Thermal Radiation of Good Conductors

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Radiation from good conductors is examined by the methods of the electrodynamical theory of thermal fluctuations. In the first part of the work, radiation in the wave zone is found, with particular attention devoted to the limiting cases of very short and very long waves. Radiation from bodies with surface anistropy is examined. In the second part of the work, the fluctuating field in the neighborhood of conducting surfaces is considered: near a metallic plane, at the focus of a parabolic mirror, and at the center of a spherical mirror. Fluctuating surface charges are calculated.

1. THERMAL FIELD IN THE WAVE ZONE

THE analysis of thermal radiation from heated bodies can be considered as a problem of phenomenological electrodynamics if, following Rytov[1,2], random extraneous fields, or equivalently, random extraneous currents with zero radius of correlation, are introduced as the source of this radiation. As has been shown by the author[2], the electrodynamic reciprocity theorem allows a substantial simplification of the problem, reducing it to quadratures, if the solution to the corresponding subsidiary diffraction problem is known. In particular, the spectral density P_ω of the flow of energy of the fluctuating field in the wave zone, per unit solid angle, is connected with the effective transversality of the absorption for a plane wave with the same polarization as the radiation field by the formula

$$P_\omega = (\Theta / 2\pi) A / \lambda^2, \qquad (1.1)$$

where $\Theta = kT$ is the temperature of the body in energy units (we limit ourselves to the classical region of frequencies $\hbar\omega \ll \Theta$), and λ is the wavelength. The direction of the quantity P_ω is the direction of the infinitely distant source of the incident plane wave.

The effective transversality of the absorption is

$$A = \frac{Q_0}{(c/8\pi)\,|E_{\text{inc}}|^2} = \frac{Q_0}{(c/8\pi)\,|H_{\text{inc}}|^2}, \qquad (1.2)$$

where Q_0 are the thermal losses of the diffracted field in the body under consideration.

For good conductors, the diffracted field inside the body differs from zero only in a thin surface layer, whose thickness we may consider small with respect to all dimensions of the body and wavelength. In this case, Joule losses may be calculated by the formulas of the theory of the strong skin effect[4]: the heat arriving at an element of the surface of the body dS is

$$dQ_0 = (c/8\pi)\,\eta\,|H_0|^2 dS; \quad \eta = \sqrt{\mu\omega/8\pi\sigma}, \qquad (1.3)$$

where H_0 is the tangential component of the magnetic vector of the diffracted field on the surface of the body, σ is the conductivity, and μ is the magnetic permeability. To a first approximation, H_0 is the same as it would be at the surface of an ideally conducting body.

Thus, for good conductors, the spectral density of radiation of interest to us can be represented in the form

$$P_\omega = \frac{\Theta}{2\pi}\,\frac{\eta}{\lambda^2 |H_{\text{inc}}|^2} \oint |H_0|^2 dS = \frac{\Theta\eta S}{2\pi\lambda^2} G. \qquad (1.4)$$

where S is the surface area of the body, and

$$G = \oint |H_0|^2 dS / |H_{\text{inc}}|^2 S \qquad (1.5)$$

is a dimensionless function of direction and polarization, generally of order 1.

Integrating Eq. (1.4) over both polarizations and all directions, we obtain the spectral density J_ω of the total radiation

$$J_\omega = \frac{\Theta\eta S}{2\pi\lambda^2} \int (G_1 + G_2)\, d\Omega, \qquad (1.6)$$

where $d\Omega$ is the element of solid angle, and G_1 and G_2 correspond to the two different polarizations. In the future, in considering solids of revolution and plane laminae, G will be denoted by $G_\parallel$ for a wave whose electric vector lies in the meridian plane (or plane of incidence), and by $G_\perp$ for a wave whose magnetic vector lies in the meridean plane.

Exact solutions of diffraction problems (for example, for a sphere or disc) are represented by functional series, the substitution of which into Eq. (1.5) gives, after completing the quadratures, a certain series for the desired function G. We will not consider these exact solutions here, but will limit ourselves to those cases in which all the dimensions of the body are either great or small with respect to the wavelength. Then approximate solutions of the diffraction problem can be used (geometrical optics, quasi-stationary approximation), yielding finite algebraic expressions for G.

1. Shortwave Radiation

If the wavelength λ is small compared to all dimensions (including radii of curvature) of an ideally conducting convex body, then in the geometrical optics approximation, the diffraction field $\mathbf{H}_0$ is zero on the shaded part of the surface of the body, and on the illuminated part is determined by the reflection formula

$$\mathbf{H}_0 = 2\,[\mathbf{h} - \mathbf{n}(\mathbf{n}\cdot\mathbf{h})]\,H_{\text{inc}} \tag{1.7}$$

where $\mathbf{n}$ is a unit vector normal to the surface of the body, and $\mathbf{h}$ is the base vector of the magnetic vector of the incident wave. Substituting (1.7) into the general formula (1.5), we obtain

$$G = \frac{4}{S}\int [1 - (\mathbf{n}\cdot\mathbf{h})^2]\,dS, \tag{1.8}$$

where the integral is taken only over the illuminated portion of the surface of the body. For bodies having a center of symmetry, the integral over the shaded portion is evidently equal to the integral over the illuminated portion, so that the formula (1.8) for such bodies can be written in the form

$$G = \frac{2}{S}\oint [1 - (\mathbf{n}\cdot\mathbf{h})^2]\,dS = 2\,[1 - \overline{(\mathbf{n}\cdot\mathbf{h})^2}],$$

where the bar denotes an average over the surface of the body. Let us consider two examples:

a) *Plane thin Lamina* (Fig. 1). For a parallel-polarized wave, the vector $\mathbf{h}$ is parallel to the lamina, and $\mathbf{n}\cdot\mathbf{h} = 0$. For a perpendicular-polarized wave, $\mathbf{n}\cdot\mathbf{h} = \sin\vartheta$. Consequently,

$$G_{\parallel} = 2;\quad G_{\perp} = 2\cos^2\vartheta. \tag{1.9}$$

For $\vartheta = 0$, as could be expected, $G_{\parallel} = G_{\perp}$; for $\vartheta = \pi/2$, $G_{\perp} = 0$ and the thermal radiation is linearly polarized.

b) *Solid of Revolution With a Center of Symmetry* (Fig. 2). Let the origin of the coordinates coincide with the center of symmetry, with the z-axis along the axis of revolution of the body. The direction cosines of the normal vector $\mathbf{n}$ are denoted as usual by α, β, γ, and the orientation of $\mathbf{h}$ by h_1, h_2, h_3. In view of symmetry, evidently,

$$\overline{\alpha\beta} = \overline{\beta\gamma} = \overline{\gamma\alpha} = 0;\quad \overline{\alpha^2} = \overline{\beta^2} = {}^1/_2\,(1 - \overline{\gamma^2}),$$

so that

$$\overline{(\mathbf{n}\cdot\mathbf{h})^2} = {}^1/_2\,(1 - \overline{\gamma^2})\,(h_1^2 + h_2^2) + \overline{\gamma^2}h_3^2.$$

For a parallel-polarized wave

$$\overline{(\mathbf{n}\cdot\mathbf{h})^2} = {}^1/_2\,(1 - \overline{\gamma^2}).$$

For a perpendicular-polarized wave

$$\overline{(\mathbf{n}\cdot\mathbf{h})^2} = {}^1/_2\,(1 - \overline{\gamma^2})\cos^2\vartheta + \overline{\gamma^2}\sin^2\vartheta.$$

Thus, for solids of revolution with a center of symmetry

$$G_{\parallel} = 1 + \overline{\gamma^2}; \tag{1.10}$$

$$G_{\perp} = 1 + \overline{\gamma^2} + (1 - 3\overline{\gamma^2})\sin^2\vartheta.$$

Just as in the case of the lamina, $G_{\parallel}$ does not depend on the angle ϑ. For a sphere, $\overline{\gamma^2} = 1/3$, and

$$G_{\parallel} = G_{\perp} = {}^4/_3. \tag{1.11}$$

For a thin disc, $\overline{\gamma^2} \approx 1$, and

$$G_{\parallel} = 2;\quad G_{\perp} = 2\cos^2\vartheta,$$

i.e., we obtain the result found earlier. For a thin extended needle, $\overline{\gamma^2} \approx 0$, and

$$G_{\parallel} = 1;\quad G_{\perp} = 1 + \sin^2\vartheta.$$

Generally, for an oblate convex solid of revolution, it is evident that $\overline{\gamma^2} > 1/3$, and consequently, according to (1.10), $G_{\perp} \leq G_{\parallel}$, and $G_{\perp}$ is maximum in the direction of the axis of revolution $\vartheta = 0$. But for a prolate convex solid of revolution, $\overline{\gamma^2} < 1/3$, $G_{\perp} \geq G_{\parallel}$, and $G_{\perp}$ is maximum in the equatorial plane $\vartheta = \pi/2$.

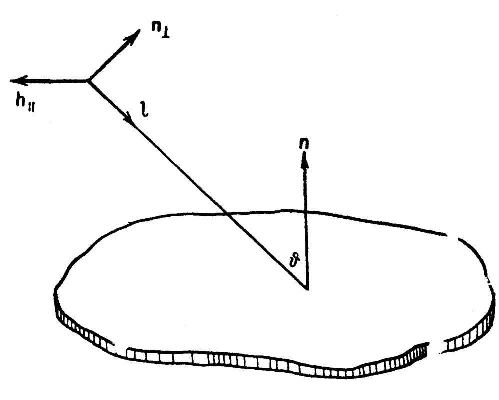

FIG. 1

In particular, for an oblate spheroid of eccentricity e

$$\overline{\gamma^2} = \frac{(\operatorname{ch}\chi + 1)(\operatorname{sh}\chi - \chi)}{(\operatorname{ch}\chi - 1)(\operatorname{sh}\chi + \chi)}\ ;\ \chi = 2\operatorname{arth} e \tag{1.12}$$

and for a prolate spheroid of eccentricity e

$$\overline{\gamma^2} = \frac{(1 + \cos\varphi)(\varphi - \sin\varphi)}{(1 - \cos\varphi)(\varphi + \sin\varphi)}\ ;\ \varphi = 2\arcsin e. \tag{1.13}$$

A graph of $\overline{\gamma}^2$ versus e, based on Eqs. (1.12) and (1.13), is given in Fig. (3). Using this graph and the general formula (1.10), the shortwave thermal radiation can be calculated for any metallic spheroid.

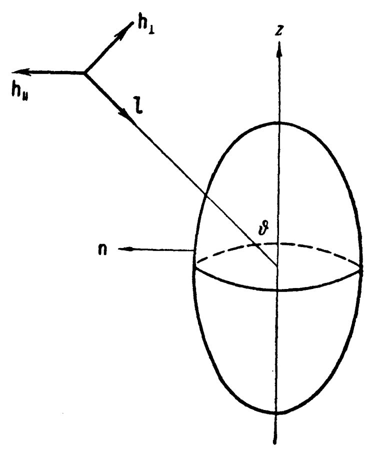

FIG. 2

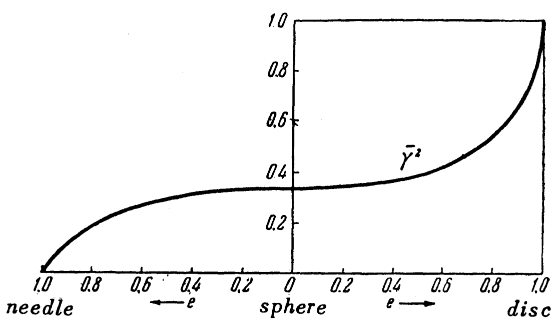

FIG. 3

Substituting (1.9) or (1.10) into the general formula (1.6), we easily find that in both cases the spectral density of the total radiation is

$$I_\omega = 16\,\Theta\gamma_i S / 3\lambda^2. \tag{1.14}$$

We will now show that formula (1.14) is valid for an arbitrary convex body. We note that this could be expected beforehand, since in the geometric optics approximation, the usual formulation of Kirchoff's law is valid, and the total thermal radiation does not depend on the form of the body, but only on its surface area.

Adding Eqs. (1.8) for the two possible mutually perpendicular polarizations h_1 and h_2 and noting that $(n \cdot h_1)^2 + (n \cdot h_2)^2 = 1 - (n \cdot l)^2$, where l is the base vector of the wave vector of the incident wave, we obtain for the summed factor $G_t = G_1 + G_2$ the following expression:

$$G_t = \frac{4}{S}\int [1 + (\mathbf{n}\cdot\mathbf{l})^2]\, dS.$$

We will integrate this equation over all directions. To each direction may be compared its direct opposite, for which an illuminated part of the surface appears as a shaded part for the original direction. Therefore,

$$\int G_t\, d\Omega = \frac{2}{S}\int \oint [1 + (\mathbf{n}\cdot\mathbf{l})^2]\, dS\, d\Omega.$$

But for arbitrary $\mathbf{n}$,

$$\int (\mathbf{n}\cdot\mathbf{l})^2\, d\Omega = 4\pi/3,$$

so that

$$\int G_t\, d\Omega = 32\pi/3. \tag{1.15}$$

Substituting (1.15) into (1.6), we again arrive at Eq. (1.14) for the total radiation of an arbitrary convex metallic body in the geometrical optics approximation.

3. Longwave Radiation

If the wavelength is large compared to the dimensions of the body, we can confine ourselves to a consideration of the diffraction problem in the quasi-stationary approximation. In the quasi-stationary approximation, the magnetic vector is $H_0 = H_{pot}$, where H_{pot} is found from the solution of the magnetostatic problem of an ideally conducting body placed in a homogeneous external magnetic field H_{inc}. Actually, the vortical addition H_{vort}, generated by the changing electric field, is determined by the equation $\operatorname{curl} H_{vort} = (2\pi i/\lambda)E$. But considering order of magnitude, $|\operatorname{curl} H_{vort}| \sim (1/a)H_{vort}$ (a is a characteristic dimension of the body, $a \ll \lambda$) and consequently, $H_{vort} \sim (a/\lambda)E \sim (a/\lambda)H_{pot}$, since $H_{pot} \sim H_{inc} \sim E$. Thus in the quasi-stationary approximation, thermal losses are due to the magnetic vector* of the incident wave.

* Taking account of the electric vector of the incident wave is essential only for very thin rods (wire antennae)having resonance properties. Thermal radiation of such rods has been considered by Rytov and the author[5].

A magnetostatic problem for a body with zero magnetic permeability corresponds formally to an ideally conducting body on the surface of which the normal component of the magnetic vector is zero. We note that when the body possesses three mutually perpendicular planes of symmetry, it is sufficient to find the factor G only for the three principal axes. Let $\mathbf{H}_{inc} = \mathbf{h} = h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k}$ where i, j, k, are the base vectors of the principal axes. Clearly,

$$\mathbf{H}_0 = h_1\mathbf{H}_{01} + h_2\mathbf{H}_{02} + h_3\mathbf{H}_{03},$$

where, for example, $\mathbf{H}_{01}$ is the magnetic field corresponding to $\mathbf{H}_{inc} = \mathbf{i}$. In view of the symmetry of the body, integrals of the form

$$\oint \mathbf{H}_{01}\mathbf{H}_{02}dS = 0,$$

and therefore, for arbitrary direction h, the function G is

$$G = h_1^2G_1 + h_2^2G_2 + h_3^2G_3. \tag{1.16}$$

As an example, we consider a body having the form of an ellipsoid with semiaxes a, b, c ($a \geq b \geq c$). Substituting the solution of the magnetostatic problem for an ellipsoid with $\mu = 0$ (see for example Ref. 6, where the detailed solution of the equivalent electrostatic problem is given), we obtain after straightforward but somewhat laborious calculations, the following expression for G_1:

$$SG_1 = \frac{2}{(1-M_1)^2}\frac{a^2}{(a^2-b^2)(a^2-c^2)} \tag{1.17}$$

$$\times\left\{\int_{c^2}^{b^2}\sqrt{\frac{(b^2-u)(u-c^2)}{u(a^2-u)}}du\right.$$

$$\times\int_{b^2}^{a^2}\sqrt{\frac{v(a^2-v)}{(v-b^2)(v-c^2)}}dv$$

$$+\int_{c^2}^{b^2}\sqrt{\frac{u(a^2-u)}{(b^2-u)(u-c^2)}}du$$

$$\left.\times\int_{b^2}^{a^2}\sqrt{\frac{(v-b^2)(v-c^2)}{v(a^2-v)}}dv\right\},$$

where

$$M_1 = \frac{abc}{2}\int_0^\infty ds/(a^2+s) \tag{1.18}$$

$$\times\sqrt{(a^2+s)(b^2+s)(c^2+s)}$$

Formulas for G_2 and G_3 are obtained from (1.17) and (1.18) by cyclic permutation in the expressions under the integrals and in the factor outside the integrals, with a subsequent change of sign if negative expressions are obtained during such permutations. We note that $M_1 + M_2 + M_3 = 1$. We will not write here the representations of the right hand side of (1.17), (1.18), as well as the area of the ellipsoid S by elliptic integrals, but will limit ourselves to the case of an ellipsoid of revolution for which the quantities of interest to us are expressed by elementary functions. Omittir the intermediate transformations, we present the final formulas.

Oblate spheroid ($a = b \geq c$) of eccentricity e:

$$G_3 = G_P \tag{1.19}$$

$$= 2(1-M)^{-2}(\chi\,\mathrm{ch}\,\chi - \mathrm{sh}\,\chi)$$

$$\div(\mathrm{ch}\,\chi - 1)(\mathrm{sh}\,\chi + \chi),$$

$$G_1 = G_2 = G_E$$

$$= 4(1+M)^{-2}(\mathrm{ch}\,\chi\,\mathrm{sh}\,\chi - \chi)$$

$$\div(\mathrm{ch}\,\chi - 1)(\mathrm{sh}\,\chi + \chi),$$

$$M = M_3 = \mathrm{cth}^2\frac{\chi}{2}\left(1 - \frac{\mathrm{arctg\,sh}(\chi/2)}{\mathrm{sh}(\chi/2)}\right);$$

where χ is associated with e by formula (1.12).

Prolate spheroid ($a \geq b = c$):

$$G_1 = G_P \tag{1.20}$$

$$= 2(1-M)^{-2}(\sin\varphi - \varphi\cos\varphi)$$

$$\div(1-\cos\varphi)(\varphi + \sin\varphi);$$

$$G_2 = G_3 = G_E$$

$$= 4(1+M)^{-2}(\varphi - \cos\varphi\sin\varphi)/(1-\cos\varphi)$$

$$\times(\varphi + \sin\varphi);$$

$$M = M_1 = \mathrm{ctg}^2\frac{\varphi}{2}\left(\frac{\mathrm{arth}\sin(\varphi/2)}{\sin(\varphi/2)} - 1\right);$$

where φ is associated with e by formula (1.13). The indices "p" and "e" in Eqs. (1.19) and (1.20) denote "pole" and "equator," respectively.

For solids of revolution, the general formula (1.16) can be written in the form

$$G = h_P^2G_P + h_E^2G_E,$$

where h_p is the projection of h on the axis of revolution, and h_e is the projection on the equatorial plane. For parallel and perpendicularly polarized waves, respectively,

$$G_{\parallel} = G_E; \; G_{\perp} = \sin^2\vartheta G_P + \cos^2\vartheta G_E. \quad (1.21)$$

For a sphere ($e = 0$), Eqs. (1.19) and (1.20) give $G_p = G_e = 3/2$, so that*

$$G_{\parallel} = G_{\perp} = {}^3/_2. \quad (1.22)$$

For a disc ($e \to 1$), it follows from (1.19) that

$$G_P = \frac{8}{\pi^2}\left(\operatorname{arth} e - \frac{1}{2}\right) \approx \frac{8}{\pi^2}\left(\ln\frac{2a}{c} - \frac{1}{2}\right);$$

$$G_E = 1$$

and

$$G_{\parallel} = 1; \; G_{\perp} = \frac{8}{\pi^2}\left(\ln\frac{2a}{c} - \frac{1}{2}\right)\sin^2\vartheta + \cos^2\vartheta.$$

For a needle ($e \to 1$), formula (1.20) gives $G_p = 1$ and $G_e = 2$, so that

$$G_{\parallel} = 2; \; G_{\perp} = 1 + \cos^2\vartheta. \quad (1.23)$$

The dependence of G_p and G_e on the eccentricity e, calculated by formulas (1.19) and (1.20), is shown in Fig. (4). Using these curves, the long wave thermal radiation of an arbitrary metallic spheroid can be calculated by Eq. (1.21).

For an oblate spheroid, $G_p > G_e$ and on the basis of Eq. (1.21), $G_{\perp}$ is maximum in the equatorial plane $\vartheta = \pi/2$. For a prolate spheroid, $G_p < G_e$, and $G_{\perp}$ is maximum in the direction of the axis of revolution $\vartheta = 0$. In the case of short wave radiation, the situation is opposite as was shown in Section 1.

In Section 1 we established that the short wave radiation of a thin plane lamina of arbitrary form possesses circular symmetry with respect to an axis perpendicular to the lamina. It is easily seen that long wave radiation possesses the same symmetry. In fact, a thin lamina with $\mu = 0$, placed in a homogeneous magnetic field does not disturb the field. Therefore, for coordinate axes x and y chosen in the plane of the lamina, $G_1 = G_2 = 1$, whence follows circular symmetry of the radiation. For the third axis z perpendicular to the lamina, the factor G_3 is of logarithmic magnitude. For example, in the case of a strongly prolate elliptical disc ($a \gg b \gg c$), Eqs. (1.17) and (1.18) give $G_1 = G_2 = 1$; $G_3 = \ln(4b/c) - 1$.

* The curve of the transition from (1.22) to (1.11) is given in Ref. (2).

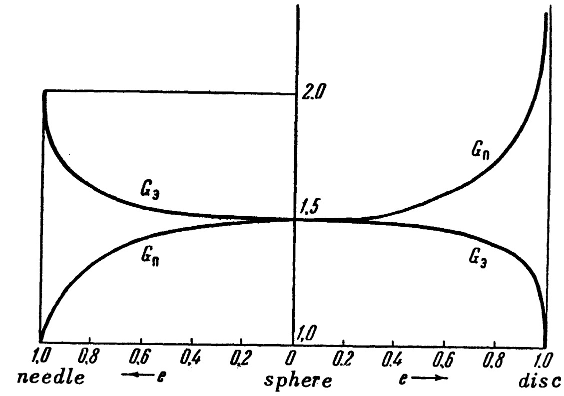

FIG. 4

Thus, for long wave radiation of any thin plane lamina

$$G_{\parallel} = 1; \; G_{\perp} = \Lambda\sin^2\vartheta + \cos^2\vartheta,$$

where the factor Λ is of the order of the logarithm of the ratio of the width of the lamina to its thickness.

At the beginning of this section it was mentioned that very thin rodlike conductors (wire antennae) possess resonance properties, thanks to which the Joule losses due to the electric vector of the incident wave may become commensurate (even in the case of small bodies) with the losses associated with the magnetic vector. To these "electrical losses" correspond the thermal antenna radiation considered in Ref. (5). As was shown in Ref. (5), for small bodies, the antenna radiation has a dipole character: it is $\parallel$ -polarized and is proportional to $\sin^2\vartheta$, where ϑ is the angle between the rod and the direction of radiation. But according to the first of Eqs. (1.23), the "magnetic" radiation considered above of the same polarization does not depend on the angle ϑ. Let us compare the spectral densities of both radiations. For example, let the rod have the form of a strongly prolate spheroid $a \gg b = c$. The spectral density of thermal "antenna" radiation of this spheroid for long waves ($\lambda \gg a$) was calculated in Ref. (5), and in the notation of the present article is

$$J^{\text{ant}} = \frac{\pi^4 a^5}{c\lambda^4}\left(\ln\frac{a}{c}\right)^2 \Theta\eta.$$

The spectral of the "magnetic" radiation of interest to us is $\parallel$-polarized according to Eq. (1.6), and the first of Eqs. (1.23) gives

$$J^{\text{mag}} = 4S\lambda^{-2}\Theta\eta = 4\pi^2 ac\lambda^{-2}\Theta\eta.$$

Thus, for a small thin needle,

$$\frac{J^{\text{ant}}}{J^{\text{mag}}} = \left(\frac{\pi}{2}\frac{a^2}{c\lambda}\ln\frac{a}{c}\right)^2.$$

3. Radiation of Bodies with Surface Anisotropy

In radio technology and radiophysics, more and more use is being made of conductors with anisotropic surfaces, characterized by a surface impedance tensor (see, for example, Ref. 7). Joule losses in such conductors are determined by the formula

$$dQ_0 = (c/8\pi)(\eta_u |H_{0u}|^2 + \eta_v |H_{0v}|^2)\, dS, \quad (1.24)$$

clearly a generalization of Eq. (1.3). u and v are orthogonal curvilinear coordinates on the surface with respect to which the surface impedance tensor is diagonal, and

$$\eta_u = \sqrt{\omega\mu_u/8\pi\sigma_v}; \quad \eta_v = \sqrt{\omega\mu_v/8\pi\sigma_u},$$

where σ_u, σ_v, μ_u, μ_v are the effective quantities corresponding to each type of laminated or similar structure.

Introducing the mean value η,

$$\eta_u = (1+\nu)\eta; \quad \eta_v = (1-\nu)\eta; \quad -1<\nu<1$$

and substituting (1.24) into the general formulas (1.2) and (1.1), we again obtain expression (1.4) for the spectral density of radiation, with the factor G given by

$$G = \oint \{(1+\nu)|H_{0u}|^2 + (1-\nu)|H_{0v}|^2\}\, dS / |H_{\text{inc}}|^2 S. \quad (1.25)$$

As an example, we consider the radiation of a sphere for which the lines u = const are parallels and the lines v = const are meridians. If the surface of the sphere is isotropic ($\nu = 0$), then $G_\parallel = G_\perp = G_{iso}$, where $G_{iso} = 4/3$ for short waves and 3/2 for long waves. With $\nu \neq 0$, a straightforward calculation based on Eq. (1.25) gives the same result in both the geometrical optics approximation and the quasi-stationary approximation:

$$G_\parallel = \frac{2-\nu}{2} G_{\text{iso}}; \quad G_\perp = \left(\frac{2-\nu}{2} + \frac{3\nu}{2}\sin^2\vartheta\right) G_{\text{iso}}$$

The agreement of the results is explained by the fact that on the illuminated part of the surface of the sphere, the magnetic field H_0 has the same geometry in both the geometrical optics and the quasi-stationary approximations.

With $\eta_u > \eta_v$ (as would be the case for example if a sphere was completely wound with bare wire, the windings of which coincided with the parallels), $\nu > 0$, and the radiation is maximum in the equatorial plane $\theta = \pi/2$. In particular, with $\eta_u \gg \eta_v$ ($\nu \approx 1$)

$$G_\parallel = {}^1/_2 G_{\text{iso}}; \quad G_\perp = {}^1/_2(1 + 3\sin^2\vartheta)\, G_{\text{iso}}.$$

In the opposite case $\eta_u < \eta_v$ the radiation will have a maximum in the direction of the axis of symmetry of the sphere, $\vartheta = 0$. In particular, with $\eta_u \ll \eta_v$ ($\nu \approx -1$)

$$G_\parallel = {}^3/_2 G_{\text{iso}}, \quad G_\perp = {}^3/_2 \cos^2\vartheta\, G_{\text{iso}}.$$

As a second example we consider the radiation of a thin plane anisotropic lamina, for which the coordinates u and v are the rectangular coordinates x and y. Let ϑ and φ be the polar angles of the wave vector of the incident wave. Then in the geometrical optics approximation, we obtain after a straightforward calculation, in place of Eq. (1.9),

$$G_\parallel = 2(1 - 2\nu\cos 2\varphi);$$

$$G_\perp = 2\cos^2\vartheta\,(1 + 2\nu\cos 2\varphi).$$

But in the quasi-stationary approximation, we will have

$$G_\parallel = 1 - 2\nu\cos 2\varphi;$$

$$G_\perp = \Lambda\sin^2\vartheta + \cos 2\vartheta\,(1 + 2\nu\cos 2\varphi).$$

2. FLUCTUATING FIELD NEAR HIGHLY CONDUCTING SURFACES

In the first part of this article, the thermal radiation of good conductors was considered in the wave zone. However, the fluctuating field near radiating bodies, at distances commensurable with or smaller than the dimensions of these bodies, also has practical interest. As was shown in Ref. (3), the problem of finding the fluctuating field at any point outside the radiating body may be reduced to quadratures by use of the electrodynamic theorem of reciprocity, if the diffracted field created by an elementary source situated at this point is known. Thus, for a uniformly heated body at the temperature Θ, the mean square spectral density of any component of the electric vector is*

$$\overline{|\mathbf{E}|^2} = 2\Theta Q_0/\pi\, |\mathbf{F}|^2, \quad (2.1)$$

* We are using the same notation as in Refs. 2 and 3. The index ω in the spectral density denotes that it corresponds to the decomposition in the spectrum according to positive frequencies. Wherever fluctuating quantities are mentioned, we have in mind spectral densities.

where $F = \int j_0 dV = i\omega p$ is a quantity characterizing the electric dipole formed by the extraneous electric currents j_0 flowing in the vanishingly small volume of integration τ, and Q_0 represents the thermal losses of the diffracted field created by these dipoles in the body under consideration. The direction of the vector F coincides with the direction of the component of the electric vector of interest to us. An equation similar to Eq. (2.1) is valid for the magnetic vector of the fluctuating field, the only difference being that the source of the diffracted field is a magnetic dipole, formed by extraneous magnetic currents j_0^m, $F = \int j_0^m dV$.

In good conductors, the diffracted field differs from zero only in a thin skin-layer. In the future we will always consider that all dimensions of the body, the wavelength, and the distance from the surface of the body to the point at which the fluctuating field is determined, are great in comparison with the thickness of the skin-layer δ. Then the thermal losses Q_0 may be calculated by the theory of the strong skin effect

$$Q_0 = (c\eta/8\pi) \oint |H_0|^2 dS. \tag{2.2}$$

Substituting (2.2) into (2.1), and introducing as a standard the mean square of the spectral densities of the field vectors in equilibrium radiation

$$\overline{|\mathbf{E}_{eq.}|^2} = \overline{|\mathbf{H}_{eq.}|^2} = 2\Theta k^2/\pi c,$$

we obtain

$$\overline{|\mathbf{E}|^2} = {}^1/_2\, \eta\, \overline{|\mathbf{E}_{eq.}|^2}\, e; \tag{2.3}$$

$$\overline{|\mathbf{H}|^2} = {}^1/_2\, \eta\, \overline{|\mathbf{H}_{eq.}|^2}\, h.$$

Here, e and h are dimensionless factors depending on the geometry of the body, the wavelength, and the coordinates of the point at which the fluctuating field is being determined,

$$e = \frac{c^2}{4\pi k^2} \oint |H_0^{el}|^2 dS; \tag{2.4}$$

$$h = \frac{c^2}{4\pi k^2} \oint |H_0^{mag}|^2 dS,$$

where H_0^{el} and H_0^{mag} are the diffracted fields of unit ($F = 1$) electric and magnetic dipoles with pertinent orientations. The factor 1/2 is introduced into Eq. (2.3) for convenience: with such normalization, the factor e for the component of the electric vector parallel to a radiating plane tends toward unity at large distances from the plane.

In the following, we will consider only those problems in which the diffracted field may be calculated by reflection formulas. In this case, $H_0 = 2H_{1\,tan}$, where H_1 is the field of a unit dipole in free space, and Eq. (2.4) may be written in the form

$$e = \frac{c^2}{\pi k^2} \int |[\mathbf{n}\cdot\mathbf{H}_1^{el}]|^2 dS; \tag{2.5}$$

$$h = \frac{c^2}{\pi k^2} \int |[\mathbf{n}\cdot\mathbf{H}_1^{mag}]|^2 dS$$

(n is a unit vector normal to the surface). The fields of the unit dipoles in free space, as is well known, are

$$\mathbf{H}_1^{el} = \frac{ik}{cR}\left(1 - \frac{i}{kR}\right)[\mathbf{f}\rho], \tag{2.6}$$

$$\mathbf{H}_1^{mag} = \frac{ik}{cR}\left\{\left(1 - \frac{3i}{kR} - \frac{3}{k^2R^2}\right)(\mathbf{f}\rho)\,\rho - \left(1 - \frac{i}{kR} - \frac{1}{k^2R^2}\right)\mathbf{f}\right\}. \tag{2.7}$$

where f is the base vector of the dipole, ρ is the base vector of the radius-vector $R = R\rho$, and the wave factor e^{-ikR} is omitted.

1. Thermal Field of a Conducting Plane

Let the origin of a cartesian system of coordinates be at the point where a dipole is situated, with the z-axis perpendicular to the radiating plane, and let R, ϑ, φ be spherical coordinates. The coordinate φ is also the polar coordinate on the plane (see Fig. 5). The element of area is $dS = r\,dr\,d\varphi = R\,dR\,d\varphi$. If the base vector f of an electric dipole is directed along the x-axis, then

$$|[\mathbf{n}\cdot\mathbf{h}_1^{el}]|^2 = \frac{k^2}{c^2R^2}\left(1 + \frac{1}{k^2R^2}\right)\cos^2\vartheta = \frac{k^2}{c^2R^2}\left(1 + \frac{1}{k^2R^2}\right)\frac{z^2}{R^2}. \tag{2.8}$$

Substituting this expression into the general formula (2.5), we obtain, after integration,

$$e_x = 1 + (1/2k^2z^2).$$

For f parallel to the z-axis,

$$|[\mathbf{n}\cdot\mathbf{h}_1^{el}]|^2 = \frac{k^2}{c^2R^2}\left(1 + \frac{1}{k^2R^2}\right)\sin^2\vartheta = \frac{k^2}{c^2R^2}\left(1 + \frac{1}{k^2R^2}\right)\frac{R^2 - z^2}{R^2}. \tag{2.9}$$

At large distances the right hand side of Eq. (2.9) decreases as $1/R^2$, so that a direct substitution of (2.9) into the equation for e_z leads to a logarithmically diverging expression. But as is well known, the reflection formulas cease to be valid at large distances; they must be multiplied by the Sommerfeld attenuation function (see for example, Ref. 8), the argument of which is the quantity $(\pi c/2\sigma\mu\lambda^2)r$, called the numerical distance. It was not necessary to take this attenuation into account in order to find e_x, because Eq. (2.8) tends toward zero sufficiently rapidly (as $1/R^4$). But if, as occurs in our case, $H_{\tan}$ falls off only as $1/R$, it becomes necessary to take attenuation into account. To do this, the square of the attenuation function must be introduced under the integral sign. However, we will not do this here, but will limit ourselves to giving a sufficiently accurate approximate calculation; without changing the expression under the integral, we will integrate over the R term giving logarithmic divergence, not up to infinity, but up to the quantity L of the order $L \sim \sigma\mu\lambda^2/c$. Doing this, we obtain

$$e_z = 2\ln\frac{L}{z} + \frac{1}{2k^2z^2}.$$

For a magnetic dipole whose base vector **f** is parallel to the x-axis,

$$|[\mathbf{n}\cdot\mathbf{h}_1^{\mathrm{mag}}]|^2 = \frac{k^2}{c^2R^2}\Big\{\Big(1 - \frac{1}{k^2R^2} + \frac{1}{k^4R^4}\Big) - \cos^2\varphi\Big[\Big(1 - \frac{5}{k^2R^2} - \frac{3}{k^4R^4}\Big) + 4\cos^2\vartheta\Big(\frac{2}{k^2R^2} + \frac{3}{k^4R^4}\Big) - \cos^4\vartheta\Big(1 + \frac{3}{k^2R^2} + \frac{9}{k^4R^4}\Big)\Big]\Big\}$$

and after integration ($\cos\vartheta = z/R$)

$$h_x = \ln\frac{L}{z} + \frac{3}{8k^4z^4},$$

where we have again replaced the infinite limit of integration by L for the logarithmic term.

Finally, in the case of a magnetic dipole parallel to the z-axis,

$$|[\mathbf{n}\cdot\mathbf{h}_1^{\mathrm{mag}}]|^2 = \frac{k^2}{c^2R^2}\Big(1 + \frac{3}{k^2R^2} + \frac{9}{k^4R^4}\Big)\cos^2\vartheta\sin^2\vartheta,$$

$$h_z = {}^1\!/_2 + 1/2k^2z^2 + 1/4k^4z^4.$$

Since in our case, in view of symmetry, $|\overline{\mathbf{E}}|^2 = 2|\overline{E_x}|^2 + |\overline{E_z}|^2$, the electric energy density of the fluctuating field of a plane is

$$w_\omega^{\mathrm{el}} = \frac{1}{2}\eta w_{\mathrm{eq.}}^{\mathrm{el}}(2e_x + e_z) \qquad (2.10)$$

$$= \eta w_{\mathrm{eq}}^{\mathrm{el}}\Big(\ln\frac{L}{z} + \frac{3}{4k^2z^2}\Big).$$

Similarly,

$$w_\omega^{\mathrm{mag}} = \frac{1}{2}\eta w_{\mathrm{eq.}}^{\mathrm{mag}}(2h_x + h_z) \qquad (2.11)$$

$$= \eta w^{\mathrm{mag}}\Big(\ln\frac{L}{z} + \frac{1}{4k^2z^2} + \frac{1}{2k^4z^4}\Big).$$

where

$$w_{\mathrm{eq.}}^{\mathrm{el}} = w_{\mathrm{eq.}}^{\mathrm{mag.}} = {}^1\!/_2\, w_{\mathrm{eq.}} \qquad w_{\mathrm{eq.}} = \Theta k^2/\pi^2 c,$$

and terms of the order of unity have been omitted on account of the indefiniteness of the quantity L. The total energy of the fluctuating field is

$$w_\omega = w_\omega^{\mathrm{el}} + w_\omega^{\mathrm{mag}} = \eta w_{\mathrm{eq.}}\Big(\ln\frac{L}{z} + \frac{1}{2k^2z^2} + \frac{1}{4k^4z^4}\Big).$$

Let us note once again that all the formulas of the present section are, by the very manner in which they were obtained, valid only for distances z large compared to the thickness of the skin-layer δ; $z \gg \delta$. Moreover, formulas containing the quantity L are valid only under the condition $z \ll L$, which is always realized in practice.

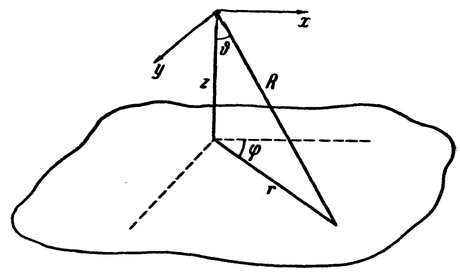

FIG. 5

Comparing Eqs. (2.10) and (2.11), we see that in the neighborhood of the plane, at distances $z \ll \lambda$, the magnetic energy considerably exceeds the electric energy

$$w^{\mathrm{mag}}/w^{\mathrm{el}} = 2/3k^2z^2 = (1/6\pi^2)(\lambda/z)^2,$$

so that $w \approx w^{\mathrm{mag}}$.

Let us now calculate the energy flow vector for thermal radiation from a plane. We will even consider a more general problem: we will find the z-component of the Poynting vector on the axis of a circular disc of large (compared with λ) but finite radius. In view of symmetry,

$$S_{\omega z} = (c/2\pi)\,\mathrm{Re}\,\overline{[\mathbf{E}\cdot\mathbf{H}^*]}_z = (c/\pi)\,\mathrm{Re}\,\overline{E_x H_y^*}. \qquad (2.12)$$

As is shown in Ref. (3), the mean value of the product of two components of the fluctuating field, A and B, (in our case, $A = E_x$, $B = H^*_y$) is

$$\overline{AB} = 2\Theta Q_{AB}/\pi F_A F_B, \qquad (2.13)$$

where Q_{AB} are the mixed thermal losses A and B of the diffracted field. For good conductors,

$$Q_{AB} = \frac{c\eta}{8\pi}\int [\mathbf{n}\cdot\mathbf{H}_{0A}]\cdot[\mathbf{n}\cdot\mathbf{H}_{0B}]\,dS. \qquad (2.14)$$

If H_{0A} and H_{0B} are the fields of unit dipoles, then the factors F_A and F_B in Eq. (2.13) must be omitted. If, moreover, the reflection formulas may be used, then finally

$$\overline{AB} = \frac{c\Theta\eta}{\pi^2}\int [\mathfrak{n}\cdot H_{1A}][\mathfrak{n}\cdot H_{1B}]\, dS, \qquad (2.15)$$

where H_{1A} and H_{1B} are the fields of unit dipoles A and B in free space. We note that if B is equal to the complex conjugate of any component of the field, then not the fields H_{0B} and H_{1B} themselves, but their complex conjugates, must be substituted into Eqs. (2.14) or (2.15).

In the case of interest to us, H_{1A} is given by Eq. (2.6) with f = i, and H_{1B} by the complex conjugate of Eq. (2.7) with f = j. Then , as is easily seen,

$$\mathrm{Re}\,\overline{E_x H_y^*} = \frac{\Theta\eta k^2}{\pi^2 c}\int \cos\vartheta\,(1 - \sin^2\vartheta\,\sin^2\varphi)\,\frac{dS}{R^2}.$$

Performing the integration and substituting the result into (2.12), we obtain

$$S_{\omega z} = (\Theta\eta k^2/\pi^2)\,[1 - \cos\Phi + {}^1/_3\,(1 - \cos^3\Phi)],$$

where Φ is the angle subtended by the radius of the disc. In particular, for an infinite plane ($\Phi = \pi/2$)

$$S_{\omega z} = 4\Theta\eta k^2/3\pi^2 = {}^4/_3\,\eta c w_{eq}. \qquad (2.16)$$

At normal incidence, the reflection coefficient of a highly conducting plane is $R = 1 - 4\eta$, so that Eq. (2.16), as could be expected, agrees with the expression given by the Kirchhoff theory of thermal radiation.

2. Thermal Field of Metallic Mirrors

Metallic mirrors, used in optics and radio technology as focussing structures, are as a rule large compared with the wavelength. Therefore, to find the fluctuating field at points whose distances from the mirror are also large compared to λ, we may limit ourselves to the geometrical optics approximation. Then only wave terms remain in Eqs. (2.6) and (2.7), and these equations take the form

$$H_1^{el} = (ik/cR)\,[\mathfrak{f}\rho]; \quad H_1^{mag} = (ik/cR)\,[[\mathfrak{f}\rho]\,\rho]. \qquad (2.17)$$

The substitution of (2.17) into the general formulas (2.5) for e and h yields the purely geometrical quantities

$$e = \frac{1}{\pi}\int [\mathfrak{n}M]^2 \frac{dS}{R^2}; \quad h = \frac{1}{\pi}\int [\mathfrak{n}N]^2 \frac{dS}{R^2}, \qquad (2.18)$$

$$M = [\mathfrak{f}\rho]; \quad N = [[\mathfrak{f}\rho]\,\rho].$$

Let us find the fluctuating field in the focus of a paraboloidal mirror. The origin of the coordinate system is placed at the focus in such a way that its axis coincides with the axis of revolution (Fig. 6). It is easy to see that for a paraboloid

$$dS/R^2 = 2\sin\frac{\vartheta}{2}\,d\vartheta\,d\varphi, \qquad (2.19)$$

$$n_x = \sin\frac{\vartheta}{2}\cos\varphi;$$

$$n_y = \sin\frac{\vartheta}{2}\sin\varphi; \quad n_z = \cos\frac{\vartheta}{2}.$$

Forming the quantities $\mathfrak{n} \times M$ and $\mathfrak{n} \times N$ of interest to us, and performing straightforward trigonometric

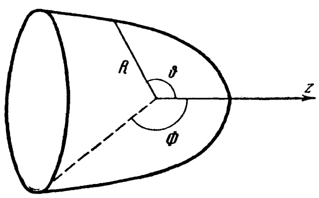

FIG. 6

transformations, we obtain, after carrying out the integration in Eq. (2.18):

$$e_x = 4\left[\left(1 - \cos\frac{\Phi}{2}\right) - \left(1 - \cos^3\frac{\Phi}{2}\right) + \frac{4}{5}\left(1 - \cos^5\frac{\Phi}{2}\right)\right]; \qquad (2.20)$$

$$e_z = 32\left[\frac{1}{3}\left(1 - \cos^3\frac{\Phi}{2}\right) - \frac{1}{5}\left(1 - \cos^5\frac{\Phi}{2}\right)\right];$$

$$h_x = 4\left[\left(1 - \cos\frac{\Phi}{2}\right) + \frac{1}{3}\left(1 - \cos^3\frac{\Phi}{2}\right) - \frac{4}{5}\left(1 - \cos^5\frac{\Phi}{2}\right) + \frac{4}{7}\left(1 - \cos^7\frac{\Phi}{2}\right)\right],$$

$$h_z = 32\left[\frac{1}{5}\left(1 - \cos^5\frac{\Phi}{2}\right) - \frac{1}{7}\left(1 - \cos^7\frac{\Phi}{2}\right)\right],$$

where Φ is the angular opening of the mirror. In particular, for an infinite paraboloid ($\Phi = \pi$)

$$e_x = 16/5 = 3.20; \quad e_z = 64/15 = 4.26;$$

$$h_x = 464/105 = 4.42; \quad h_z = 64/35 = 1.83.$$

Curves calculated on the basis of Eq. (2.20) are given in Fig. (7).

Furthermore, from Eq. (2.20)

$$2e_x + e_z = 2h_x + h_z$$
$$= 8\left[\left(1 - \cos\frac{\Phi}{2}\right) + \frac{1}{3}\left(1 - \cos^3\frac{\Phi}{2}\right)\right],$$

so that at the focus

$$w_\omega^{el} = w_\omega^{mag} = \frac{1}{2} w_\omega$$
$$= 2\eta w_{eq.}\left[\left(1 - \cos\frac{\Phi}{2}\right) + \frac{1}{3}\left(1 - \cos^3\frac{\Phi}{2}\right)\right].$$

In particular, for an infinite paraboloid $w_\omega = (16/3)\eta w_{eq.}$

As a second example we consider the fluctuating field at the center of a spherical mirror with an angular opening $\Phi \leq \pi/2$ (Fig. 8). In this case, $dS/R^2 = \sin\vartheta d\vartheta d\varphi$. After simple calculations we obtain

$$e_x = h_x = (1 - \cos\Phi) + {}^1/_3\,(1 - \cos^3\Phi), \quad (2.21)$$
$$e_z = h_z = 2\,[(1 - \cos\Phi) - {}^1/_3\,(1 - \cos^3\Phi)].$$

In particular, for a hemisphere $e_x = e_z = h_x = h_z = 4/3$. Curves based on Eq. (2.21) are given in Fig. (9).

In the case of a spherical mirror, again,

$$w_\omega^{el} = w_\omega^{mag} = {}^1/_2\, w_\omega,$$
$$\text{a } w_\omega = 2\eta w_{eq.}(1 - \cos\Phi) = 4\eta w_{eq.}\Omega/4\pi,$$

where $\Omega = 2\pi(1 - \cos\Phi)$ is the solid angle of the mirror.

For parabolic and spherical mirrors with a small angular opening ($\Phi \ll 1$) Eqs. (2.20) and (2.21) evidently give the same expressions

$$e_x = h_x = \Phi^2; \quad e_z = h_z = {}^1/_2\Phi^4.$$

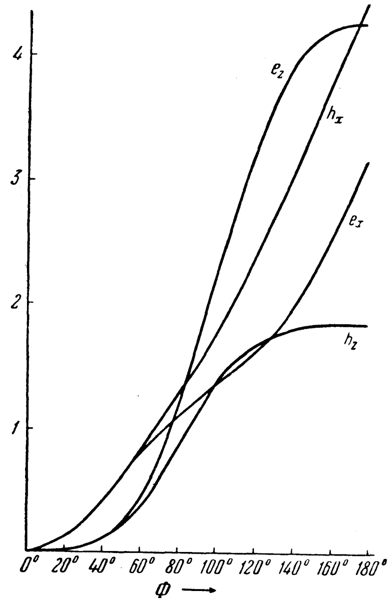

FIG. 7

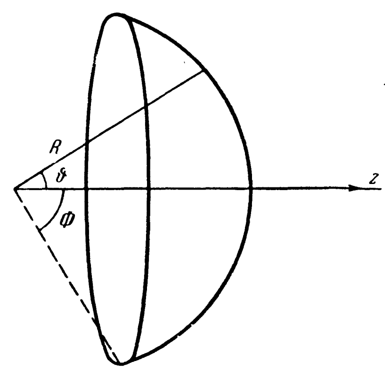

FIG. 8

3. Fluctuating Surface Charges

As is well known, in the solution of high-frequency electrodynamical problems in regions containing good conductors with a strongly expressed skin effect, we may limit ourselves to the consideration of the electromagnetic field only exterior to these conductors. It is only necessary to require that on the surface of good conductors, the external field vectors satisfy the approximate Leontovich boundary conditions[9]. Finding the thermal radiation of such bodies may also be treated (see Ref. 2) as an external electrodynamic boundary value problem: the sources of the fluctuating field are the random surface extraneous fields entering into the Leontovich boundary conditions, introduced into the theory as the equivalent of random volume fields. The external radiation of the body is due to the volume extraneous fields distributed in the skin-layer. Hence, the replacement of volume sources by surface sources corresponds formally to the transition to the limit $\delta \to 0$. Therefore, within the limits of such a "surface" treatment of the fluctuating fields using the inhomogeneous Leontovich conditions, it is natural to introduce the random surface charge densities

$$\sigma = (1/4\pi)\,(E_n)_{sur},$$

which are characterized by the correlation function

$$\overline{\sigma_a\sigma_b^*} = (1/16\pi^2)\,\overline{(E_{an}E_{bn}^*)}_{\text{sur}}, \qquad (2.22)$$

where a and b are points on the surface of the conductor.

Here, we will find this correlation function for distances small in comparison with all dimensions of the body and with the wavelength. Then, clearly, the surface of the body may be replaced by a plane, Eq. (2.22) may be written in the form

$$\overline{\sigma_a\sigma_b^*} = (1/16\pi^2)\,\overline{(E_{az}E_{bz}^*)}_{z=0} \qquad (2.23)$$

and the right hand side of (2.23) may be calculated by a formula of the type (2.15)

$$\overline{E_{az}E_{bz}^*} = \frac{c\Theta\eta}{\pi^2}\int [\mathbf{nH}_{1a}]\,[\mathbf{nH}_{1b}^*]\,dS. \qquad (2.24)$$

Furthermore, in the expressions (2.6) for $H_{1a,b}$ it is possible to leave only quasi-stationary terms

$$\mathbf{H}_{1a} = [\mathbf{n}\rho_a]/cR_a^2; \quad \mathbf{H}_{1b} = [\mathbf{n}\rho_b]/cR_b^2,$$

where we have taken account of the fact that in our case $\mathbf{f}_a = \mathbf{f}_b = \mathbf{n}$.

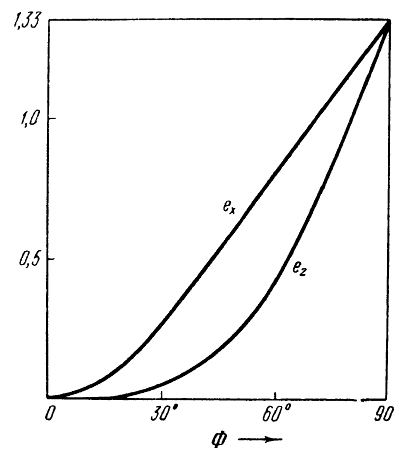

FIG. 9

We will first calculate the right hand side of Eq. (2.24) with z not equal to zero. Let D be the distance between the points a and b at a distance z from the plane; r_a and r_b are the projections of the vectors $\mathbf{R}_a$ and $\mathbf{R}_b$ on the plane, ψ is the angle between these projections (Fig. 10), r and φ are polar coordinates on the plane, the origin of which is midway between the projections of the points a and b. Then $|[\mathbf{n}\times\rho]| = \sin\vartheta = r/R$, so that

$$\overline{E_{az}E_{bz}^*} = \frac{\Theta\eta}{\pi^2 c}\int \frac{r_a r_b\cos\psi}{R_a^3R_b^3}\,dS. \qquad (2.25)$$

Using the theorem concerning the square of a side of a scalene triangle, we easily find

$$r_a r_b \cos\psi = r^2 - D^2/4;$$

$$R_aR_b = [(r^2 + {}^1\!/_4 D^2 + z^2)^2 - r^2D^2\cos^2\varphi]^{1/2},$$

and the calculation of the integral in (2.25) gives

$$\overline{E_{az}E_{bz}^*} = (\Theta\eta/2\pi c)\,z\,(z^2 + D^2/4)^{-3/2}. \qquad (2.26)$$

But

$$\lim_{z\to 0}\frac{z}{(z^2 + D^2/4)^{3/2}} = 8\pi\delta\,(\mathbf{r}_a - \mathbf{r}_b),$$

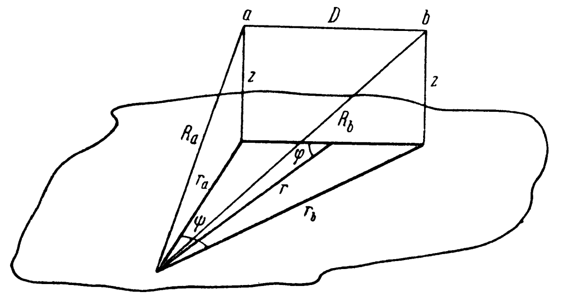

FIG. 10

where $\delta(\mathbf{r}_a - \mathbf{r}_b)$ is the two-dimensional δ-function. Now substituting the limiting expression (2.26) into (2.23), we obtain the correlation function of interest to us

$$\overline{\sigma_a\sigma_b^*} = (\Theta\eta/4\pi^2 c)\,\delta\,(\mathbf{r}_a - \mathbf{r}_b). \qquad (2.27)$$

The fluctuating charge relating to the area of the surface S is $q = \int_S \sigma\,dS$. On the basis of Eq. (2.27), its mean square is

$$\overline{|q|^2} = \int_S\cdot\int_S \overline{\sigma_a\sigma_b^*}\,dS_a dS_b = \frac{\Theta\eta}{4\pi^2 c}\,S$$

or, transforming to the spectral densities of the decomposition according to positive frequencies (the remaining random quantities in this paragraph were spectral densities of the interval $-\infty < \omega < +\infty$), we have

$$q_\omega^2 = (\Theta\eta/2\pi^2 c)\,S.$$

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Translated by D. Lieberman
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ERRATA TO VOLUME 4

	reads	should read
P. 218, column 2, Eq. (10)	$\ldots \xi(\sqrt{3}+2).(2-\sqrt{3})$	$\ldots \xi(\sqrt{2}+2)/(2-\sqrt{3}) \ldots$
P. 219, column 1, Eq. (11)	$\ldots (t\xi)^{\sqrt{3/2}} \ldots$	$\ldots (t\xi)]^{\sqrt{3/2}} \ldots$
P. 219, column 1, Eq. (12)	$y^2 = \rho^{2/3}$	$y^2 - \rho^{2/3} >> 1$
P. 223, column 1, Eq. (45)	$\ldots (E_0 \mu^{3/4})^{\sqrt{3/4}}$	$\ldots (E_0 \mu^{3/4})^{\sqrt{3}/4}$
P. 223, column 2, Eq. (46)	$\ldots \mu^{3\sqrt{3/4}} \ldots$	$\ldots \mu^{3\sqrt{3/4}} \ldots$
P. 225, column 1, 3 lines above Eq. (1.1)	transversality	cross section
P. 225, column 1, 3 lines above Eq. (1.2)	transversality	cross section
P. 256, column 1, Eq. (37)	$\ldots \frac{55\sqrt{3}}{48} \ldots$	$\ldots \frac{55}{\sqrt{3}\,48} \ldots$
P. 289, column 2, Eq. (2)		$I = \sum_n \frac{1}{2n+1} A_n \sum_{\nu=-n}^{n} \frac{1}{1+i\omega\tau} Y_{n\nu}(n_1) Y_{n\nu}(n_2)$
P. 377, column 1, last line	$\delta_{35} = \eta - 21 \times \eta^5$	$\delta_{35} - 21\,\eta^5$
P. 436–7	Figures 2 and 3 should be exchanged.	
P. 449, column 1, last Eq.	$\ldots Y_{lm}\, \varphi \sigma\, \alpha$	$\ldots Y_{lm}\, \varphi_{\sigma}\, \alpha$
P. 449, column 2, Eq. (12)	$\ldots W(l, j, \sigma\, 1;\, j) \ldots$	$\ldots W(l, j, \sigma\, 1;\, \sigma\, j) \ldots$
P. 451, column 1, Eq. (7)	$\ldots D_{\alpha\beta}^{(1)}(p, 0, \lambda'\, \lambda) = \ldots$	$\ldots D_{\alpha\beta}^{(1)}(p, \omega_0, \lambda', \lambda) = \ldots$
P. 541, column 1, Eq. (28)	$M_{++}^{*\,\mathrm{monex}}$	$M_{+}^{*\,\mathrm{monex}}$
P. 543, column 2, Eq. (35)	$\ldots \int_{\rho^2 - \tau^2 + l_0^2}$	$\ldots \int_{\rho^2 < \tau^2 + l_0^2} \ldots$