \sqrt{3/2\pi e^{-3y}e^{-\rho}y^{-1/4}}.$$

Putting $y^{-1/2} = 1$, which usually introduces an error of the order of 2, we have

$$\varepsilon \approx (\varepsilon_0/3)(\Delta^2/t\xi)^{2/3}; \quad u^2 \sim t/\xi. \tag{29}$$

3. Let us evaluate the applicability limits of the unidimensional solution. For this purpose let us write Eq. (1) as :

$$4\frac{\partial}{\partial t}(\varepsilon u^2) + \frac{\partial\varepsilon}{\partial\xi} + 8\frac{\partial}{\partial x_2}(\varepsilon u^2\vartheta) = 0; \tag{30}$$

$$\frac{\partial\varepsilon}{\partial t} + \frac{\partial}{\partial\xi}\left(\frac{\varepsilon}{u^2}\right) + 4\frac{\partial}{\partial x^2}(\varepsilon\vartheta) = 0; \tag{31}$$

$$4\frac{\partial}{\partial t}(\varepsilon u^2\vartheta) + 4\frac{\partial}{\partial x_1}(\varepsilon u^2\vartheta) + \frac{\partial\varepsilon}{\partial x_2} = 0. \tag{32}$$

The following was assumed in the derivation of these equations: $u_0 \sim u_1 = u$; $u_2 = u_3 = u\vartheta$; $\partial/\partial x_2 = \partial/\partial x_3$ *. In addition, the term discarded in (32) is considerably smaller than the term remaining. Under these assumptions, the fourth equation of (1) becomes identical wthh (32).

Equations (31) and (32) differ from the equations for the unidimensional problem in their last terms. We shall assume that in (32) these terms are small compared with the remaining ones. We shall therefore write equation (32), which relates small components of the tensor T, in an approximate form, depending on what section of the liquid is under investigation. Let us consider first the leading edge, for which $x_1 \sim t$. Since $\epsilon(t=0) = \epsilon(x_1 = 0) = \epsilon_0$ and $u(t=0) = u(x_1 = 0) \sim 1$, then $\partial/\partial t \sim \partial/\partial x_1$ and Eq. (32) can be rewritten as:

$$8(\partial/\partial t)(\varepsilon u^2\vartheta) + \partial\varepsilon/\partial x_2 = 0. \tag{32a}$$

We seek a solution in the form

$$u^2 = u_1^2(t,\xi) + \varphi_1(t,\xi);$$

$$\varepsilon = e^{-x_2/a}[\varepsilon_1(t,\xi) + \varphi_2(t,\xi)]$$

(ϵ_1 and u_1 satisfy the equations describing the unidimensional motion), corresponding to an

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*The Landau solution is derived from the asymptotic expansion in Ref. 5.

*The equalities $u_2 = u_3$ and $\partial/\partial x_2 = \partial/\partial x_3$ follow from the axial symmetry.

initial exponential radial distribution of the energy density. Assuming $\varphi_1 \ll u_1^2$, and $\varphi_2 \ll \epsilon_1$ and making use of the relative smallness of the last terms of (30) and (31), we obtain the following system:

$$4\frac{\partial}{\partial t}(\varphi_1\varepsilon_1+\varphi_2 u_1^2)+\frac{\partial\varphi_2}{\partial\xi}-\frac{1}{a^2}\int\varepsilon_1 dt=0, \quad (33a)$$

$$\frac{\partial\varphi_2}{\partial t}+\frac{\partial}{\partial\xi}\left[\frac{\varphi_2}{u_1^2}-\frac{\varepsilon_1\varphi_1}{u_1^4}\right]-\frac{1}{2a^2u^2}\int\varepsilon_1 dt=0, \quad (33b)$$

$$\vartheta=\frac{1}{8a\varepsilon_1 u_1^2}\int\varepsilon_1 dt. \quad (33c)$$

We next seek a solution in the form

$$\varphi_1=(C_1\xi t/a^2)\,u_1^2, \quad \varphi_2=(C_2\xi t/a^2)\,\varepsilon_1.$$

Such a solution satisfies the initial conditions $\varphi(t=0)=\varphi(\xi=0)=0$. Using (14) we get

$$\frac{11-4\sqrt{3}}{2\sqrt{3}} \times (C_1+C_2)\,t^{(11-6\sqrt{3})/2\sqrt{3}}\,\xi^{-(4\sqrt{3}+5)/2\sqrt{3}}\,\Delta^{-\sqrt{3}} - \frac{2\sqrt{3}+4}{\sqrt{3}}C_2 t^{2(2-\sqrt{3})/\sqrt{3}} \times \xi^{-(3\sqrt{3}+4)/\sqrt{3}} - \frac{\sqrt{3}}{2(2-\sqrt{3})}t^{2(2-\sqrt{3})/\sqrt{3}}\,\xi^{-(3\sqrt{3}+4)/\sqrt{3}}=0, \quad (34a)$$

$$2(2-\sqrt{3})\,C_2 t^{(4-3\sqrt{3})/\sqrt{3}}\,\xi^{-2(2+\sqrt{3})/\sqrt{3}} - (11+4\sqrt{3})(C_2-C_1)\,t^{(5-4\sqrt{3})/2\sqrt{3}} \times \xi^{-(11+6\sqrt{3})/2\sqrt{3}}\,\Delta^{\sqrt{3}} - \frac{3}{2-\sqrt{3}}t^{(5-4\sqrt{3})/2\sqrt{3}}\,\xi^{-(11+6\sqrt{3})/2\sqrt{3}}\,\Delta^{\sqrt{3}}=0. \quad (34b)$$

Equations (34) cannot be solved for $C=$ constant. However, we can assume that the last equation in (10) is approximately satisfied in the vicinity of the leading edge. Then, expressing $t^{(2-\sqrt{3})/2}$ [in the first terms of (34a) and (34b)]* in terms of ξ, in accordance with (10), we obtain a simple equation from which it follows that

$$\varphi_1 \sim (\xi t/3a^2)\,u_1^2; \quad \varphi_2 \sim (\xi t/4a^2)\,\varepsilon_1. \quad (35)$$

We next consider a section in which the condition $\rho^2/3 \ll y^2$ is satisfied. In this case it is convenient to write Eq. (32) as :

$$4\frac{\partial}{\partial t}(\varepsilon u^2\vartheta)+2\frac{\partial}{\partial\xi}(\varepsilon\vartheta)+\frac{\partial\varepsilon}{\partial x_2}=0. \quad (32b)$$

In the region under investigation $t \sim \xi$ and $\partial/\partial t \sim \partial/\partial\xi$, and therefore we have in lieu of (33)

$$4\frac{\partial}{\partial t}(\varphi_1\varepsilon_1+\varphi_2 u_1^2)+\frac{\partial\varphi_2}{\partial\xi}-\frac{4}{9a^2}\int\varepsilon_1 dt=0; \quad (36a)$$

$$\frac{\partial\varphi_2}{\partial t}+\frac{\partial}{\partial\xi}\left[\frac{\varphi_2}{u_1^2}-\frac{\varepsilon_1\varphi_1}{u_1^4}\right]-\frac{2}{9a^2u_1^2}\int\varepsilon_1 dt=0; \quad (36b)$$

$$\vartheta=\frac{1}{6a\varepsilon_1 u_1^2}\int\varepsilon_1 dt. \quad (36c)$$

Using (31), equations (36) can be readily solved by putting, as before,

$$\varphi_1=(\overline{C}_1\xi t/a^2)\,u_1^2; \quad \varphi_2=(\overline{C}_2\xi t/a^2)\,\varepsilon_1,$$

upon substitution we obtain

$$\varphi_1 \sim -(\xi t/4a^2)\,u_1^2; \quad \varphi_2 \sim (\xi t/2a^2)\,\varepsilon_1. \quad (35a)$$

The condition $t\xi/a^2 \lesssim 1$ is close to the analogous condition derived by L. D. Landau, but with the substantial difference that in Ref. 1 the condition is stated in the form of a strong inequality. The formal similarity is not surprising since both derivations employed the relationship $u^2 \sim t/\xi$, which, as we verified, is approximately satisfied whether $y^2 \sim \rho^2/3$ or $y^2 \gg \rho^2/3$.

4. Let us next examine the temperature limits of the applicability of the unidimensional motion. First of all, let us express the parameter $y = 1/4 \ln(\epsilon/\epsilon_0)$ in terms of the enrgy E_0 of the incident particle. The initial energy density is

$$\varepsilon_0=\overline{E}_0/V, \quad (36)$$

*We note that the exponent $(2-\sqrt{3})/2$ is considerably smaller than unity.

where $\bar{E}_0 = \sqrt{2E_0}$ is the total energy in the center-of-mass system; let us also assume, in agreement with Fermi[2], $V = (4\pi/3\mu^3)\sqrt{2/E_0}$; substituting the expression for V into (36) we get

$$\varepsilon_0 = (3/4\pi)E_0\mu^3. \qquad (37)$$

From simple thermodynamic considerations (see, for example, Ref. 4 or 7) we can ascertain that at a temperature T the boson energy density is

$$\varepsilon = (6{,}5/2\pi^2)a_1T^4, \qquad (38)$$

and the fermion energy density is

$$\varepsilon = (5{,}7/2\pi^2)a_2T^4, \qquad (39)$$

where a_1 and a_2 are the numbers of the internal degrees of freedom of the particles; $a_1 = 3$ for pi-mesons and $a_2 = 8$ for nucleon-antinucleon pairs. Strictly speaking, Eqs. (38) and (39) are valid if T is many times greater than the rest mass $\mathfrak{M}$ of the particles under consideration. However, by applying the rigorous expressions for the energy density[4] it can be shown that even if $T \sim \mathfrak{M}$ the values obtained from (38) and (39) differ from the true ones by approximately 10%. Hereinafter we shall restrict ourselves to pi-mesons and to temperatures $T \gtrsim \mu$, using therefore Eq. (38). Equations (38) and (39) yield

$$y = {}^1\!/_4 \ln(4T^4/E_0\mu^3). \qquad (40)$$

Let us consider next the leading edge. According to (12)

$$t\xi/a^2 = (2/E_0)e^{-4y} = \mu^3/2T^4. \qquad (41)$$

We thus obtain $t\xi/a^2 = 1$ at $T \sim 1.3\,\mu$ and $t\xi/a^2 \sim 3$ at $T = \mu$. Equation (37) indicates that as far as the leading edge is concerned, the motion of the liquid can be said to be quasi-unidimensional at temperatures up to $T \gtrsim \mu$. The quantity $t\xi/a^2$ is greatly dependent on the temperature, and at $T_k < \mu$ the lateral breakup would therefore influence the energy characteristics quite substantially.

It is interesting to note that Eq. (41) does not depend on the energy. This shows that the diminishing breakup angles and the longer path covered by the liquid with increasing E_0 compensate each other.

Consider next the region of the slowest particles. Using (31) and (40), we readily obtain for this region

$$t\xi/a^2 = (1/4\cdot3^{3/2})E_0^{1/2}\mu^{9/2}T^{-6}. \qquad (42)$$

For an energy $E_0 = 10^5$, we obtain $t\xi/a^2 = 1$ if $T \sim 2.5\,\mu$, but $t\xi/a^2 \sim 5$ if $T \sim 2\mu$.

Apparently the quantity $t\xi$ increases continuously with diminishing ρ. Although this statement has not been rigorously proven, the following estimates make its truth quite likely. We examined

$$\frac{\partial(t\xi)}{\partial\rho} \sim \frac{\partial}{\partial\rho}\left[\left(\frac{\partial\chi}{\partial y}\right)^2 - \left(\frac{\partial\chi}{\partial\rho}\right)^2\right]$$

for the leading edge [see Eq. (10)] and for the logarithmic approximations (25) and (26). In both cases we found

$$\partial(t\xi)/\partial\rho < 0.$$

Consequently, the lower the energy of the escaping particles, the more pronounced the lateral breakup, with the lowest temperature at which the quasi-unidimensional solution is valid in all regions being given approximately by Eq. (42).

In conclusion, let us tabulate the values of $\xi t/a^2$ for various values of E_0 and T, numerically determined from the exact Eqs. (3) to (6).

$E_0 = 3\cdot10^5$; $T = 2\mu$		$E_0 = 4\cdot10^4$; $T = 1{,}5\mu$		$E_0 = 10^3$; $T = \mu$		$E_0 = 2\cdot10^4$; $T = \mu$	
u_0	$\xi t/a^2$	u_0	$\xi t/a^2$	u_0	$\xi t/a^2$	u_0	$\xi t/a^2$
49	0.2	11	0.8	11	3,2	49	3,2
24	0,35	5.5	1,0	5,5	4.4	24	4,5
10	0.7	3,0	1,2	3	4,7	10	8
3	1,2	1.4	1.6	1,4	6,2	3	16
1,4	2.0	1.1	2,0	1,1	8,0	1,4	25

Comparing the data in the last line of the table against those obtained from Eq. (42) we see that the latter apparently gives a value that is approximately 2–3 times too high. The motion thus is quasi-unidimensional at $T \sim 1.5\,\mu$ to 2μ. It is interesting to check the validity of the quasi-unidimensional approximation at $T \sim \mu$. As can be seen from the Table [cf. also (41)], the regions adjacent to the leading edge satisfy approximately the criterion $t\xi / a^2 \sim 1$, but the quasi-unidimensional nature is lost in the region of the slower particles. Nevertheless, a noticeable difference does take place as E_0 changes from 10^3 to 2×10^4. At $E_0 \sim 10^3$ we get for the slow particles $\xi t / a^2 \sim 6$ to 8 and consequently $\varphi_1 \sim u_1^2$ and $\varphi_2 \sim \epsilon_1$ [cf. (35a)]. We can therefore expect the quasi-unidimensional approximation to give the correct order of magnitude for these particles. One cannot expect more in this region, for the condition $u >> 1$ is no longer satisfied here. Consequently, it becomes probable that for $T \sim \mu$ (or $E_0 \sim 10^3$) the quasi-unidimensional approach will also yield greater accuracy than the conical breakup. A somewhat different situation arises at higher energies, where we can have $\varphi_1 >> u_1^2$ and $\varphi_2 >> \epsilon_1$.

5. Let us analyze the temperature dependence* of the energy $\overline{E}_m$ of the fastest particle.[8] The velocity u_m of this particle is determined from the relationship

$$\rho_m = -\sqrt{3}\,y, \tag{43}$$

which is satisfied in the self-modeling motion region (see Ref. 5). Substituting the value of y from Eq. (40) and using $u = e\rho/2$ we get

$$u_m = {}^1/_2\,(E_0\mu^3/4)^{\sqrt{3}/4}\,T^{-\sqrt{3}}. \tag{44}$$

To obtain E_m (in center-of-mass system) we shall assume that the particles bound to the element move isotropically in the system. Then, assuming that the secondary particles are mesons, we have

$$\overline{E}_m = 2\beta T_k u_m = \beta\,(E_0\mu^3/4)^{\sqrt{3}/4}\,T^{-(\sqrt{3}-1)}, \tag{45}$$

where $\beta \sim 2$ (for $T_k \sim \mu$, β=1.8).[7]

The transverse components can be neglected in the transition to the laboratory system, and therefore

$$\overline{E}_m = 2^{-1/2(\sqrt{3}-1)}\,\beta E^{(2+\sqrt{3})/4}\mu^{3\sqrt{3}/4}/T^{(\sqrt{3}-1)}. \tag{46}$$

We thus obtain for $E_0 = 10^3$

$\overline{E}_m \sim 0.35 E_0 \quad (T_k = \mu);$

$\overline{E}_m \sim 0.25 E_0 \quad (T_k = 2\mu);$

$\overline{E}_m = 0.1 E_0 \quad (T_k = 3\mu).$

It follows from an analysis[9] of experimental data[10,11] that at these energies one of the particles carries off a considerable fraction ($\gtrsim 1/2$) of the primary energy. One can therefore expect within the framework of the hydrodynamic theory to have $T_k \leq \mu$ *. This conclusion agrees with the deduction made essentially by Belen'kii[12] who compared the dependence of the fraction of energy carried off by heavy particles on T_k with Grigorov's[13] and Zatsepin's[9] experimental data. The relationship $T_k \lesssim \mu$ leads to an important deduction. Very simple estimates[4] show that if $T_k = 1.5\,\mu$ and $\sigma = (1/\mu)^2/3$, the mean free path is approximately three times greater than the linear dimensions of the system, $1/\mu$. In this case the free breakup should therefore occur even at $T_k \sim 1.5\,\mu$, contradicting the condition $T_k \lesssim \mu$. One can conclude from this that at $T_k \sim \mu$ the interaction cross section of the secondary particles is of the order of $1/\mu^2$

In conclusion, we must emphasize the very preliminary character of this deduction. It is based essentially on the study of the behavior of the fastest particle, a study that cannot be carried out conclusively within the framework of pure hydrodynamics. One should rather examine the entire leading edge as a whole (cf. Ref. 7). The situation becomes even more complicated by the fact that the experimental data on hand indicate[9,13] that as a rule the fastest particle is a nucleon. If this fact is confirmed at sufficiently high energies ($\geq 10^{12}$ ev), the only ones for which the hydro-

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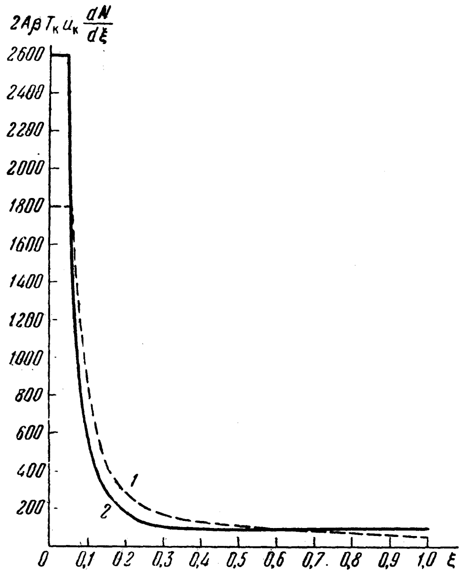
*The behavior of the fastest particle at $T \sim \mu$ was studied previously by Gerasimova and Chernavskii.[8]

*Although some discrepancy between the experimental and theoretical data remains at $T_k = \mu$ there are many factors (thermal motion[7], presence of heavy particles[12]) that should tend to decrease this discrepancy.

dynamic treatment can be deemed valid, it will become necessary to resort to nonstatistical factors to explain this fact. A possible way out, within the spirit of the hydrodynamic conception, is to refine the initial conditions of the problem (to assume, for example, that the first particle escaping from the volume is a nucleon, which acts so to speak as a "piston" for the entire remaining system*).



$1 — T_k = 3{,}0\ \mu$, $2 — T_k = 1{,}5\ \mu$

6. In this section, let us consider the energy distribution. Since the number dN of particles in an element of liquid is proportional to the entropy dS contained in the element, we can write

$$dN \sim s_k u_k R_k^2 d\xi, \qquad (47)$$

where s_k and R_k are the entropy density and the transverse dimensions of the element at the instant of free breakup.

Inasmuch as $s_k \sim T_k^3 = \text{const.}$, we have

$$dN = A u_k R_k^2 d\xi, \qquad (48)$$

A= const; $R_k = 1 + t_k^2 / a^2 u_k^2$; we can put approximately

$$R_k = 1 + t_k \xi_k / a_k.$$

*The value of $\overline{E}_m$ depends also on the volume V, of which only the order of magnitude is known. Using (37), (43), and (45), it is easy to show that $\overline{E}_m \sim V^{-\sqrt{3/4}}$.

The constant A is determined from the equality

$$2A\beta T_k \int u_k dN = \overline{E}_0/2. \qquad (49)$$

The energy distribution thus depends on the function $u_k(\xi_k)$. Approximate estimates can be made from equations (7), (8), (11), (25), (26), and (31).

By way of illustration, the figure shows the numerically-computed energy distributions at $E_0 = 2 \times 10^4$ and at temperatures $T_k = 1.5\mu$ and $T_k = 3\mu$.

In conclusion, the author acknowledges his debt to L. D. Landau, S. Z. Belen'kii, and I. M. Khalatnikov for discussing individual problems touched upon in this investigation.

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ERRATA TO VOLUME 4

	reads	should read
P. 218, column 2, Eq. (10)	$\ldots \xi(\sqrt{3}+2)(2-\sqrt{3})$	$\ldots \xi(\sqrt{2}+2)/(2-\sqrt{3})\ldots$
P. 219, column 1, Eq. (11)	$\ldots (t\xi)^{\sqrt{3/2}}\ldots$	$\ldots (t\xi)]^{\sqrt{3/2}}\ldots$
P. 219, column 1, Eq. (12)	$y^2 = \rho^{2/3}$	$y^2 - \rho^{2/3} >> 1$
P. 223, column 1, Eq. (45)	$\ldots (E_0\mu^{3/4})^{\sqrt{3/4}}$	$\ldots (E_0\,\mu^{3/4})^{\sqrt{3}/4}$
P. 223, column 2, Eq. (46)	$\ldots \mu^{3\sqrt{3/4}}\ldots$	$\ldots \mu^{3\sqrt{3/4}}\ldots$
P. 225, column 1, 3 lines above Eq. (1.1)	transversality	cross section
P. 225, column 1, 3 lines above Eq. (1.2)	transversality	cross section
P. 256, column 1, Eq. (37)	$\ldots \frac{55\sqrt{3}}{48}\ldots$	$\ldots \frac{55}{\sqrt{3}\,48}\ldots$
P. 289, column 2, Eq. (2)		$I=\sum_n \frac{1}{2n+1} A_n \sum_{\nu=-n}^{n} \frac{1}{1+i\omega\tau} Y_{n\nu}(n_1)\, Y_{n\nu}(n_2)$
P. 377, column 1, last line	$\delta_{35} = \eta - 21\times\eta^5$	$\delta_{35} - 21\,\eta^5$
P. 436–7	Figures 2 and 3 should be exchanged.	
P. 449, column 1, last Eq.	$\ldots Y_{lm}\,\varphi\sigma\,\alpha$	$\ldots Y_{lm}\,\varphi_\sigma\,\alpha$
P. 449, column 2, Eq. (12)	$\ldots W(l, j, \sigma\, 1;\, j)\ldots$	$\ldots W(l, j, \sigma\, 1;\, \sigma\, j)\ldots$
P. 451, column 1, Eq. (7)	$\ldots D_{\alpha\beta}^{(1)}(p, 0, \lambda'\,\lambda)=\ldots$	$\ldots D_{\alpha\beta}^{(1)}(p, \omega_0, \lambda', \lambda)=\ldots$
P. 541, column 1, Eq. (28)	$M_{++}^{*\,\mathrm{monex}}$	$M_{+}^{*\,\mathrm{monex}}$
P. 543, column 2, Eq. (35)	$\ldots \int_{\rho^2 - \tau^2 + l_0^2}$	$\ldots \int_{\rho^2 < \tau^2 + l_0^2}\ldots$