Investigation of Stability of Electron Motion in Cyclic Accelerators when Quantum Effects are Included*

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The general case of motion of electrons in cyclic accelerators has been investigated, taking account of quantum effects, when axial as well as radial oscillations are possible. We show that the quantum effects can be included by quantizing the adiabtaic invariants according to the Bohr-Sommerfeld method. The effect of quantum fluctuations on radial-phase oscillations in a synchroton is also investigated. Finally, the problem of quantum excitation of macroscopic oscillations is discussed.

1. ADIABATIC INVARIANTS AND EQUILIBRIUM ORBIT

IN the present paper, we want to investigate electron motion in a magnetic field according to quantum theory, and include the possibility that axial as well as radial oscillations can be produced. In this case it is simplest to solve the problem in cylindrical coordinates $r=\sqrt{x^2+y^2}$, z, φ.

We shall first consider the motion of electrons in cylindrical systems like the betatron, where the variation of the magnetic field H over the region around the stationary orbit ($r=R_0$ = const, $z=0$) is given by

$$H = \text{const}\cdot r^{-q}, \qquad (1)$$

and its average value satisfies the Wideroe condition

$$\overline{H}(R_0) = \frac{2}{R_0^2}\int_0^{R_0} rH(r)\,dr = 2H(R_0), \qquad (2)$$

which has been investigated in detail by Terletskii.[5] In addition, the magnetic field H must satisfy the equations:

$$\operatorname{div}\mathbf{H} = 0, \qquad \operatorname{curl}\mathbf{H} = 0. \qquad (3)$$

In order to satisfy conditions (2) and (3), the vector potential is taken in the form

$$A_x = -\tfrac{1}{2}y\overline{H}, \quad A_y = \tfrac{1}{2}x\overline{H}, \qquad (4)$$

and the average value $\overline{H}$ set equal to

$$\tfrac{1}{2}\overline{H}(r, z) = \frac{H(r)}{(2-q)}\sum_{i=0}^{\infty} b_i\left(\frac{z^2}{r^2}\right)^i + \frac{H(r)(1-q)}{(2-q)}\left(\frac{r}{R_0}\right)^{q-2}, \qquad (5)$$

$$b_i = (-1)^i(q-2)^2q^2(q+2)^2\ldots(q+2i-2)^2/(2i)!\,(q-2)(q+2i-2). \qquad (6)$$

$H(r)$ is given by formula (1). The last term on the right of (5) is chosen so that, on the one hand, it does not affect the value of the magnetic field in the region of the stable orbit ($z=0$), since

$$H(r) = \frac{1}{r}\frac{\partial\, \tfrac{1}{2}r^2\overline{H}}{\partial r}, \qquad (7)$$

and on the other hand so that conditions (1) and (2) are satisfied for $r=R_0$ and $z=0$. It will be sufficient to take the first two terms in the expansion of the sum in (5):

$$\tfrac{1}{2}\overline{H}(r, z) \approx \frac{H(r)}{(2-q)}\left[1 + \frac{q(2-q)}{2}\frac{z^2}{r^2}\right] + \frac{H(r)(1-q)}{(2-q)}\left(\frac{r}{R_0}\right)^{q-2}. \qquad (8)$$

In our case, the Lagrangian and the generalized momenta are

$$L = -mc^2[1 - c^{-2}(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2)]^{1/2} - (e/2c)\overline{H}r^2\dot{\varphi}, \qquad (9)$$

$$p_\varphi = (e/c)r^2(H - \tfrac{1}{2}\overline{H}), \qquad (10)$$
$$p_r = (E/c^2)\dot{r}, \qquad p_z = (E/c^2)\dot{z}$$

Here we have made use of the fact that for the stationary orbit

$$\beta E = eHR_0, \qquad (11)$$

and the electron energy is

$$E = mc^2/\sqrt{1-\beta^2}, \qquad (12)$$

where $c\beta$ is the electron velocity and m its rest mass.

* The present paper is a continuation of a series of papers[1-4] by the authors on the quantum theory of the radiating electron, which we shall cite as I-IV.

We shall show that the electron trajectory, including quantum effects, can be found using the Bohr quantum theory. The final results agreee in first approximation with the results of rigorous quantum theory. This method is very convenient for practical computations when the energy is not too high, since it is very simple and descriptive.

We introduce the three adiabatic invariants associated with the azimuthal quantum number $l=0, +1, +2 \ldots$, the radial quantum number $s=0, 1, 2 \ldots$, and the axial quantum number $k=0, 1, 2 \ldots$

$$\oint p_\varphi d\varphi = 2\pi\hbar l, \quad \oint p_r dr = 2\pi\hbar s, \quad \oint p_z d_z = 2\pi\hbar k. \tag{13}$$

According to the Ehrenfest adiabatic principle, the adiabatic invariants remain unchanged during slow variations of the magnetic field. In particular, we see from (10) and (13) that on the stationary orbit $[H(R_0) = \frac{1}{2}\bar{H}(R_0)]$ in the absence of radial and axial oscillations all the adiabatic invariants become zero, i.e., $l=s=k=0$ (the "ideal orbit").

2. OSCILLATION OF THE ELECTRON ABOUT THE INSTANTANEOUS EQUILIBRIUM ORBIT, ACCORDING TO CLASSICAL THEORY

We shall investigate the stability of the motion when the adiabatic invariants are different from zero.

In the relativistic case, the energy is related to the momenta by

$$E^2/c^2 - m^2c^2 = p_r^2 + p_z^2 + V(r, z), \tag{14}$$

where the function $V(r,z) = (\hbar l/r + er\bar{H}/2c)^2$ can be considered to be the expression for some effective potential energy (cf. Ref. 6). The radius R of the instantaneous equilibrium orbit, for which there are no radial oscillations ($s=0$) and no axial oscillations ($k=0$) can be found from the equations

$$\left.\frac{\partial V(r,z)}{\partial r}\right|_{r=R} = \frac{\partial V(r,z)}{\partial z} = 0. \tag{15}$$

We thus obtain

$$z = 0, \quad R = \left[\frac{nc\hbar\,(2-q)}{eH(R_0)\,R_0^q\,(1-q)}\right]^{1/(2-q)} \tag{16}$$

$$= \left[R_0^{2-q} + \frac{lc\hbar\,(2-q)}{eH(R_0)\,R_0^l\,(1-q)}\right]^{1/(2-q)}$$

$$\approx R_0 + \frac{lc\hbar}{eH(R_0)\,R_0\,(1-q)},$$

where the quantum number

$$n = l + l_0, \quad l_0 = R_0^2 eH(R_0)(1-q)/c\hbar(2-q)$$

is not an adiabatic invariant since it follows from (16) that, even when $\hbar l \neq 0$, with increasing magnetic field the radius R of the instantaneous equilibrium orbit approaches the radius R_0 of the stationary orbit ($R \to R_0$).

When $s \neq 0$ and $k \neq 0$, using Eqs. (10) and (14) we get the following approximate equations describing the radial and axial oscillations:

$$\frac{E}{c^2}\frac{d^2}{dt^2}(r-R) + \frac{E}{c^2}\omega_1^2(r-R) = 0, \tag{17}$$

$$\frac{E}{c^2}\frac{d^2}{dt^2}z + \frac{E}{c^2}\omega_2^2 z = 0, \tag{18}$$

where, for sufficiently large values of H, we may set [7]

$$\omega_1 = [(c^4\hbar^2/E^2)(e^2H^2/c^2\hbar^2)(1-q)]^{1/2} \tag{19}$$

$$= \omega_0\sqrt{1-q}, \quad \omega_2 = \omega_0\sqrt{q},$$

where $\omega_0 = c/R$ is the angular velocity of rotation of the electron in a circular orbit. From this is clear that the oscillations will be stable for $0 < q < 1$. We use (13) to determine the amplitudes A and B. We then get

$$A^2 = 2\hbar cs/e\sqrt{1-q}H = A_0^2 H(0)/H, \tag{20}$$

$$B^2 = 2\hbar ck/e\sqrt{q}H = B_0^2 H(0)/H, \tag{21}$$

where $H(0)$ is the magnetic field at $t=0$. It is then clear that as the magnetic field is increased adiabatically, the amplitudes A and B of the radial and axial oscillations gradually decrease. Substituting the solutions of equations (17) and (18) in (14), we find the expression for the energy of the electron

$$E = \left\{e^2H^2(R)\,R^{2q}\left[\frac{nc\hbar\,(2-q)}{eH(R)\,R^q\,(1-q)}\right]^{2(1-q)/(2-q)} + eH(R)\sqrt{1-q}\,2c\hbar s + eH(R)\sqrt{q}\,2c\hbar k + m^2c^4\right\}^{1/2}.$$

Differentiating with respect to the adiabatic invariants, we again get the values of the circular frequencies

$$\omega_0 = \partial E / \partial \hbar l = c / R,$$

$$\omega_1 = \partial E / \partial \hbar s = \sqrt{1-q}\,\omega_0,$$

$$\omega_2 = \partial E / \partial \hbar k = \sqrt{q}\,\omega_0.$$

3. CRITERION FOR APPEARANCE OF QUANTUM EFFECTS

As was shown by Ivanenko and one of the authors [8] (cf. also Reference 9), the spectral angular distribution of the radiation is given, in the classical case, by the expression

$$W_\nu(\theta)\,d\nu \sin\theta d\theta = \frac{e^2\nu^2 c}{3\pi^2 R^2}\left\{\varepsilon^2 K_{2/3}\left(\frac{\nu}{3}\varepsilon^{3/2}\right) + \varepsilon\cos^2\theta K_{1/3}\left(\frac{\nu}{3}\varepsilon^{3/2}\right)\right\} d\nu \sin\theta d\theta. \quad (22)$$

Integrating this expression over angles, we get a formula which quite accurately describes the spectral distribution of intensity of the radiation over the whole frequency range:

$$W_\nu d\nu = \frac{9\sqrt{3}}{8\pi} W_{\rm cl}\, y dy \int_y^\infty dx K_{5/3}(x), \quad (23)$$

where $d\nu=1$, $\epsilon=1-\beta^2 \sin^2\theta$ and the quantity y is proportional to the harmonic number of the radiation: $y = 2/3\,\nu\,(mc^2/E)^3$. Integrating (23) over all frequencies, we get

$$\int_0^\infty W_\nu d\nu = (2e^2 c / 3R^2)(E/mc^2)^4 = W_{\rm cl}.$$

The integral intensity of the radiation, taking quantum corrections into account is

$$W = W_{\rm cl}\{1 - (55\sqrt{3}/16)(\hbar / mcR_0)(E/mc^2)^2\}.$$

(cf. I and II; later this formula was also obtained in Reference 10). From this we see that quantum corrections to the integral intensity will be comparable to the corresponding classical quantities only in the region of very high energies, $E \sim E_{1/2}$, where

$$E_\mu = mc^2(mcR_0/\hbar)^\mu. \quad (24)$$

This condition ($\mu = 1/2$) for the appearance of quantum corrections refers to the quantum number l or n. Despite the limitations on this condition, some authors assume that for $E << E_{1/2}$ we can in general neglect quantum corrections (cf., for example, Ref. 11), and assert that all attempts to set a more stringent condition for quantum corrections than the condition $E \sim E_{1/2}$ will fail. We cannot agree with this viewpoint. For this purpose, let us study the change of the other adiabatic invariants $\hbar s$ and $\hbar k$ as a result of radiation.

The amplitudes of harmonic oscillations can change non-adiabatically if: a) there is a sudden change in the position of the center of oscillation, or b) some momentum is suddenly imparted to the oscillating point. Suppose a material point is carrying out a harmonic oscillation along the x-axis. Then the square of the oscillation amplitude can be found from the relation $D_0^2 = x^2 + p^2 c^4 / E^2\omega^2$, where $p = (E/c^2)\dot{x}$ is the momentum of the particle. The changes in the quantities x and p are given by

$$x = D_0 \sin(\omega t + \varphi),\ p = (E\omega/c^2) D_0 \cos(\omega t + \varphi),$$

where φ is a phase which, in general, depends on the initial conditions.

Let us suppose that at a certain instant of time the center of oscillation is shifted by an amount Δx, or the momentum is changed by an amount Δp. Then the square amplitude of the oscillation will be

$$D^2 = (x-\Delta x)^2 + (p+\Delta p)^2 c^4 / E^2\omega^2 = D_0^2 - 2D_0\sin(\omega t+\varphi)\Delta x + \Delta x^2 + (c^4/E^2\omega^2)[\Delta p^2 + 2\Delta p D_0 (E\omega/c^2)\cos(\omega t + \varphi)].$$

If we assume that the excitation of oscillations occurs statistically independently (cf. Ref. 12)*, then, by averaging the last expression over φ, we get

$$(\Delta D)^2 = (\Delta x)^2 + (c^4/E^2\omega^2)(\Delta p)^2. \quad (25)$$

From formula (16) we see that the radiation of a photon with energy $\hbar\nu\omega_0$ leads to a decrease of the radius of the trajectory by the amount

$$\Delta R = \frac{1}{2-q}\frac{\nu}{n} R = \frac{1}{1-q}\frac{\Delta E}{E} R, \quad (26)$$

where $\Delta E \approx \nu c\hbar/R$. We can get an analogous expression for the change in radius by introducing, on the right side of Eq. (17) which describes the radial oscillations in the classical case, the fluctuation force

$$F_r = \frac{1}{R}\left[\sum_i \mu(t, t_i)\hbar\omega_i - \int_0^t W dt\right]$$

* This result will be established more rigorously later, using quantum mechanics (cf. Section 4).

(t_i is the time of emission, $\mu = 1$ for $t > t_i$, $\mu = 0$ for $t < t_i$). On the other hand, as we see from (20), quantum radial transitions can occur if the square amplitude of the oscillation changes by at least the amount $\Delta(A^2) \sim \hbar c/eH = R^2/n$ ($\Delta s = 1$). Thus we obtain the condition $(\Delta R)^2 > \Delta(A^2)$ or $\nu^2/n \gg 1$ for production of radial oscillations as a result of quantum fluctuations of the radius. Substituting the harmonic number ν corresponding to the maximum of the radiation, $\nu \sim (E/mc^2)^3$, and setting n equal to $n \sim RE/\hbar c$, we find the condition for appearance of radial quantum oscillations to be $E > E_{1/5}$, where $E_{1/5}$ is given by formula (24) with $\mu = 1/5$. This condition was already found in the first papers on the quantum theory of the radiating electron[13] (cf. also I and Ref. 14).

It is just as easy to get the condition for appearance of axial quantum oscillations. Axial oscillations are produced as a result of the recoil of the electron when it emits a photon. When a photon is emitted, the z component of the momentum changes by an amount $\Delta p_z = (\hbar\omega_0/c)\nu\cos\theta$. According to (25), the increase in the amplitude of axial oscillation due to emission of one quantum is

$$(\Delta B_1)^2 = (R^2/q)(\Delta E/E)^2\cos^2\theta \sim R^2\nu^{4/3}/n^2. \quad (27)$$

We note that (27) can also be obtained from the classical equation (18) for the axial oscillations, by substituting on the right the corresponding fluctuation force

$$F_z = \sum_i \Delta p_z \delta(t - t_i).$$

On the other hand, for a minimal change of the adiabatic invariant ($\Delta k = 1$), the increase of the square amplitude is equal, according to (21) to

$$\Delta(B^2) \sim \hbar c/eH = R^2/n. \quad (28)$$

From (27) and (28) we see that quantum corrections for the axial oscillations must be considered when $(\Delta B_1)^2 > \Delta(B^2)$, i.e. $\nu^{4/3}/n > 1$ or $E > E_{1/3}$, where $E_{1/3}$ is given by (24) with $\mu = 1/3$. This condition was given in the first edition of the monograph of Ivanenko and one of the authors.[15]

4. DEVELOPMENT OF RADIAL AND AXIAL OSCILLATIONS ACCORDING TO QUANTUM THEORY

According to the criteria which we have established in the preceding paragraph, quantum effects should first influence the radial oscillations (when $E \sim E_{1/5}$), then the axial oscillations (when $E \sim E_{1/3}$), and will only affect the total intensity of the radiation in the region of very high energies (when $E \sim E_{1/2}$).

Quantum excitation of radial oscillations should begin to play an important role in the energy region $E > E_{1/5}$, since according to the classical theory the amplitude of oscillation then tends to zero.

As we see from formulas (20), (21), (26) and (27), when a photon is radiated the quantum numbers n, s, and k change by the amounts

$$\Delta l = l' - l = -\nu, \quad (29)$$

$$\Delta s = \frac{1}{2(1-q)^{3/2}}\frac{\nu^2 c\hbar}{R^2 eH(R)},$$

$$\Delta k = \frac{1}{2\sqrt{q}}\frac{\nu^2 c\hbar}{R^2 eH(R)}\cos^2\theta,$$

But according to classical theory, the quantum numbers s and k should remain constant. The expressions (29) were obtained by a semiclassical method (since in considering the radiation of an individual photon, we averaged over the phase φ). We shall show that a more rigorous quantum treatment will lead us to the same expression (29). Actually we can treat the oscillations along r and z as oscillations at non-relativistic velocities, but with the relativistic mass E/c^2, and having frequencies $\omega_1 = \sqrt{1-q}\,c/R$ and $\omega_2 = \sqrt{q}\,c/R$ respectively.[1] Thus the wave functions will have the forms

$$\Psi_s = \left(\frac{\lambda_1}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^s s!}}e^{-\lambda_1(r-R)^2/2}H_s[\sqrt{\lambda_1}(r-R)],$$

$$\Psi_k = \left(\frac{\lambda_2}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^k k!}}e^{-\lambda_2 z^2/2}H_k(\sqrt{\lambda_2}z),$$

$$\lambda_1 = \omega_1 E/\hbar c^2 = \sqrt{1-q}\,eH(R)/c\hbar,$$

$$\lambda_2 = \omega_2 E/\hbar c^2 = \sqrt{q}\,eH(R)/c\hbar,$$

while the radius of the instantaneous equilibrium orbit $R = R(l)$ is given by equation (16). Let us find the probability of transition $W_{ss'}$ of an electron from a state s to state s', with a change of azimuthal quantum number from n to $n' = n - \nu$. The transition probability is equal to the square of the matrix element

$$W_{ss'} = \left| \left(\frac{\lambda_1}{\pi 2^{s+s'} s!\, s'!} \right)^{1/2} \int_0^\infty dr \exp\left\{ -\frac{\lambda_1}{2} [r - R(n)]^2 \right.\right.$$

$$\left.\left. - \frac{\lambda_1}{2} [r - R(n-\nu)]^2 \right\} H_s [\sqrt{\lambda_1}(r - R(n))]\, H_{s'} [\sqrt{\lambda_1}(r - R(n-\nu))] \right|^2$$

$$= \left| \frac{e^{-\xi_1/2}}{\sqrt{\pi 2^{s+s'} s!\, s'!}} \int_{-\infty}^{\infty} dx e^{-x^2} H_s\left(x - \sqrt{\frac{\xi_1}{2}}\right) H_{s'}\left(x + \sqrt{\frac{\xi_1}{2}}\right) \right|^2 = I^2_{ss'}(\xi_1),$$

$$\xi_1 = {}^1/_2 \lambda_1 [R(n) - R(n-\nu)]^2 = \nu^2 / 2n(2-q)\sqrt{1-q}.$$

The functions $I_{ss'}(\xi_1)$ are related to the Laguerre polynomi als $Q_s^{s'-s}(\xi_1)$:

$$I_{s's}(\xi) = \frac{\xi^{(s'-s)/2} e^{-\xi/2}}{\sqrt{s!\, s'!}} Q_s^{s'-s}(\xi). \qquad (30)$$

Thus, in a single transition, the change of the quantum number s is

$$\Delta s = \sum_{s'=0}^{\infty} (s'-s)\, I^2_{ss'}(\xi_1) \qquad (31)$$

$$= \xi_1 = c\nu^2\hbar / 2(1-q)^{3/2} eHR^2.$$

Earlier we obtained this last expression by a semiclassical method [cf. Eq. (29)].

By a similar method we get the transition probability $W_{kk'}$ from a state k to state k', when the z component of the photon momentum is

$$\Delta p_z = \hbar(\omega_0/c)\nu\cos\theta;$$

$$W_{kk'} = \left| \left(\frac{\lambda_2}{\pi 2^{k+k'} k!\, k'!} \right)^{1/2} \int_{-\infty}^{\infty} dz \right.$$

$$\times \exp\left\{ -\lambda_2 z^2 - iz \frac{\omega_0}{c} \nu\cos\theta \right\}$$

$$\left. \times H_k(\sqrt{\lambda_2} z) H_{k'}(\sqrt{\lambda_2} z) \right|^2$$

or

$$W_{kk'} = I^2_{kk'}(\xi_2),$$

where $\xi_2 = \omega_0^2 \nu^2 \cos^2\theta / 2c^2\lambda_2$, and the functions $I_{kk'}(\xi_2)$ are given by (30). When a photon is emitted, as a result of the recoil the adiabatic invariant k which measures the amplitude of the axial oscillations changes by an amount

$$\Delta k = \sum_{k'=0}^{\infty} (k'-k)\, I^2_{kk'}(\xi_2) \qquad (32)$$

$$= \xi_2 = \nu^2 c\hbar \cos^2\theta / 2\sqrt{q} eHR^2,$$

which coincides precisely with the corresponding expression (29) which we found by a semiclassical method. Thus the average over phases in the semiclassical method gives the quantum result, which justifies the general theorem of statistical independence of emission of photons in the quantum case.

As we see from formulas (31) and (32), the change of the adiabatic invariants s and k per unit time as a result of quantum transitions will be

$$\frac{dl}{dt} = -\int_0^\infty d\nu \int_0^\pi \sin\theta d\theta \frac{\nu W_\nu(\theta)}{\hbar\nu\omega_0},$$

$$\frac{ds}{dt} = \int_0^\infty d\nu \int_0^\pi \sin\theta d\theta \frac{c\nu^2\hbar}{2(1-q)^{3/2} eHR^2} \frac{W_\nu(\theta)}{\hbar\nu\omega_0}, \qquad (33)$$

$$\frac{dk}{dt} = \int_0^\infty d\nu \int_0^\pi \sin\theta d\theta \frac{c\nu^2\hbar\cos^2\theta}{2\sqrt{q} eHR^2} \frac{W_\nu(\theta)}{\hbar\nu\omega_0},$$

where the quantity $W_\nu(\theta)$ is given by (22). Introducing the variable

$$x = (\nu/3)\varepsilon^{3/2} = (\nu/3)(1 - \beta^2\sin^2\theta)^{3/2}$$

and carrying out the integrations (cf. II), we obtain expressions giving the total change of the radius and also of the amplitudes of radial and free axial oscillations:

$$R^{2-q} = R_0^{2-q} - \frac{2}{3}\frac{e^2 c}{eH}\frac{2-q}{1-q}\int_0^t \left(\frac{E}{mc^2}\right)^4 \frac{1}{R} dt, \qquad (34)$$

$$A^2 = A_0^2 \frac{H(0)}{H(t)} \qquad (35)$$

$$+ \frac{55}{24\sqrt{3}} \frac{e^2\hbar R}{(1-q)^2 mE} \int_0^t \left(\frac{E}{mc^2}\right)^6 \frac{1}{R^2} dt,$$

$$B^2 = B_0^2 \frac{H(0)}{H(t)} + \frac{13}{24\sqrt{3}} \frac{e^2\hbar R}{qmE} \int_0^t \left(\frac{E}{mc^2}\right)^4 \frac{1}{R^2} dt. \qquad (36)$$

The first term in Eqs. (35) and (36) corresponds to the classical contraction and the second to the quantum broadening. Formula (35) for the radial oscillations was obtained by us earlier, both for a uniform field (cf. III, $q = 0$)*, and for a field having axial symmetry (cf. IV, $q \neq 0$). In the present paper we have generalized this formula to the case where axial oscillations are also possible, and have also determined the influence of quantum effects on the magnitude of the axial oscillations. From the last formulas we see that quantum effects should have a very strong influence only on the free radial oscillations.

As we see from formulas (20) and (35), because of the quantum character of the radiation the percentage increase of the mean square radial deviation from the instantaneous equilibrium orbit $\overline{\Delta R^2_{qu}} = \overline{x^2} = 1/2A^2$ will vary according to the law (cf. also III and IV)

$$\left(\frac{\Delta R_{\rm qu.}}{R}\right)^2 = \frac{55\sqrt{3}}{48}\frac{e^2\hbar c}{(1-q)^2 mER}\int_0^t\left(\frac{E}{mc^2}\right)^6\frac{1}{R^2}dt. \quad (37)$$

5. RADIAL-PHASE OSCILLATIONS IN A SYNCHROTRON WHEN QUANTUM EFFECTS ARE INCLUDED

In the preceding paragraphs we have shown that fluctuations of the radius, which are associated with the quantum character of the emission, lead to excitation of a special type of radial oscillations. Therefore, for the motion of an electron in a synchrotron, we must still estimate the effect of these fluctuations on the so-called radial-phase oscillations.

In a synchrotron, the increase in energy of the electron results from passage of the electron through an accelerating gap. In a single turn the energy increases by the amount $eV\sin\varphi$ where V is the amplitude of the voltage across the accelerating gap, and φ is the phase at which the electron crosses the gap. On the other hand, in one turn the electron loses the energy $I = (4\pi e^2/3R)(E/mc^2)^4$ as a result of radiation. Thus the average increase per unit time is

$$dE/dt = \omega(eV\sin\varphi - I)/2\pi, \quad (38)$$

where ω is the angular velocity of rotation of the electron. We shall denote quantities referring to the instantaneous equilibrium orbit by the index s. These quantities are related by the equations

$$E_s = eH_sR_s, \quad (\omega_s/2\pi)\Delta E_s = (ec/\omega_s)\dot{H}_s.$$

The increase in energy of the electron will be given by

$$(\omega_s/2\pi)\Delta E_s = dE_s/dt = \omega_s(eV\sin\varphi_s - I_s)/2\pi. \quad (39)$$

We note that in the extreme relativistic region the energy loss per turn reaches values far exceeding the average energy given to the electron per turn, i.e.,

$$\Delta E_s \ll eV\sin\varphi_s \approx I_s. \quad (40)$$

Denoting the deviation from the equilibrium phase by $\psi = \varphi - \varphi_s$, we have

$$\omega - \omega_s = \dot{\psi}, \qquad \Delta R/R_s = \Delta E/(1-q)E_s = -\dot{\psi}/\omega_s, \quad (41)$$

where $\Delta R = R - R_s$, $\Delta E = E - E_s$. Subtracting (39) from (38), we get

$$\frac{d}{dt}\Delta E = \frac{eV\cos\varphi_s}{2\pi}\omega_s\psi - \frac{I - I_s}{2\pi}\omega_s + \frac{\Delta E_s}{2\pi}\dot{\psi}. \quad (42)$$

Using the fact that in the ultrarelativistic case, at a given time, $E = eHR = \text{const}\cdot R^{1-q}$, i.e., $I = \text{const}\cdot R^{3-4q}$, we find

$$I - I_s = -(3-4q)I_s\dot{\psi}/\omega_s.$$

In addition, using (40) we can drop the last term on the right of (42) and set $eV\cos\varphi_s \approx I_s\cot\varphi_s$. Then we get the equation for the radial-phase oscillations

$$\ddot{\psi} + \gamma\dot{\psi} + \Omega^2\psi = 0. \quad (43)$$

The damping coefficient γ and the angular frequency of the radial-phase oscillations, Ω, are equal respectively to

$$\gamma = (W_s/E_s)(3-4q)/(1-q), \qquad \Omega^2 = \omega_s W_s \operatorname{ctg}\varphi_s/(1-q)E_s,$$

where the quantity $W_s = I_s(c/2\pi R_s)$ is the energy loss through radiation, per unit time, for an electron on the stationary orbit.

From this it is clear that the radial-phase oscillations will be damped for $q < 3/4$; if $q > 3/4$ the motion will be unstable. From equations (41) and (43) we find the following differential equation

* For a uniform field, Eq. (35) was obtained rigorously in III, using the Dirac equation.

for ΔR:

$$d^2\Delta R / dt^2 + \gamma d\Delta R / dt + \Omega^2 \Delta R = 0. \quad (44)$$

The solution of this equation has the form:

$$\Delta R = e^{-\gamma t/2} \cos \Omega t \Delta A. \quad (45)$$

For high energies, the quantity ΔA will be the oscillation amplitude which results from the quantum character of the radiation. We then get for the mean square percentage shift in radius the value:

$$(\Delta R / R_s)^2 = {}^1/_2 (\Delta A / R_s)^2 e^{-\gamma t}. \quad (46)$$

Suppose that the radiation occurs at some time t_i. Then we get for the square of the ratio $\Delta R/R_s$ the value

$$\left(\frac{\Delta R}{R_s}\right)^2 = \frac{1}{(1-q)^2} \frac{(\Delta E_\gamma)^2}{E_s^2} = \frac{1}{2} \sum_\nu e^{-\gamma(t-t_i)} W_\nu dt, \quad (47)$$

where $\Delta E_\gamma = \hbar \nu c/R_s$, while W_ν is given by (23). Summing the right side of (47) over ν, and assuming that during the damping time $\tau = 1/\gamma$ the energy E_s can be taken to be constant, we get the equation found by Sands[16],

$$\left(\frac{\Delta R}{R}\right)^2 = \frac{55}{48\sqrt{3}} \frac{1}{(1-q)^2} \frac{\hbar}{mcK_s^3} \frac{e^2}{mc} \left(\frac{E_s}{mc^2}\right)^5 \tau. \quad (48)$$

In this case, Eq. (37) should be used for $t < \tau$, and its limiting form [cf. Eq. (48)] for $t >> \tau$.

We note that relation (48) can be obtained from our more general formula (37) [cf. also III and IV] if we limit the duration of the free betatron oscillations to the damping time τ.

In conclusion, we note that the amplitudes of the radial oscillation and radial momentum will be of order

$$A \sim \sqrt{c\hbar s/eH}, \quad (49)$$

$$P \sim (E\omega/c^2)A \approx \sqrt{eH\hbar s/c}.$$

If the quantum number s changes by unity as the result of emission of radiation, we get an uncertainty in these amplitudes equal to

$$\Delta A \approx (\partial A/\partial s)\Delta s \sim A/s, \quad \Delta P \sim P/s, \quad (50)$$

since it is impossible by present theory to determine when the emission occurred. In the classical approximation $E << E_{1/5}$, the value of s does not change, so that this indeterminacy cannot occur. But in the quantum case $E >> E_{1/5}$, a change of the quantum number s will occur several times in the course of a single period of oscillation. The change in s leads to indeterminacies in the coordinates, Δx and Δp, which cannot be given by (50), since for $s >> 1$ we would obtain the inequality $\Delta x \Delta p \sim \hbar/s << \hbar$, which contradicts the uncertainty relation $\Delta x \Delta p \sim \hbar$. Thus we have two limiting cases giving uncertainties in coordinate and momentum,

$$\Delta x \sim \Delta A \sim A/s, \quad \Delta p \sim \hbar/\Delta x \sim P$$

$$\Delta p \sim \Delta P \sim P/s, \quad \Delta x \sim \hbar/\Delta p \sim A$$

Neither of the two cases permits a classical approximation. To investigate the quantum excitation of macroscopic radial oscillations, forming a sort of "macro-atom", we can use either the rigorous methods of quantum theory as was done in Section 4, or the classical equations (17) and introduce fluctuation forces (cf. Section 3).

In the equivalence of the two methods we are inclined to see a connection between quantum methods and the theory of fluctuations, where there are Markoff processes whose basis is the statistical independence of successive processes (cf., for example, the theory of the Brownian motion, and also Welton's theory which qualitatively explains the Lamb shift by considering fluctuations of the virtual photons). In any case, in the present example both theories actually give the same quantitative results.

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ERRATA TO VOLUME 4

	reads	should read
P. 218, column 2, Eq. (10)	$\ldots \xi(\sqrt{3}+2)(2-\sqrt{3})$	$\ldots \xi(\sqrt{2}+2)/(2-\sqrt{3})\ldots$
P. 219, column 1, Eq. (11)	$\ldots (t\xi)^{\sqrt{3/2}}\ldots$	$\ldots (t\xi)]^{\sqrt{3/2}}\ldots$
P. 219, column 1, Eq. (12)	$y^2 = \rho^{2/3}$	$y^2 - \rho^{2/3} \gg 1$
P. 223, column 1, Eq. (45)	$\ldots (E_0\mu^{3/4})^{\sqrt{3/4}}$	$\ldots (E_0\,\mu^{3/4})^{\sqrt{3}/4}$
P. 223, column 2, Eq. (46)	$\ldots \mu^{3\sqrt{3/4}}\ldots$	$\ldots \mu^{3\sqrt{3/4}}\ldots$
P. 225, column 1, 3 lines above Eq. (1.1)	transversality	cross section
P. 225, column 1, 3 lines above Eq. (1.2)	transversality	cross section
P. 256, column 1, Eq. (37)	$\ldots \frac{55\sqrt{3}}{48}\ldots$	$\ldots \frac{55}{\sqrt{3}\,48}\ldots$
P. 289, column 2, Eq. (2)		$I = \sum_n \frac{1}{2n+1} A_n \sum_{\nu=-n}^{n} \frac{1}{1+i\omega\tau} Y_{n\nu}(n_1) Y_{n\nu}(n_2)$
P. 377, column 1, last line	$\delta_{35} = \eta - 21 \times \eta^5$	$\delta_{35} - 21\,\eta^5$
P. 436–7	Figures 2 and 3 should be exchanged.	
P. 449, column 1, last Eq.	$\ldots Y_{lm}\,\varphi\sigma\,\alpha$	$\ldots Y_{lm}\,\varphi_\sigma\,\alpha$
P. 449, column 2, Eq. (12)	$\ldots W(l, j, \sigma 1;\ j)\ldots$	$\ldots W(l, j, \sigma 1; \sigma j)\ldots$
P. 451, column 1, Eq. (7)	$\ldots D_{\alpha\beta}^{(1)}(p, 0, \lambda'\lambda) = \ldots$	$\ldots D_{\alpha\beta}^{(1)}(p, \omega_0, \lambda', \lambda) = \ldots$
P. 541, column 1, Eq. (28)	$M_{++}^{*\,\mathrm{monex}}$	$M_{+}^{*\,\mathrm{monex}}$
P. 543, column 2, Eq. (35)	$\ldots \int_{\rho^2 - \tau^2 + l_0^2}$	$\ldots \int_{\rho^2 < \tau^2 + l_0^2} \ldots$