However, we can choose the condition that violates causality in some fashion which does not impose it in the interval. Here, generally speaking, one chould expect that the dispersion relations will no longer be maintained.

We have therefore come to the following conclusion: if the experimental data are in contradiction with the dispersion relations, then this will mean that at small distances, the propagation of signals with velocities exceeding that of light can go on; at the same time, in accord with experimental data with dispersion relations, we cannot exclude the violation of causality at small distances, in particular the propagation of the reaction between two points lying not inside the light cone but inside the hyperboloid appears to be possible.

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On a Singularity of the Field of an Electromagnetic Wave Propagated in an Inhomogeneous Plasma

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The effect of field growth is investigated in a region where the plasma dielectric constant becomes zero. The problem of the absorption influence is fully explained. The relationship between this effect and plasma resonance is established.

IN solving the problem of propagation of electromagnetic waves in an inhomogeneous plano-stratified medium the simplest case is that of normal incidence. Under the conditions of complete reflection it is most convenient to use the linear approximation of the dielectric constant $\epsilon(z)$ in the neighborhood of its zero (point of reflection). In fact, the consideration of this simplest case enables one to explain completely the field of a standing wave in the reflection region (see Ref. 1, Sec. 66). An analogous situation occurs for oblique incidence.

Zhekulin[2] carried out a detailed investigation of solutions describing the oblique incidence of radio waves on a plano-stratified isotropic ionosphere. In such a medium waves with different polarizations of the electric vector E (perpendicular and parallel to the incidence plane) are propagated independently of each other. It turns out that the reflection problem of the wave, with an electric vector perpendicular to the incidence plane, does not differ in principle from the well known case of normal incidence. They differ only in the displacement of the incident wave reflection level. However, the equation describing the wave with a different polarization of the electric vector is of a more specific type; in this case, the point at which the dielectric constant of the medium ϵ, $\epsilon(z)$ becomes zero is a singularity. Zhekulin showed that the condition for the solution to become zero at infinity (in the region of negative values for ϵ) is not compatible with the requirement that the field be finite at the zero of ϵ. In trying to remove this solution singularity, the author replaces (without any basis) the function $\epsilon(z)$, which changes sign at a definite point, by a positive function which does not become zero anywhere.

Forsterling and Wuster[3,4] discussed in greater detail the singularities of the field for oblique incidence of the wave. From the analysis of approximate solutions, valid in a relatively small neighborhood of the zero of the function ϵ, it was established that the field component E_z becomes infinite as $1/\epsilon$ at this point, while the component E_y has a logarithmic singularity. The abrupt growth of field strength in the region where ϵ takes on infinitesimally small values leads to the conclusion that the field description with the aid of the usual dielectric constant becomes impossible. This is clear from the fact that the motion of the electrons cannot be harmonic under the influence of a field with an abrupt space innomogeneity. This case can be accounted for by the fact that the equations describing the field in the medium become nonlinear, and, during the propagation of a wave with a definite frequency, there waves arise in such a medium of other frequencies (higher harmonics).[4]

However, in the above mentioned papers, there remained unexplained the problems of the amplitude in a growing field within an absorbing medium, and the physical nature of this singularity in a medium without absorption.

This paper is devoted to a detailed discussion of these problems.

1. SINGULARITIES OF THE ELECTROMAGNETIC WAVE FIELD WITH OBLIQUE INCIDENCE ON A LINEAR LAYER

Let us consider the problem of the electromagnetic wave incidence on an inhomogeneous medium, the properties of which are a function of a single coordinate z only. We shall assume that the normal to the wave front of the plane wave lies in the y,z plane. In such a case the equation of interest to us, describing the wave with its components H_x, E_y E_z, is written in the form (see, for instance, Refs. 1,2)

$$\frac{d^2w}{dz^2}-\frac{1}{\varepsilon'(z)}\frac{d\varepsilon'(z)}{dz}\frac{dw}{dz}+k_0^2(\varepsilon-q^2)w=0. \quad (1)$$

The function $w(z)$ is related to the H_x component by the relationship

$$H_x=w(z)\,e^{i(\omega t+k_0 qy)}. \quad (2)$$

The remaining field components are determined from the equations

$$E_y=\frac{1}{i'k_0\varepsilon'(z)}\frac{\partial H_x}{\partial z},\quad E_z=-\frac{1}{ik_0\varepsilon'(z)}\frac{\partial H_x}{\partial y}. \quad (3)$$

In Eq. (1) and in formulas (2), (3) there are also found the following symbols: $k_0=\omega/c$ is the wave number in vacuum; $q=\sin\theta_0$, where θ_0 is the value of the incidence angle at the boundary of the inhomogeneous layer. The dielectric constant of the ionized gas $\epsilon^1(z)$ is expressed by the formula (see Ref. 1, Sec. 57)

$$\varepsilon'(z)=1-4\pi e^2N(z)/m\omega^2(1-i\nu_{\rm eff}/\omega),$$

where e,m is the charge and mass of an electron, N is electron density, $\nu_{\rm eff}$ is effective number of electron collisions with neutral molecules. Assuming a small absorption, that is $\nu_{\rm eff}/\omega \ll 1$, one can write the approximation

$$\varepsilon'(z)=\varepsilon(z)-(i\omega_0^2/\omega^2)\,\nu_{\rm eff}/\omega.$$

Here $\omega_0^2=4\pi e^2N/m$ is the natural frequency of the plasma oscillations. Furthermore, by assuming that the absorption varies little with altitude, one can take the imagniary part of $\epsilon^1(z)$ as equal to its constant value at the point where $\omega=\omega_0$. In such a case

$$\varepsilon'(z)=\varepsilon(z)-i\nu_{\rm eff}/\omega,$$

and for the linear dependence $\epsilon(z)=1-\omega_0^2$

$$\varepsilon'(z)=-\alpha z-i\nu,\qquad \nu=\nu_{\rm eff}/\omega. \quad (4)$$

The origin of coordinates is chosen so that with $z>0$ $\epsilon(z)<0$, and with $z<0$ $\epsilon(z)>0$.

The differential equation (1) is now written in the form

$$\frac{d^2w}{dz^2}-\frac{1}{x}\frac{dw}{dz}-\rho^2(x+q^2)w=0, \quad (5)$$

where $x=\alpha z+i\nu$, and $\rho=k_0/\alpha$. It is easy to observe that the form of Eq. (5) does not change when going through $\nu=0$.

The only difference for the problem in which absorption is taken into account is that here the "mathematical" point of reflection $x=-q^2$ corresponds to complex values of the z-coordinate: in the complex plane x, the real values of z

lie on a straight line passing in the upper half-plane parallel to the real axis at a distance equal to ν.

Furthermore, let us note that for a medium with slowly changing properties the parameter of Eq.(5) $\rho >> 1$. Thus, in the F-layer of the ionosphere $(\alpha \sim 10^{-7})\rho \sim 3 \times 10^4$ for the frequencies $\omega \sim 10^8$.

In Refs. 3,4 it is shown that the solution of Eq. (5), satisfying the necessary physical requirements, takes on some value not equal to zero at a point where (ϵ^1 / z) becomes zero $(x = 0)$. Therefore, in accordance with (3), the vertical component of the electric field

$$E_z = -(qw / s')\, e^{i(\omega t + k_0 q y)} \qquad (6)$$

becomes infinite at this point. The nature of this singularity depends on the behavior of $\epsilon^1 (x)$. Thus, for a linear layer, E_z becomes infinite as $1/x$ and the E_y component has a logarithmic singularity.[4]

This singularity is located on the real axis only for $\nu = 0$. In taking into account absorption the maximum value of E_z will be equal to

$$|E_z|_{z=0} = q\,|w(0)|/\nu$$

and can be very large for sufficiently small values of ν. With this, the magnitude of the field depends to a considerable extent on the values taken on by the function ω (0). This function depends on the angle of incidence and thus determines the magnitude $(Ez)_{z=0}$ in the entire interval of values for the parameter $q = \sin \theta_0$.

First, let us attempt to determine the form of this function for large angles of incidence when the reflection point $x = -q^2$ and the particular point $x = 0$ are separated by a considerable distance. For the sake of convenience we shall investigate [instead of Eq. (5)] the equation

$$(d^2u / dx^2) - [\rho^2 (x + q^2) + 3/4x^2]\, u = 0, \qquad (7)$$

which is satisfied by the function

$$u(x) = w(x) / \sqrt{x}. \qquad (8)$$

We shall assume that the distance between the reflection point $x = -q^2$ and the special point $x = 0$ of the Eq. (7) is much larger than a wavelength. For a medium with slowly changing properties $(\rho >> 1)$ this takes place for values of q^2 which are not too small. Then the approximate solution of Eq. (7) (valid everywhere except for a small region around the point $x = 0$ and representing a standing wave to the left of $x = -q^2$) can be written in the form[5]:

$$u = \sqrt{\pi\rho/2}\; e^{i\pi/2} \sqrt{s/s'}\, H^{(1)}_{1/3}(is), \qquad (9)$$

$$s = \rho \int_{-q^2}^{x} \sqrt{x + q^2}\, dx \qquad (10)$$

$$= \frac{2}{3}\rho (x + q^2)^{3/2}; \quad s' = ds/dx,$$

where $H_{1/3}$ is Hankel's function of order 1/3. The constant which appears in the solution of Eq. (9) is chosen so that the amplitude of the incident wave field at the boundary of the inhomogenesous layer $[\epsilon^1 (z) = 1]$ is equal to unity.

Another approximate solution, valid to the right of the reflection point, can be obtained by using the method proposed in Ref. 5. Let us introduce a new independent variable

$$\xi = \rho \int_0^x \sqrt{x + q^2}\, dx = {}^2/_3\, \rho\, [(x + q^2)^{3/2} - q^3].$$

It is easily shown that the function

$$u^* = B \sqrt{\frac{\xi}{d\xi/dx}}\, H_1^{(1)}(i\xi) \qquad (11)$$

satisfies the equation

$$\frac{d^2u^*}{dx^2} - \Big[\rho^2 (x + q^2) \qquad (12)$$

$$+ \frac{3}{4}\left(\frac{1}{\xi}\frac{d\xi}{dx}\right)^2 - \frac{5}{16(x + q^2)^2}\Big]\, u^* = 0.$$

For small $x\xi = \rho q x$ Eq.(12) also has the same singularity at the point $x = 0$ as the fundamental Eq. (7). Besides, for large values of the parameter ρ, Eqs. (12) and (7) differ very little from each other, if one excludes from consideration some region around the point $x = -q^2$ where the function $(x + q^2)^{-2}$ begins to grow abruptly. Consequently, the solutions of these equations in the region $x > -q^2$ will also differ little from each other. At the same time, Eq. (11) will approximate the solution which approaches zero as $x \to +\infty$ [in the region of negative values of $\epsilon (z)$]. We note that the solution which was obtained and investigated in Ref. 4 can be obtained from Eq. (11) by assuming that for small x, $\xi = \rho q x$.

We obtained the approximate solutions (9) and (10) giving the asymptotic behavior of the desired solution (for $\rho \to \infty$) for the various regions of the variable x values: to the left of $x = 0$ [Eq. (9)] and to the right of $x = -q^2$ [Eq. (11)]. In the interval $= -q^2 < x < 0$ both approximations are correct, which enables one to combine these solutions so that they would yield the same particular solution of our problem. The necessary calculations are materially simplified if one uses the fact that both approximate expressions have the same asymptotic behavior in the above mentioned interval. Actually, using the asymptotic representations for the functions $H_{1/3}^{(1)}(is)$ in (9) and $H_1^{(1)}(i\xi)$ in Eq. (11) we obtain for $s >> 1$ to the right of the point of $x = -q^2$

$$u \approx -\sqrt{\rho/s'}\,e^{-s+i\pi/4} \tag{13}$$

and to the left of x–0 for $|\xi| >> 1$,

$$u^* \approx -B\sqrt{2/\pi\xi'}\,e^{|\xi|}, \tag{14}$$

expressions accurately agreeing with each other up to the constant multiplier, because $ds' = d\xi'$ and

$$s = \rho\int_{-q^2}^{x}\sqrt{x+q^2}\,dx = s_0 - |\xi|; \quad s_0 = \frac{2}{3}\rho q^3 \tag{15}$$

in the interval $-q^2 < x < 0$. Comparing (13) and (14), we find the value of the constant B

$$B = \sqrt{\pi\rho/2}\,e^{i\pi/4 - s_0}. \tag{16}$$

Thus on the basis of (11) and (16) we can now write the final formula, which gives the behavior of function $w(x)$ in the region $x > -q^2$, in the following form:

$$w(x) = \sqrt{x}u^* = \sqrt{\pi\rho/2}\cdot\sqrt{x\xi/\xi'}\,e^{i\pi/4 - s_0}H_1^{(1)}(i\xi). \tag{17}$$

The remaining components of the field can be calculated from formulas (2) and (3).

Here we shall be interested primarily in the behavior of the E_z component in the vicinity of the $\epsilon'(z)$ zero. Formula (17) shows that $w(x)$ converges to a finite value for $x \to 0$:

$$w(0) = \sqrt{2/\pi\rho}\,e^{-s_0 - i3\pi/4}/q \tag{18}$$

and consequently, with $|x| << 1$, the behavior of $|E_z|$ according to (6) is given by the formula

$$|E_z| = \sqrt{2/\pi\rho}\,e^{-s_0}/|x| \tag{19}$$

with a maximum value (in a medium with an absorption $x = \alpha z + i\nu$)

$$|E_z|_{z=0} = \sqrt{2/\pi\rho}\,e^{-s_0}/\nu. \tag{20}$$

It is to be remembered that the final formulas are applicable only for large angles of incidence; Eq. (18) for $q \to 0$ gives an obviously incorrect result. However, for the upper ionosphere layers ($\rho >> 1$) the approximation formulas are applicable up to angles of incidence θ_0 of the order 4–5°, and, as it can be easily verified, under these conditions, the effect of field growth near the point $z = 0$ will be negligible ($s_0 >> 1$). The field will take on large values only for unrealistically small ν_{eff}. At the same time, the presence of the singularity at the point where $\epsilon = 0$ does not affect the behavior of the field in the region below the reflection point, that is, the reflection of the wave having an E_z component, and under these conditions of oblique incidence takes place in the same manner as in the case of the wave with an electric vector perpendicular to the plane of incidence.

Furthermore, let us note that in the case of normal incidence ($q = 0$) $E_z = 0$. Consequently, for some small angle of incidence the growth of the field will be a maximum. In connection with this, the behavior of the function $w(0, q)$ for all incidence angles is of interest. Earlier we determined the form of this function for values of q that were not too small [Eq. (18)]. Besides, it is easy to obtain the value of this function for $q = 0$. Actully, the solution of equation (5) converging to zero for $x \to +\infty$ can be written in the form[6]

$$w(x, 0) = 2\sqrt{\rho/3\pi}\,xK_{2/3}\left({}^2\!/_3\,\rho x^{3/2}\right), \tag{21}$$

where $K_{2/3}$ is MacDonald's function. The constant multiplier in the solution of (21) is chosen so that it includes the incident wave with an amplitude equal to unity at the boundary of the inhomogeneous layer. Using the derivative from the Airy function and its relationship with Bessel's functions[7] let us write the solution of (21) in another form.

$$w(x,\ 0) = -2\rho^{-1/6}v'(t);\quad (t = \rho^{2/3}x).$$

Consequently, the value of the function $w(0,q)$ for $q = 0$ (normal incidence) will be equal

$$w(0,\ 0) = -2\rho^{-1/6}v'(0). \tag{22}$$

Let us look for the approximate expression of this function in the entire range of values of the parameter q. First we shall find the approximate expression $w(x, q)$ for small values of x.

In order to do this we shall make explicit the behavior of the function u satisfying Eq. (7). If one looks for its solution in the form

$$u = u_1(s(x))\,u_2(x),\ s(x) = {}^2/_3\,\rho\,(x + q^2)^{3/2}, \tag{23}$$

then for u_1 and u_2 we obtain the equation

$$\frac{d^2u_2}{dx^2} + 2\frac{du_2}{dx}\frac{ds}{dx}\frac{u_1'}{u_1} + \left[\frac{u_1''}{u_1}\left(\frac{ds}{dx}\right)^2 + \frac{u_1'}{u_1}\left(\frac{d^2s}{dx^2}\right) - \rho^2(x+q^2) - \frac{3}{4x^2}\right]u_2 = 0, \tag{24}$$

in which we require that

$$\frac{u_1''}{u_1}\left(\frac{ds}{dx}\right)^2 + \frac{u'_1}{u_1}\frac{d^2s}{dx^2} - \rho^2(x+q^2) = 0$$

(here the primes mean differentiation with respect to s). Using Eq. (23), we shall express the last in equation in the form

$$\frac{d^2u_1}{ds^2} + \frac{1}{3s}\frac{du_1}{ds} - u_1 = 0.$$

The required particular solution converging to zero for $s \to +\infty$ is

$$u_1 = s^{1/3}K_{1/3}(s). \tag{25}$$

Let us substitute this expression in Eq. (24). Then

$$\frac{d^2u_2}{dx^2} + 2f(x)\frac{du_2}{dx} - \frac{3}{4x^2}u_2 = 0, \tag{26}$$

where $f(x)$ denotes the function

$$f(x) = (ds/dx)\,u_1'/u_1 = -\rho\,(x+q^2)^{1/2}K_{2/3}(s)/K_{1/3}(s).$$

Limiting the investigation of the solution to a small region around the point $x = 0$, and using the fact that $f(x)$ when compared to the solution itself is a slowly varying function we can consider it as a constant and equal to the value for $x = 0$. Then, denoting the positive constant by

$$-f(0) = q\rho K_{2/3}(s_0)/K_{1/3}(s_0) = b; \quad (s_0 = {}^2/_3\,\rho q^3), \tag{27}$$

we obtain the following approximate equation for $u_2(x)$:

$$d^2u_2/dx^2 - 2b\,du_2/dx - \frac{3}{4x^2}u_2 = 0.$$

Its solution can be written by means of known functions.[6] If one chooses the particular solution

$$u_2 = \sqrt{xb}\,K_1(bx)\,e^{bx} \tag{28}$$

(K_1 is MacDonald's function of the first order), which for $bx << 1$ approaches a constant value, then the final desired function $w(x_2\ q)$ is written thus:

$$w(x,\ q) = \sqrt{x}u = As^{1/3}K_{1/3}(s)\sqrt{bx^2}e^{bx}K_1(bx), \tag{29}$$

where the arbitrary constant A must be chosen so that approximation (29) would agree in the small region around the point $x = 0$ with the particular solution in which the amplitude of the incident wave is equal to unity at the boundary of the inhomogeneous layer. If $s_0 >> 1$, then from (29) for $w(0, q)$ we get (18) accurate up to the constant.

A simple calculation gives

$$A = 2\,({}^2/_3\,\rho)^{1/6}/\pi \tag{30}$$

and finally for the function $w(0,q)$

$$w(0,\ q) = (2/\pi)\,({}^2/_3\,\rho)^{1/6}\,s_0^{1/3}K_{1/3}(s_0) \times \{K_{1/3}(s_0)/[q\rho K_{2/3}(s_0)]\}^{1/2}. \tag{31}$$

If the Airy function is used,

$$v(t) = \sqrt{t/3\pi}K_{1/3}\left(\frac{2}{3}t^{3/2}\right); \quad (31)$$

$$v'(t) = -t(3\pi)^{-1/2}K_{2/3}(^2/_3\, t^{3/2})$$

(see, for instance, Ref. 7) then $w(q)$ can be written in the following form which is convenient for computations:

$$w(q) = 2\sqrt{2/\pi}\,\rho^{-1/6}v(t)\sqrt{v(t)/-v'(t)}; \quad (32)$$

$$t = q^2\rho^{2/3}.$$

From the calculations it can be seen first that function (32) must be a good approximation to the true value of $w(q)$ for angles of incidence which are not close to zero. A relatively large error can be expected for $q \to 0$. However, a comparison of this function with its exact expression for $q = 0$ (22) shows that formula (32) gives a relatively good description of the true behavior of $w(q)$ even for very small incidence angles. In fact, the ratio of the limiting values is given by

$$w(0)_{\text{exact}}/w(0)_{\text{approx}} = \sqrt{\pi/2}\,(-v'(0)/v(0))^{3/2} \approx 0.8,$$

and, consequently, one can be sure that the error from the approximate expression (32) does not exceed 20% in the interval of q values from zero to some small magnitude beyond which the error of this approximation is negligible.

With Eq. (32) we can now describe the behavior of the field component E_z of interest to us in the neighborhood of the point where the dielectric constant of the medium become zero ($x = 0$). In accordance with (6), for small values of (x),

$$|E_z| = \frac{|qw(q)|}{|x|},$$

$$|qw(q)| = 2\sqrt{2/\pi\rho}\sqrt{tv(t)}\sqrt{v(t)/-v'(t)},$$

($t = \rho^{2/3}q^2$) or, introducing a new parameter $\tau = \sqrt{t} = \rho^{1/3}q$,

$$|qw(q)| = \frac{4\tau v(\tau^2)}{\sqrt{2\pi\rho}}\sqrt{\frac{v(\tau^2)}{-v'(\tau^2)}} = \frac{\Phi(\tau)}{\sqrt{2\pi\rho}}. \quad (33)$$

Consequently

$$|E_z| = \Phi(\tau)/\sqrt{2\pi\rho}\,|x|; \qquad (x = \alpha z + i\nu) \quad (34)$$

and takes on a maximum value for $z = 0$ equal to

$$|E_z|_{z=0} = \Phi(\tau)/\sqrt{2\pi\rho}\,\nu. \quad (35)$$

The dependence of the maximum magnitude $|E_z|$ on the angle of incidence is thus determined by the function $\Phi(\tau)$, the graph of which is shown in Fig. 1; here we also introduce (parallel to the parameter τ axis) a scale in degrees for the incidence angle θ_0 for $\alpha = 10^{-7}$ and $\omega = 2\pi \times 10^7$. It is essential to note that $\Phi(\tau)$ takes on a value of the order of unity for a very narrow interval of the incidence angle values, with the curve maximum equal to 1.2 corresponding to $\theta_0 = 1.5°$, and when $\theta_0 = 5°$, $\Phi(\tau) \sim 10^{-4}$.

Let us evaluate, by means of Eq. (35), the values reached by the field $|E_z|$ in an isotropic plasma.[1] For the data of the E layer in the ionosphere one can take $\alpha = 10^{-6}$, $\omega = 6\pi \times 10^6$ ($\lambda_0 = 100$ m). The maximum magnitude $|E_z|$ for $\nu_{\text{eff}} = 10^5$ will be $|E_z|_{z=0} \approx 3$, and for $\nu_{\text{eff}} = 10^4$: $|E_z|_{z=0} \approx 30$.

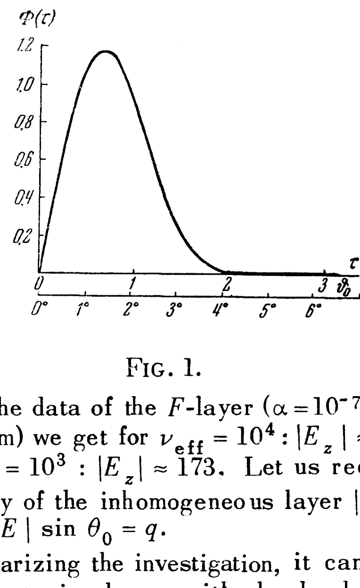

FIG. 1.

For the data of the F-layer ($\alpha = 10^{-7}$, $\omega = 2\pi \times 10^7$, $\lambda_0 = 30$m) we get for $\nu_{\text{eff}} = 10^4$: $|E_z| \approx 17.3$, and for $\nu_{\text{eff}} = 10^3$: $|E_z| \approx 173$. Let us recall that the boundary of the inhomogeneous layer $|E| = 1$ and $|E_z| = |E| \sin\theta_0 = q$.

Summarizing the investigation, it can be said that for an isotopic plasma with slowly changing properties the growth effect of the field in the region of small $\epsilon(z)$ values plays no role in the case of large incidence angles, but becomes appreciable for small angles $\theta_0 \sim 2$–$3°$ ($\nu_{\text{eff}} \leq 10^3$. In the latter case the presence of the point at which $\epsilon = 0$ materially changes the solution form beyond the reflection point, and the growth of the field intensity of the standing wave is not smoothed out by the existing absorption.

2. THE APPROXIMATE CALCULATION OF THE EFFECT OF PLASMA WAVES

We have limited ourselves heretofore to a discussion of collision effects whose calculation leads naturally to a finite value for the field of the electromagnetic wave at the point where $\epsilon = 0$. But in a medium with a small absorption, there remains the anomalous behavior of corresponding solutions and the unexplained problem of the true behavior of the field, since in an inhomogeneous plasma an important role can be played by other factors; their calculations, like the calculation of collisions, will lead to the removal of the singularity discussed previously . The possibility of the existence of other factors (besides collisions and nonlinear phenomena) leading to a finite value of the field intensity at the zero of $\epsilon(z)$ becomes apparent from the following considerations.

The characteristic behavior of the vertical field component E_z of the electromagnetic wave propagated in a plano-stratified ionized medium suggests the idea of relating this phenomenon with definite resonance characteristics inherent in a quasineutral plasma. It is well known (Ref. 1, Sec. 63) that the frequency determined from the condition $\epsilon = 1-4\pi e^2 N/m\omega^2 = 0$ is the frequency of the so-called plasma oscillations. It is completely clear that the singular behavior of the field **E** in the neighborhood of the point where $\epsilon = 0$ is intimately connected with the excitation of these oscillations. In such a case, the function representing the change in $|E_z|^2$ as dependent on the coordinate x (Fig. 2) is a type of a resonance curve. The resonance takes place at the point $x = 0$ [$\epsilon(0) = 0$], that is, where the frequency of the incident wave ω coincides with the natural frequency of the plasma ω_0 .

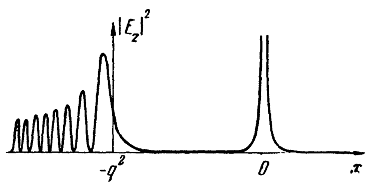

FIG. 2.

Of course, such a dependence of $|E_z|^2$ on a coordinate is characteristic for an idealized problem in which we neglect all sorts of energy dissipation in the standing electromagnetic wave. Naturally, the calculation of collisions results in the elimination of the infinite values for the field. At the same time the behavior of the vertical component in the vicinity of the resonance point is given by the function [see Eq. (34)] .

$$|E_z|^2 = A\,[(\alpha z)^2 + (\nu_{\text{eff}}/\omega)^2]^{-1}; \quad A = \text{const.}$$

The width of this "resonance curve", as in the oscillation circuit, is determined by absorption and is equal to $z' = \nu_{\text{eff}}/\omega\alpha$.

However, in an inhomogeneous plasma, another energy dissipation mechanism is possible, namely, the generation of plasma waves. The presence of an abruptly changing longitudinal component of the electric field results in the appearance of a space inhomogeneity in the electron gas. In each section of the medium excited in this manner, the electrons experience oscillations (under the influence of the wave's electric field), the amplitude of which increases with the approach to the resonance point ($x = 0$). Actually, these local oscillations are not independent: every change in electron density in one of the medium's regions is transmitted to its neighbor through electronic pressure; the calculation of the latter results in the appearance of plasma waves which carry away some fraction of the standing electromagnetic wave energy. Finally, the energy associated with the plasma wave is used up in the heating of the gas.

In the more general formulation of the problem we must take into account the possible emergence of plasma waves which, undoubtedly, will result in the elimination of the singularity in the solution and in the finite field value to the resonance point. The following approximate calculations explain the effect of plasma waves on the behavior of the electric field's vertical component in the vicinity of this point.

For the oblique incidence of an electromagnetic wave on a plano-layered isotopic medium the properties of which depend on the coordinate z, the field equations can be written in the form

$$\partial H_z/\partial y - \partial H_y/\partial z = ik_0(E_x + 4\pi P_x), \qquad (36a)$$

$$\partial E_x/\partial z = -ik_0 H_y; \quad \partial E_x/\partial y = ik_0 H_z;$$

$$\partial E_z/\partial y - \partial E_y/\partial z = -ik_0 H_x, \qquad (36b)$$

$$\partial H_x/\partial z = ik_0(E_y + 4\pi P_y),$$

$$\partial H_x/\partial y = -ik_0(E_z + 4\pi P_z);$$

$$\operatorname{div}(\mathbf{E} + 4\pi\mathbf{P}) = 0 \qquad (36c)$$

(it is assmed that the field is a function of time according to the law $e^{i\omega t}$ and is independent of the x-coordinate). To the system (36) one must also add the relationship between the electric field strength E and the polarization P. This relation- can be easily obtained for the problem at hand by using the equation for the motion of electrons under the influence of the field E and electron pressure

$$mN\ddot{\mathbf{r}} = -\varkappa T\nabla N + eN\mathbf{E} \qquad (37)$$

($\varkappa$ is Boltzmann's Constant, T is absolute temperature, N is electron concentration) and the equation of continuity

$$(\partial N/\partial t) + \operatorname{div} N\dot{\mathbf{r}} = 0. \qquad (38)$$

Equations (37) and (38) are for processes harmonic in time. By taking into account the fact that

$$\partial N/\partial t \approx \partial n/\partial t = i\omega n; \quad \nabla N \approx \nabla n$$

(where n is small deviation of electron concentration from its equilibrium value), these equations can be written in the form

$$\mathbf{P} = (\varkappa Te/m\omega^2)\nabla n \qquad (39)$$

$$-(e^2N/m\omega^2)\mathbf{E}; \; en + \operatorname{div}\mathbf{P} = 0.$$

Moreover, in writing these formulas, the well known definition of the polarization vector was used, $\mathbf{P} = eN\mathbf{r}$. In addition, taking into account (36c), let us write the relationship between P and E in the following coordinate form

$$P_x = -(e^2N/m\omega^2)E_x; \qquad (40)$$

$$P_y = \frac{\varkappa T}{m\omega^2 4\pi}\left(\frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial y\,\partial z}\right) - \frac{e^2N}{m\omega^2}E_y;$$

$$P_z = \frac{\varkappa T}{m\omega^2 4\pi}\left(\frac{\partial^2 E_y}{\partial y \partial z} + \frac{\partial^2 E_z}{\partial z^2}\right) - \frac{e^2N}{m\omega^2}E_z.$$

It is not difficult to see that the system (36), in combination with (40) resolves itself into two independent equation systems. It appears then that the wave with components E_x, H_y, H_z satisfies the same equations as in the problem considered in Sec. 1. The calculation of electron pressure results in changes of the second and third equations (36b) only:

$$\frac{\partial H_x}{\partial z} = ik_0\left[\varepsilon E_y + \frac{\beta^2}{k_0^2}\left(\frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial y\,\partial z}\right)\right]; \qquad (41)$$

$$-\frac{\partial H_x}{\partial y} = ik_0\left[\varepsilon E_z + \frac{\beta^2}{k_0^2}\left(\frac{\partial^2 E_y}{\partial y\,\partial z} + \frac{\partial^2 E_z}{\partial z^2}\right)\right],$$

where $\epsilon = 1 - 4\pi e^2 N/m\omega^2$ is the usual dielectric constant of the plasma (the absorption is neglected) and $\beta^2 = k_0^2\, \varkappa T/m\omega^2$. The parameter

$$\beta = (1/c)\sqrt{\varkappa T/m} \sim v/c \qquad (42)$$

has an order of magnitude equal to the ratio of the electron thermal velocity to the velocity of light and represents under ordinary ionospheric conditions a rather small quantity; for instance, at temperatures of the order of 500° K: $\beta = 3\times10^{-4}$.

Let us look for the solution in the form

$$H_x = w(z)\,e^{ik_0qy}, \quad E_z = u(z)\,e^{ik_0qy},$$

$$E_y = v(z)e^{ik_0qy}.$$

Then the system (41) with the elimination of E_y is reduced to the following two interrelated differential equations of the second order:

$$\frac{d^2w}{dz^2} - \frac{d\varepsilon/dz}{\varepsilon - \beta^2q^2}\frac{dw}{dz} \qquad (43)$$

$$+ k_0^2(\varepsilon - q^2)\,w = \beta^2 q\,\frac{d\varepsilon/dz}{\varepsilon - \beta^2q^2}\frac{du}{dz},$$

$$(\beta^2 d^2u/dz^2) + k_0^2(\varepsilon - \beta^2q^2)\,u = qk_0^2(\beta^2 - 1)\,w.$$

Thus, the calculation of electron thermal motions results in equations of higher order.[9] The solutions of system (43) describe normal waves of two types, which in a few special cases, and also in the regions of small interaction, enable us to consider the wave field as the superposition of electromagnetic and plasma waves. Thus, for $q = 0$ (normal incidence) system (43) is resolved into two independent equations. The first equation gives the same results as Eq. (1) and the solution describes electromagnetic waves; the second gives a wave equation for plasma waves from which it is easy to obtain the known formula of the index of refraction for these waves [$n = (\epsilon/\beta)^{1/2}$](see, for example, Ref. 1, Sec. 63). Besides, as the investigation of system (43) shows, the subdivision of the field into plasma and electromagnetic waves is possible in regions relatively far removed from the interaction region (vicinity of the point where $\epsilon = 0$).

It is easy to show that the solutions of system (43) will be analytical functions and a singularity appears in them when the small parameter β^2 approaches zero, β^2 being the coefficient of the highest order derivative in the fourth order equation equivalent to system (43).

We note that for small β^2 the first equation of system (43) differs little from Eq. (1), obtained without taking into account the thermal motions of electrons. The second equation for $\beta^2 \to 0$ turns into an algebraic relation

$$u(z) = -qw(z)/\varepsilon(z).$$

It can be shown by solving equations (43) by the method of successive approximations that the value of component E_z at the points where $\epsilon(z)$ becomes zero will have an order of magnitude equal to

$$|E_z|_{z=0} \approx qw(q)(\beta/\rho)^{-2/3}. \qquad (44)$$

The appearance of the multiplier $(\beta/\rho)^{-2/3}$ in formula (44) can be explained as follows. Let us write the equation for electrons moving under the influence of the E_z component by taking into account collisions and the pressure gradient (for processes dependent on time according to the law $e^{i\omega t}$)

$$-\omega^2 Nmr + i\omega\nu_{\text{eff}}\, Nmr = eNE_z + \varkappa T\partial N/\partial z. \qquad (45)$$

If, in addition, one assumes the continuity equation (38) and sets $\partial N/\partial z \approx kn$; $\partial N/\partial t \approx i\omega n$; div $N\mathbf{r} \approx kNr$, where $1/k$ is some characteristic dimension of the wave field, then Eq. (45) can be written in the form

$$-\omega^2 Nmr + i\omega\nu_{\text{eff}}\, Nmr = eNE_z + \varkappa Tk^2 Nr,$$

from which, by comparing the terms $\omega\nu_{\text{eff}} Nmr$ and $\varkappa Tk^2 Nr$, it is easy to see that the quantity $(\varkappa T/m\omega^2)k^2$ plays the role of ν_{eff}/ω (by assuming collisions), on the assumption of the analogous effect of the pressure gradient (of the plasma waves).

Consequently, in the formula

$$|E_z| \approx qw(q)/|\alpha z + i\nu_{\text{eff}}/\omega| \qquad (46)$$

in the last case $(\varkappa T/m\omega^2)k^2$ will replace ν_{eff}/ω. Assuming, furthermore, that the dimension of the field inhomogeneity (by reducing the effect of plasma waves to some effective collision number) has an order of magnitude of $1/k$, we obtain the relationship

$$k = (m\omega^2\alpha/\varkappa T)^{1/3},$$

and the value of the field E_z at the resonance point ($z = 0$) will have an order of magnitude given by

$$|E_z| \approx qw(q)/(\varkappa T k^2/m\omega^2) = qw(q)(\beta/\rho)^{-2/3}.$$

Consequently, the effect of plasma waves in our problem can be compared to the analogous effect of absorption, and some effective number of collisions can be introduced:

$$\nu_{\text{pl}}/\omega = (\beta/\rho)^{2/3}. \qquad (47)$$

For the data of E-layer ($\beta = 3\times10^{-4}$, $\alpha = 10^{-6}$, $\lambda_0 = 100$ m, $\rho = 2\pi\times10^2$) $\nu_{\text{pl}} \sim 10^3$.

For the F-layer ($\beta = 3\times10^{-4}$, $\lambda_0 = 30$m) we obtain $\nu_{\text{pl}} = 3.7\times10^2$ ($\alpha = 10^{-7}$); $\nu_{\text{pl}} = 1.7\times10^3$ ($\alpha = 10^{-6}$).

From these numbers it is seen that the calculation of the plasma wave effect is just as effective in some cases ($\alpha \sim 10^{-6}$) as the calculation of collisions ($\nu_{\text{eff}} \leq 10^3$). However, in the lower ($\nu_{\text{eff}} \geq 10^4$) ionospheric layers the absorption effect, connected with collisions, is predominant (See Ref. 1).

Finally, let us note that the assumption of nonlinear effects results in negligible corrections. Thus, in formula (46) along with ν_{eff}/ω there appears the magnitude $\alpha\int_0^t v_z\,dt$ where v_z is the speed of electrons in the z directions. This addition has an order of magnitude $\alpha eE_z/m\omega^2 \sim 10^{-5}E_z$ (for $\alpha = 10^{-7}$, $\omega = 2\pi\times10^7$ and is comparable to ν_{eff}/ω in the F-layer only for a field intensity $E_z \sim 1$ cgs unit at the resonance point. If the amplification coefficient of the field is taken to be 10^2, then at the boundary of the inhomogeneous layer the wave must have an intensity $E_z \sim 10^{-2}$ cgs unit of the order of several v/cm. Consequently, for ionospheric field intensities the role of nonlinear effects is negligible.

In conclusion I express deep gratitude to V. L. Ginzburg for suggesting the problem and for his help in the investigation.

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Quantum Field Theory with Causal Operators*

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A mathematical formalism is set up for the space-time description of field theory, starting from an action principle. It is proved that such a formalism necessarily leads to a representation by means of external sources.

1. INTRODUCTION

THE unsatisfactory state of quantum field theory makes urgent the task of investigating and reformulating the mathematical basis of the theory. Recently a number of papers have appeared, dealing with the space-time description of field theory. A consistently 4-dimensional treatment uses operators of a different type from those of the ordinary 3-dimensional formalism. The former type of operators we shall call "causal". They were introduced simultaneously by several authors authors,[1-3] working from different points of view.

Coester[1] found it necessary to define operators having the properties of causal operators, in order to construct a quantization scheme for Feynman amplitudes. Gol' fand considered these operators under the name of "quasi-fields," using them as a convenient device for calculating expectation-values of the S-matrix. The present author[2] defined causal operators as operators which refer to a definite direction of time, and which satisfy requirements connected with the principle of causality.

It has been proved[1-3] that the use of causal instead of ordinary field-operators does not change the results of the usual scattering-matrix theory. Equations for the state-vector were also introduced[1] into the causal operator formalism. But the form of these equations was fixed by requiring that the results of the causal operator theory should coincide with the results of the usual theory, and this cannot be regarded as a satisfactory basis for a logically constructed formalism.

The present paper contains a systematic development of the theory of causal operators, starting from an action principle. In Sec. 2 the action principle is formulated, and the basic difference between the causal operator theory and the usual theory is explained. The difference arises from our considering the situation from a 4-dimensional point of view. One consequence of this is the nonexistence of equations of motion for causal operators.* In Sec. 3 the state-vector is studied in a representation in which a complete set of commuting observables can only be constructed by means of causal operators. In this case the action principle leads to equations for the state vector which are identical (after some changes in notation) with the

* The content of this paper was presented at the All-Union Conference on quantum electrodynamics and the theory of elementary particles on April 2, 1955.

* This feature was not made clear in the author's second paper (see Ref. 2), where it was suggested that equations of motion of causal operators should exist. Hence, Eqs. (3) and (5) of Ref. 2 refer to the solution of a boundary-value problem, not for the causal operators, but for particular matrix elements of these operators. Equation (5), obtained by combining (3a) with (3b), is incorrect, since the matrix elements (3a) and (3b) refer to different functionals. In any case, Eq. (5) was never used in the rest of the paper.