Charge Renormalization for an Arbitrary, Not Necessarily Small, Value of e_0

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RECENTLY Taylor[1], combining the results of Landau, Abrikosov and Khalatnikov[2] with the general theory of Gell-Mann and Low[3], attempted to prove in general that the only case in which the present electrodynamics does not lead to a contradiction is that with the renormalized charge e_c equal to zero (since otherwise the "seed" charge e_0 turns out to be imaginary and the interaction operator is non-Hermitian). In accordance with Ref. 3, Taylor writes the quantity $e_c^2 d_c(e_c^2, \xi)$ [d_c is the renormalized propagation function of the photon, and $\xi = \ln(-k^2/m^2)$] in the form of some function of only one variable*:

$$e_c^2 d_c = \Phi(\lambda_c), \ \lambda_c(\xi) = e_c^2 \Big/ \Big[1 - \frac{e_c^2}{3\pi}\xi - e_c^2 f_c(e_c^2)\Big], \quad (1)$$

where $f_c(e_c^2)$ is some unknown function. Comparing Eq. (1) with the results of Ref. 2, Taylor shows that $\lim_{y\to 0} y f_c(y) = 0$, and

$$\Phi(\lambda_c) \approx \lambda_c, \quad \text{if} \quad \lambda_c \to +0. \quad (2)$$

With $\xi = L$, $L = \ln(\Lambda^2/m^2)$, where Λ is the cut-off limit for momentum, Eq. (1) determines the charge renormalization

$$e_0^2 = e_0^2 d(L) = e_c^2 d_c(L) = \Phi[\lambda_c(L)]. \quad (1')$$

If in Eq. (1) e_c^2 is regarded as an arbitrary fixed quantity, then for $L \to \infty$, $\lambda_c \to -3\pi/L$. Taylor obtains the result stated above by assuming that (2) holds also for $\lambda_c \to -0$:

$$\Phi(\lambda_c) \approx \lambda_c, \ \lambda_c \to -0, \quad (2')$$

i.e., that $\lambda_c \Phi(\lambda_c)$ is a function of λ_c continuous at zero. Indeed, for $L \to \infty$, $\lambda_c \approx -3\pi/L \to -0$, it follows from (1′) and (2′) that

$$e_0^2 \approx -3\pi L, \quad (3)$$

i.e., e_0 turns out to be an imaginary quantity.

Unfortunately, it is so far quite impossible to find any basis for the assumption on the continuity of the function $\lambda_c \Phi(\lambda_c)$ at the point $\lambda_c = 0$, so that Taylor's whole proof remains without foundation. Even the reverse appears more probable: that the function $\Phi(\lambda)$ has an essential singularity at $\lambda = 0$, for example, of the type $\exp(1/\lambda)$; this would correspond to the fact that all expansions in powers of e_c^2 are apparently asymptotic series. In any case one can display many functions $\Phi(\lambda)$ for which the condition (2) is fulfilled (i.e., the relation (1) of Gell-Mann and Low goes over for $e_c^2 \to 0$ into the formula $d_c = [1 - (e_c^2 \xi/3\pi)]^{-1}$ of Landau, Abrikosov and Khalatnikov), but (2′) and consequently also (3), are invalid**.

If, however, we consider all quantities before renormalization and confine ourselves to a simpler problem than the one attacked by Taylor, the whole discussion can be carried through quite rigorously. In fact, we assume that the "seed" coupling constant e_0^2 is an arbitrary fixed quantity. Moreover, $e_c^2 = e_c^2(e_0^2, L)$. We shall show that under these conditions $e_c^2 \to 0$ for $L \to \infty$ not only for $e_0^2 \ll 1$, but for arbitrary $e_0 \gtrsim 1$.

Indeed, according to Gell-Mann and Low[3], we have***

$$e_0^2 d(e_0^2, L - \xi) = \Phi(\lambda_0), \ \lambda_0(\xi)$$
$$= e_0^2 \Big/ \Big[1 + \frac{e_0^2}{3\pi}(L - \xi) - e_0^2 f_0(e_0^2)\Big], \quad (4)$$

where $\Phi(\lambda)$ is the same function as in (1), and $f_0(e_0^2)$ is some unknown function. For $e_0 \to 0$ this equation goes over into the relation

$$d = \Big[1 + \frac{e_0^2}{3\pi}(L - \xi)\Big]^{-1} \quad (5)$$

of Ref. 2 only if

$$\lim_{y \to 0} [y f_0(y)] = 0 \quad (6)$$

and, moreover, the relation (2) must be satisfied. [The condition (6) must be fulfilled in any case: otherwise, for $e_0^2 \to 0$, Φ would depend on a quantity different from $e_0^2[1 + (e_0^2/3\pi)(L - \xi)]^{-1}$, and (5) would be in contradiction with (4)]. It must be noted that Eqs. (2) and (6) are not independent conditions; one follows from the other. In fact, the functions $f_0(y)$ and $\Phi(y)$ can be expressed in terms of each other, since they are connected by the condition

$$d(e_0^2, 0) = 1, \quad (\xi = L),$$

which, by Eq. (4) gives

$$e_0^2 = \Phi\left[\frac{e_0^2}{1 - e_0^2 f_0(e_0^2)}\right] = \Phi\left[\frac{1}{e_0^{-2} - f_0(e_0^2)}\right]$$

or

$$f_0(y) = 1/y - [\Phi^{-1}(y)]^{-1}, \tag{7}$$

where $\Phi^{-1}(y)$ is the function inverse to $\Phi(y)$. If (6) is satisfied, it follows that $\Phi^{-1}(y) \to y$ for $y \to 0$. But the function inverse to $\Phi^{-1}(y) \approx y$ will be $\Phi(y) \approx y$; i.e., we get Eq. (2). In an analogous way, Eq. (6) follows at once from (2) and (7).

From Eqs. (4) and (2) it follows at once that $e_c^2 \to 0$ for $L \to \infty$. Indeed, let us suppose that e_0^2 has a fixed (and arbitrary) value, and $L - \xi \to \infty$. Then, by Eq. (4), $\lambda_0(\xi) \to 3\pi/(L - \xi)$, i.e., according to Eq. (2),

$$e_0^2 d(e_0^2, L - \xi) \approx 3\pi/(L - \xi)$$

with increasing accuracy as $L - \xi$ is made larger. Since $e_c^2 d_c \equiv e_0^2 d$,

$$e_c^2 d_c(e_c^2, \xi) \approx 3\pi/(L - \xi), \quad L - \xi \to \infty.$$

For $\xi \to 0$, when $d_c \approx 1$, this equation gives

$$e_c^2 \approx 3\pi/L \to 0, \quad L \to \infty,$$

which was to be proved.

We note that if in Eq. (4) we regard e_0^2 as dependent on L, (as Taylor indeed assumed), the proof does not go through, since as L increases the quantity $e_0^2(L) f_0[e_0^2(L)]$ in Eq. (4) can change in such a way that λ_0 will not decrease, and will in general not be small for $L \to \infty$.

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* Cf. Ref. 3, Eq. (5.6). Account is taken of the relation $(-k^2/m^2)\Phi(e_c^2) = \exp(-3\pi/\lambda_c)$ if the function f_c of Eq. (1) is related to the function φ of Ref. 3 in the following way: $f_c = (1/e_c^2) + \ln\varphi(e_c^2)$ (the quantities k^2 and e^2 of Gell-Mann and Low are here denoted by $-k^2$ and e_c^2).

** For example, as Landau has remarked, for the function $[\ln(1 + e^{1/\lambda})]^{-1}$ the relation (2) holds, but the condition (2′) does not: for $\lambda \to -0$

$$[\ln(1 + e^{1/\lambda})]^{-1} \approx e^{1/|\lambda|}.$$

*** Cf. Eq. (B.11) of Ref. 3; note that

$$\frac{-k^2}{\lambda^2} G(e_0^2) = \exp(-3\pi/\lambda_0),$$

if in Eq. (4) $f_0(e_0^2) = e_0^{-2} + \ln G(e_0^2)$. Therefore,

$$F\left[\frac{-k^2}{\lambda^2} G(e_0^2)\right] \equiv \Phi(\lambda_0),$$

if $\Phi(y)$ is determined by the function F of Ref. 3 by the equation $\Phi(y) = F(e^{-3\pi/y})$.

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[3] M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

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A Polarization Method for Measuring the Velocities of Particles with Intrinsic Magnetic Moment

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MEASUREMENT of the velocities of particles in a beam (spectrometry) is a typical problem of physical experiment. A polarization method can be used for particles which possess an intrinsic magnetic moment. The spectrometer resembles the type which is used to determine the magnetic moments of individual particles. In the path of the beam there is placed a polarizer, a device for rotating the plane of polarization, an analyzer and, finally, a particle detector. The device for spin rotation can be constructed in such a way as to change the orientation only for particles which possess a given energy. The analyzer removes the remaining particles. The detector readings correspond to the number of particles of the given energy in the beam spectrum.

For neutral atoms with spin it is technically