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Quantum Field Theory with Causal Operators*

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A mathematical formalism is set up for the space-time description of field theory, starting from an action principle. It is proved that such a formalism necessarily leads to a representation by means of external sources.

1. INTRODUCTION

THE unsatisfactory state of quantum field theory makes urgent the task of investigating and reformulating the mathematical basis of the theory. Recently a number of papers have appeared, dealing with the space-time description of field theory. A consistently 4-dimensional treatment uses operators of a different type from those of the ordinary 3-dimensional formalism. The former type of operators we shall call "causal". They were introduced simultaneously by several authors authors,[1-3] working from different points of view.

Coester[1] found it necessary to define operators having the properties of causal operators, in order to construct a quantization scheme for Feynman amplitudes. Gol' fand considered these operators under the name of "quasi-fields," using them as a convenient device for calculating expectation-values of the S-matrix. The present author[2] defined causal operators as operators which refer to a definite direction of time, and which satisfy requirements connected with the principle of causality.

It has been proved[1-3] that the use of causal instead of ordinary field-operators does not change the results of the usual scattering-matrix theory. Equations for the state-vector were also introduced[1] into the causal operator formalism. But the form of these equations was fixed by requiring that the results of the causal operator theory should coincide with the results of the usual theory, and this cannot be regarded as a satisfactory basis for a logically constructed formalism.

The present paper contains a systematic development of the theory of causal operators, starting from an action principle. In Sec. 2 the action principle is formulated, and the basic difference between the causal operator theory and the usual theory is explained. The difference arises from our considering the situation from a 4-dimensional point of view. One consequence of this is the nonexistence of equations of motion for causal operators.* In Sec. 3 the state-vector is studied in a representation in which a complete set of commuting observables can only be constructed by means of causal operators. In this case the action principle leads to equations for the state vector which are identical (after some changes in notation) with the

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* This feature was not made clear in the author's second paper (see Ref. 2), where it was suggested that equations of motion of causal operators should exist. Hence, Eqs. (3) and (5) of Ref. 2 refer to the solution of a boundary-value problem, not for the causal operators, but for particular matrix elements of these operators. Equation (5), obtained by combining (3a) with (3b), is incorrect, since the matrix elements (3a) and (3b) refer to different functionals. In any case, Eq. (5) was never used in the rest of the paper.

equations expressing the Schwinger functional in terms of external sources. The appearance of external sources seems to be inevitable in a space-time description of field theory. This seems to be connected with the inapplicability of a principle of stationary action. In Sec 4 the state-vector is studied by expressing it as a generalized Fock functional. The action principle leads to the equations for the state-vector which were proposed by Coester.[1] In Sec. 4 equations are also derived for Feynman amplitudes. In Sec. 5 the energy-momentum vector of the causal operator theory is defined.

2. STARTING EQUATIONS. THE ACTION PRINCIPLE.

The nucleon field operators are denoted by $\chi(x)$ and $\overline{\chi}(y)$, the meson field operators by $\Phi(x)$. We assume as already known that the causal operators $\chi(x)$, $\overline{\chi}(y)$ and $\Phi(z)$ anticommute or commute for all positions of the points x, y, z.

$$\{\chi(x), \overline{\chi}(y)\} = 0; \; [\Phi(x), \Phi(y)] = 0; \; [\chi(x), \Phi(z)] = 0, \text{etc.} \tag{2.1}$$

It is assumed that the matrixes χ, $\overline{\chi}$ and Φ are not diagonal in the particle occupation-numbers. We consider what consequences follow from the assumption that the basic field operators are χ, $\overline{\chi}$ and Φ with the properties (2.1).

In consequence of Eq. (2.1), we can construct a complete set of commuting observables ξ by means of the operators χ, $\overline{\chi}$ and Φ at the various points of space-time. Therefore, in the causal operator formalism, a state labelled by the set of eigenvalues ξ' has a well-defined space-time behavior. This fact is basic to the development of the theory of the operators χ, $\overline{\chi}$, Φ.

We suppose a complete set of state-vectors $\Omega(\xi')$ to be labelled by the eigenvalues ξ' of the set of observables ξ which constitute a space-time description of the system. The most general problem of the theory is to describe the state Ω of the interacting fields over the whole of space-time. Once Ω is known, the probabilties of all scattering processes and the properties of all stationary states are determined. The state-vector

$$\Omega = \int C(\xi')\, d\xi' \Omega(\xi') \tag{2.2}$$

will be fixed if the coefficients $C(\xi') = [\Omega(\xi'), \Omega]$ can be calculated. The coefficients $C(\xi')$ cannot be assigned arbitrarily, and the equations restricting their values follow from an action principle.

Consider an infinitesimal variation $\delta C(\xi')$ produced by infinitesimal variations of ξ' throughout space-time. The action principle is expressed by the equation

$$\delta C(\xi') = i(\Omega(\xi'), \delta W \cdot \Omega), \tag{2.3a}$$

where W is the action operator. This formulation of the principle is equivalent to the action principle of Feynman.[4] Since $\delta C(\xi') = [\delta\Omega(\xi'), \Omega]$, we can define an infinitesimal transformation operator $G_{\xi'}$, by*

$$(\delta\Omega(\xi'), \Omega) = (\Omega(\xi'), G_{\xi'}\Omega),$$

and then the action principle (2.3a) becomes

$$\delta_C \Omega \equiv G_{\xi'}\Omega = i\delta W \cdot \Omega. \tag{2.3b}$$

The variation $\delta_C \Omega$ in Eq. (2.3b) is the variation produced by varying the coefficients $C(\xi')$.

We assume that the action has the same form as in the usual theory, but expressed in terms of the operators χ, $\overline{\chi}$ and Φ. It is convenient to write the expression for W in the form of a double integral, with $\delta^4(x-y) = \delta(x-y)\,\delta(x_0 - y_0)$,

$$W = \int L(x, y)\, d^4x d^4y;$$

$$L(x, y) = \frac{1}{2} i\delta^4(x-y) \times \{\overline{\chi}(y) D(x)\chi(x) - \chi(y) D(-x)\overline{\chi}(x) - {}^1/_2 g\Phi(x)\gamma_5(y)[\overline{\chi}(y)\chi(x) - \chi(y)\overline{\chi}(x)]\} - \delta^4(x-y)\left[\left(\frac{\partial\Phi(x)}{\partial x_\nu}\right)^2 + \mu^2\Phi^2(x)\right]. \tag{2.4}$$

Here we use the notation

$$D(x) = i(\gamma_\lambda(x)\,\partial/\partial x_\lambda + m),$$

and $\gamma_\lambda(x)$ means that the matrix γ_λ operates on the spinor referring to the point x; thus,

$$\gamma_\lambda(x)\; \overline{\chi}(y)\chi(x) = \overline{\chi}(y)\gamma_\lambda\chi(x), \quad \text{etc.}$$

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* We always denote by P_μ the energy-momentum vector. To avoid confusion, the infinitesimal transformation operator is denoted by $G_{\xi'}$ instead of by the customary $P_{\xi'}$.

The representation (2.4) of the action W avoids the ambiguities which arise in considering products of causal operators referring to equal times. The operator $L(x, y)$ in Eq. (2.4) is symmetric in the coordinates x, y, and this is necessary in order that $L(x, y)$ should be invariant under charge-conjugation. The interaction term in $L(x, x)$ contains a normal product of nucleon field operators, in this formalism just as in the usual theory.

It is clear from Eq. (2.3b) that different choices of $\Omega(\xi')$ will in general correspond to different representations of the equations of motion for Ω. These representations reduce to two main types: (a) when the observables ξ are constructed directly from the operators χ, $\bar{\chi}$, Φ and (b) when the construction of the ξ involves the separation of the field-operators χ, $\bar{\chi}$, Φ into emission and absorption parts. In the first case the state-vector Ω is represented by a Schwinger functional,[5] and in the second case by a generalized Fock functional.[6]

3. THE STATE-VECTOR Ω AS A SCHWINGER FUNCTIONAL

If the operators ξ are constructed from the χ, $\bar{\chi}$ and Φ, then the variations $\delta\xi$ may be expressed in terms of operators $\delta\chi$, $\delta\bar{\chi}$, $\delta\Phi$. Then $G_{\xi'}$ In Eq. (2.3b) contains only these variations of the causal operators. We assume, as in the 3-dimensional formalism[7], that $\delta\chi$, $\delta\bar{\chi}$ and $\delta\Phi$ satisfy the relations

$$\{\delta\chi(x), \bar{\chi}(y)\} = 0; \qquad (3.1)$$
$$\{\delta\bar{\chi}(x), \chi(y)\} = 0; \quad [\delta\Phi(x), \Phi(y)] = 0, \text{ etc.}$$

Since W is constructed from the operators χ, $\bar{\chi}$, Φ, Eq. (2.3b) is equivalent to

$$(G_\chi + G_{\bar{\chi}} + G_\Phi)\,\Omega = i(\delta_\chi W + \delta_{\bar{\chi}} W + \delta_\Phi W)\,\Omega, \qquad (3.2)$$

where G_χ, $G_{\bar{\chi}}$ and G_Φ are operators with the properties

$$[G_\chi, \chi] = \delta\chi; \; [G_{\bar{\chi}}, \bar{\chi}] = \delta\bar{\chi}; \; [G_\Phi, \Phi] = \delta\Phi, \qquad (3.3)$$

and

$$\delta_\chi W = [G_\chi, W]; \; \delta_{\bar{\chi}} W = [G_{\bar{\chi}}, W]; \; \delta_\Phi W = [G_\Phi, W]$$

are the variations of W produced by the variations of χ, $\bar{\chi}$ and Φ.

To construct $G_{\xi'}$ we introduce operators $\pi(x)$, $\bar{\pi}(x)$ and $\Pi(x)$ determined by the following commutation rules:

$$\{\pi(x), \chi(y)\} = (1/i)\,\delta^4(x-y); \qquad (3.4)$$
$$\{\bar{\pi}(x), \bar{\chi}(y)\} = i\delta^4(x-y);$$
$$[\Pi(x), \Phi(y)] = (1/i)\,\delta^4(x-y).$$

The operators π, $\bar{\pi}$ and Π are the 4-dimensional analogs of the canonically conjugate momenta in the usual 3-dimensional treatment. The operators G_χ, $G_{\bar{\chi}}$ and G_Φ then have the form

$$G_\chi = i\int \delta\chi(x)\,\pi(x)\,d^4x;$$
$$G_{\bar{\chi}} = -i\int \delta\bar{\chi}(x)\,\bar{\pi}(x)\,d^4x;$$
$$G_\Phi = i\int \delta\Phi(x)\,\Pi(x)\,d^4x.$$

Substituting these expressions into Eq. (3.2), using Eqs. (2.4) and (3.1), and remembering that the variations $\delta\chi$, $\delta\bar{\chi}$ and $\delta\Phi$ are independent, we obtain the system of equations for Ω

$$D(x) - g\gamma_5\Phi(x)]\,\chi(x)\,\Omega = i\bar{\pi}(x)\,\Omega; \qquad (3.5)$$
$$[D(-x) - g\gamma_5(x)\,\Phi(x)]\,\bar{\chi}(x)\,\Omega = i\pi(x)\,\Omega;$$
$$\{i(\Box - \mu^2)\,\Phi(x) + {}^1/_2\, g\,[\bar{\chi}(y)\,\gamma_5\chi(x) - \gamma_5\chi(y)\,\chi(x)]_{x=y}\}\,\Omega = i\Pi(x)\,\Omega.$$

A formal solution of Eq. (3.5) is easy to find if we assume that Ω can be represented in the form

$$\Omega \equiv \Omega[\chi, \bar{\chi}, \Phi] = \sum_{n,m,k} \int F(x\ldots|y\ldots|z\ldots) \times \chi(x)\ldots\bar{\chi}(y)\ldots\Phi(z)\ldots d^4x\ldots d^4y\ldots d^4z\ldots\Omega^0, \qquad (3.6)$$

where Ω^0 satisfies the conditions

$$\pi(x)\,\Omega^0 = 0; \; \bar{\pi}(x)\,\Omega^0 = 0; \; \Pi(x)\,\Omega^0 = 0, \qquad (3.7)$$

and the function $F(x\ldots|y\ldots|z\ldots)$ is antisymmetric in the coordinates $x\ldots, y\ldots$ and symmetric in the coordinates $z\ldots$. Then in Eq. (3.5) $\Phi(x)$ is to be considered as the operation of multiplying by a certain function $\Phi'(x)$ while $\Pi(x)$ is the operation of functional differentiation

$$\Pi(x)\Omega = -i(\delta\Omega/\delta\Phi'(x)).$$

In the same way $\pi(x)$ and $\overline{\pi}(x)$ in Eq. (3.5) are to be considered as functional differential operators, and they may be formally replaced by $-i\delta/\delta\chi(x)$ and $i\delta/\delta\overline{\chi}(x)$. Under these conditions, the formal solution of Eq. (3.5) is

$$\Omega[\chi, \overline{\chi}, \Phi] = e^{iW}\Omega^0. \qquad (3.8)$$

From Eq. (3.8) we can also obtain formal solutions for other cases.

The meaning of the state-vector Ω is most easily explained by examining the connection between Ω and the Schwinger functional Z which is a functional of external sources, $\eta(x)$, $\overline{\eta}(x)$ associated with the nucleon field, and $J(x)$ with the meson field. The functional Z is determined by the equations[5,8]

$$\left\{D(x) - ig\gamma_5\frac{\delta}{\delta J(x)}\right\}\frac{\delta Z}{\delta\overline{\eta}(x)} = -\eta(x)Z; \qquad (3.9)$$

$$\left\{D(-x) - ig\gamma_5\frac{\delta}{\delta J(x)}\right\}\frac{\delta Z}{\delta\eta(x)} = \overline{\eta}(x)Z;$$

$$i(\Box - \mu^2)(\delta Z/\delta J(x)) = J(x)Z + gi\,\mathrm{Sp}\,\gamma_5(\delta^2 Z/\delta\eta(x)\,\delta\overline{\eta}(x)),$$

and the conditions

$$Z = 1, \ \delta Z/\delta\eta = \delta Z/\delta\overline{\eta} = \delta Z/\delta J = 0 \qquad (3.10)$$

$$\text{for } J = \eta = \overline{\eta} = 0.$$

Comparing Eq. (3.9) with (3.5), we see that Eq. (3.9) is nothing else than the equation (3.5) for the state-vector $\Omega(\pi, \overline{\pi}, \Pi)$, in the representation in which $\Pi(x)$ is the operation of multiplication by $\Pi'(x)$ and $\Phi(x)$ is the operation of functional differentiation

$$\Phi(x)\Omega[\pi, \overline{\pi}, \Pi] = i\frac{\delta}{\delta\Pi'(x)}\Omega[\pi, \overline{\pi}, \Pi].$$

The symbolic representation of $\chi(x)$ and $\overline{\chi}(x)$ is

$$\chi(x) = -i\delta/\delta\pi(x); \ \overline{\chi}(x) = i\delta/\delta\overline{\pi}(x).$$

The role of the external sources is thus played by the "conjugate momenta": $\pi(x) \sim \eta(x)$, $\overline{\pi}(x) \sim \overline{\eta}(x)$, $\Pi(x) \sim J(x)$. The initial conditions (3.7) for the functional $\Omega(\chi, \overline{\chi}, \Phi)$ in the $\chi, \overline{\chi}, \Phi$ representation correspond to the conditions (3.10) for the functional $\Omega(\pi, \overline{\pi}, \Pi) = Z(\eta, \overline{\eta}, J)$ in the $\pi, \overline{\pi}, \Pi$ representation. Thus $\Omega(\pi, \overline{\pi}, \Pi)$ is identical with the generating functional of the Green's functions for the system of interacting fields.

The appearance of external sources in the present formalism is at first glance unexpected, since the original action operator (2.4) does not involve external sources. But in a consistent space-time treatment the appearance of external sources is inevitable, as the following more general argument shows.

In the usual (either classical or quantized) 3-dimensional treatment of field theory, there exists a principle of stationary action. This principle states that variations of the field-quantities (the external sources being held constant), within the 4-dimensional volume bounded by hyperplanes t = constant in the remote past and future, do not change the action and do not influence the development of the system in time. In our 4-dimensional treatment of field theory, the development of the system in time is not considered, since the time-development is already included in the basic assumption of the 4-dimensional character of the wave-functional $\Omega(\xi)$. Since we assume the possibility of a 4-dimensional specification of a state, we necessarily assume that variations of the field-quantities (for example the variations $\delta\chi$, $\delta\overline{\chi}$ and $\delta\Phi$) at any point of space-time will influence the state-vector Ω. In other words, we are here speaking of variations which are forbidden from the point of view of the principle of stationary action. Hence, in the basic statement of the problem of giving a space-time treatment of field-theory, the principle of stationary action is necessarily abandoned. A particular consequence of this is the nonexistence of equations of motion for the field operators in the 4-dimensional treatment. There are no equations for $\chi, \overline{\chi}, \Phi$, but only equations such as Eq. (3.9) and (4.9) which restrict the possible choice of state-functionals.

Our mathematical formalism can then be equivalent to an ordinary 3-dimensional treatment, only if the 3-dimensional formalism allows variations which violate the principle of stationary action. Such variations are variations of external sources or of external parameters, and the space-time treatment is therefore necessarily and closely linked with the consideration of external sources and external invariant parameters.

From Eq. (2.3) we can in the usual way deduce the equation describing the development of the system as a particular external parameter is varied. For example, let the coupling constant g of the field interaction vary. Then the equation for $\Omega(g)$, which replaces the equation for the state-vector in the usual interaction representation, becomes

$$i\partial\Omega(g)/\partial g = W_{12}^{0}\Omega(g), \tag{3.11}$$

with W_{12}^0 independent of g. $W_{12} = gW_{12}^0$ is the interaction term in the operator (2.4). Equation (3.11) has the formal solution

$$\Omega(g) \equiv S\Omega(0) = e^{iW_{12}}\Omega(0), \tag{3.12}$$

where $\Omega(0)$ is the functional describing noninteracting fields. Expressions of the form of Eq. (3.12) were obtained in several earlier investigations[1,2,3,8].

4. THE STATE-VECTOR AS A GENERALIZED FOCK FUNCTIONAL

We consider Eq. (2.3) in the representation in which the state-vector Ω is a generalized Fock functional.[6] We denote by $a^+_\rho(x)$, $b^+_\rho(x)$ the 4-dimensional creation operators of the nucleon field, and by $c^+(x)$ those of the meson fields, ρ being a spinor index. The commutation relations between these creation operators and the absorption operators $a_\rho(x)$, $b_\rho(x)$, $c(x)$, unlike the usual commutation relations, have on the right-hand side a 4-dimensional δ-function:

$$\{a_\rho(x),\, a_\sigma^+(y)\} = \delta_{\rho\sigma}\delta^4(x-y);\quad \{b_\rho(x),\, b_\sigma^+(y)\} = \delta_{\rho\sigma}\delta^4(x-y);\quad [c(x),\, c^+(y)] = \delta^4(x-y). \tag{4.1}$$

All other commutators vanish. The transformation properties of the operators $a(x)$, $b(x)$ and $c(x)$ are fixed by requiring that the quantities $[\Omega', a(x)\Omega]$, $[\Omega', b(x)\Omega]$ transform like a bispinor and a conjugate bispinor, while $[\Omega', c(x)\Omega]$ transforms like a pseudoscalar.*

The operators a, b, c and $a^+ b^+$, c^+ act on the functional Ω, which may be considered as a superposition of eigen-functionals of the number-operator of nucleons $a^+(x)\,a(x)d^4x$, the number-operator of antinucleons $\int b^+(x)\,b(x)\,d^4x$, and that of mesons $\int c^+(x)\,c(x)\,d^4x$:

$$\Omega = \sum \Omega_{nmk} = \sum_{nmk} (n!\,m!\,k!)^{-1/2} \times \int d^4x_1 \dots d^4x_n d^4y_1 \dots d^4y_m d^4z_1 \dots d^4z_k \times f(x_1 \dots x_n \,|\, y_1 \dots y_m \,|\, z_1 \dots z_k) \times a^+(x_1) \dots a^+(x_n)\, b^+(y_1) \dots b^+(y_m) \times c^+(z_1) \dots c^+(z_k)\,\Omega_0. \tag{4.2}$$

Here Ω_0 is the normalized functional of the vacuum state, determined by the conditions $a(x)\,\Omega_0 = 0$, $b(x)\,\Omega_0 = 0$, $c(x)\,\Omega_0 = 0$, $(\Omega_0, \Omega_0) = 1$. The function $f(x_1 \dots x_n \,|\, y_1 \dots y_m \,|\, z_1 \dots z_k)$, which plays the role of the coefficients $C(\xi')$ in Eq. (2.2), is the Feynman amplitude for n nucleons with coordinates $x_1, \dots, x_n$, m antinucleons with coordinates $y_1, \dots, y_m$, and k-mesons with coordinates $z_1, \dots, z_k$. The function $f(x \dots | y \dots | z \dots)$ is antisymmetric in the x and y coordinates, and symmetric in the z coordinates. Coester[1] introduced a functional Ω of the form of Eq. (4.2).

The scalar product (Ω, Ω') of the generalized functionals Ω and Ω' is defined in analogy to the three-dimensional case[6]. If $f(x \dots | y \dots | z \dots)$ are the functions in the expansion (4.2) of Ω, and if

$$\Omega'^+ = \sum_{nmk} \int \Omega_0^+ c(z_k) \dots b(y) \dots a(x)\, f'^*(x \dots | y \dots | z \dots) \tag{4.3}$$

is the functional conjugate to Ω', then

$$(\Omega', \Omega) = \sum_{nmk} \int f'^*(x_1 \dots x_n \,|\, y_1 \dots y_m \,|\, z_1 \dots z_k) \times f(x_1 \dots x_n \,|\, y_1 \dots y_m \,|\, z_1 \dots z_k) \times dx_1 \dots dx_n dy_1 \dots dy_m dz_1 \dots dz_k. \tag{4.4}$$

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* One may call the operator $a(x)$ a bispinor, in the same sense in which the Dirac matrices γ_ν are sometimes said to constitute a vector.

To obtain the explicit form of Eq. (2.3) for a generalized Fock functional Ω, one must establish the correspondence between the creation and absorption operators and the causal operators χ, $\bar{\chi}$ and Φ. The correspondence is based on the supposition that the operators χ, $\bar{\chi}$ and Φ can be separated into creation parts χ^c, $\bar{\chi}^c$ and Φ^c and absorption parts χ^a, $\bar{\chi}^a$ and Φ^a, all the absorption parts commuting or anticommuting with each other, and all the creation parts likewise. This supposition is essential to the argument. It limits the possible form of the operators χ, $\bar{\chi}$ and Φ, and restricts the form of the commutation relations between the creation and absorption parts of these operators. From the conditions (2.1) and (4.1) we deduce that these commutation relations must be

$$\{\chi^a(x),\ \bar{\chi}^c(y)\} = \sigma_F(x, y);\qquad [\Phi^a(x),\ \Phi^c(y)] = d_F(x, y);\qquad \{\bar{\chi}^a(x),\ \chi^c(y)\} = -\sigma_F(y, x) \tag{4.5}$$

where the quantities σ_F and d_F are functions and not field-operators. The functions $\sigma_F(x, y)$ and $d_F(x, y)$ are given by

$$\sigma_F(x, y) = (\Omega_0,\ \chi(x)\bar{\chi}(y)\Omega_0) = -(\Omega_0,\ \bar{\chi}(y)\chi(x)\Omega_0);$$

$$d_F(x, y) = (\Omega_0,\ \Phi(x)\Phi(y)\Omega_0) = (\Omega_0,\ \Phi(y)\Phi(x)\Omega_0).$$

These equations show that σ_F and d_F are Green's functions of causal, or Feynman, type.

The commutation relations (4.3) will be obeyed if we assume $\chi^a(x) \equiv a(x)$, $\bar{\chi}^a(x) \equiv b(x)$, $\Phi^a(x) \equiv c(x)$, and represent the operators χ, $\bar{\chi}$ and Φ in the form

$$\chi(x) = a(x) - \int \sigma_F(x, y)\, b^+(y)\, d^4y;\qquad \bar{\chi}(x) = b(x) + \int a^+(y)\, \sigma_F(y, x)\, d^4y;\qquad \Phi(x) = c(x) + \int d_F(x, y)\, c^+(y)\, d^4y. \tag{4.6}$$

This representation of the causal operators was already given by Coester[1]. The choice of the functions σ_F and d_F, with the operators χ, $\bar{\chi}$ and Φ given by Eq. (4.6), fixes the meaning of the basic states to which the functionals Ω_{nmk} are referred. Usually d_F and σ_F are referred to the noninteracting fields, in which case $d_F = \tfrac{1}{2}\Delta_F$ and $\sigma_F = -\tfrac{1}{2}S_F$; however, renormalization is then impossible. To obtain a renormalizable theory one must choose[9] $\sigma_F = -\tfrac{1}{2}S'_F$ and $d_F = \tfrac{1}{2}\Delta'_F$.

Remembering that

$$f(x_1 \ldots x_n \mid y_1 \ldots y_m \mid z_1 \ldots z_k) = (n!\ m!\ k!)^{-1/2}(\Omega_0,\ c(z_k)\ldots c(z_1)\, b(y_m) \ldots b(y_1)\, a(x_n) \ldots a(x_1)\,\Omega), \tag{4.7}$$

we find the expression for an infinitesimal transformation operator

$$G = -\int [\delta a(x)\, a^+(x) + \delta b(x)\, b^+(x) + \delta c(x)\, c^+(x)]\, d^4x. \tag{4.8}$$

Since the variations $\delta a(x)$, $\delta b(x)$ and $\delta c(x)$ are independent, the action principle (2.3) is equivalent to the following system of equations for Ω:

$$\{D(x)\chi(x) - g\gamma_5\Phi(x)\chi(x)\}\,\Omega = -b^+(x)\,\Omega;$$
$$\{D(-x)\bar{\chi}(x) - g\Phi(x)\bar{\chi}(x)\gamma_5\}\,\Omega = a^+(x)\,\Omega;$$
$$\{i(\Box - \mu^2)\Phi(x) + gN[\bar{\chi}(x)\gamma_5\chi(x)]\}\,\Omega = -c^+(x)\,\Omega. \tag{4.9}$$

These equations were originally proposed by Coester[1].

Equations (4.9) imply an infinite system of coupled equations for the functions $f(x\ldots \mid y\ldots \mid z\ldots)$. These coupled equations are a generalization of the equations obtained in Fock's method of functionals[6]. They differ from Fock's equations in two ways: first, not only the meson field but also the nucleon field is treated by means of second quantization; second, the amplitudes are Feynman amplitudes of four-dimensional type*. Inserting the expansion (4.2) of the functional Ω into the first equation (4.9), we obtain the equations for the coefficients $f(x_1 \ldots x_n \mid y_1 \ldots y_m \mid z_1 \ldots z_k)$ with $(k, n, m = 0, 1, 2, \ldots \infty)$,

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* The equations for the lowest Feynman amplitudes were obtained by Zimmerman (see Ref. 9) without using a four-dimensional method of quantization.

$$\sqrt{n+1}\,D(x)\,f(xx_1\ldots x_n\mid y_1\ldots y_m\mid z_1\ldots z_k) \tag{4.10}$$
$$-\frac{(-1)^n}{\sqrt{m}}[\{D(x)\sigma_F(x,y_1)$$
$$-\delta^4(x-(y_1)\}f(x_1\ldots x_n\mid y_2\ldots y_m\mid z_1\ldots z_k)]_y$$
$$=g\gamma_5(x)\{\sqrt{n+1}\sqrt{k+1}$$
$$\times f(xx_1\ldots x_n\mid y_1\ldots y_m\mid xz_1\ldots z_k)$$
$$-(-1)\,\frac{\sqrt{k+1}}{\sqrt{m}}$$
$$\times[\sigma_F(x,y_1)f(x_1\ldots x_n\mid y_2\ldots y_m\mid xz_1\ldots z_k)]_y$$
$$+[f(xx_1\ldots x_n\mid y_1\ldots y_m\mid z_2\ldots z_k)d_F(x,z_1)]_z$$
$$-\frac{(-1)^n}{\sqrt{mk}}[f(x_1\ldots x_n\mid y_2\ldots y_m\mid z_2\ldots z_k)$$
$$\times\sigma_F(x,y_1)d_F(x,z_1)]_{y,z}\}.$$

The factor $(-1)^n$ arises from the fact that the operator $b^+(y)$ must be computed through n operators a in order to arrive at the beginning of the b^+ operators. The notation $[\ \ ]_{y,z}$ means that the expression inside the brackets must be symmetrized in the meson coordinates (z) and antisymmetrized in the antinucleon coordinates (y); for example,

$$[\sigma_F(x,y_1)f(\ldots\mid y_2\ldots y_m\mid\ldots)]_y$$
$$=\sigma_F(x_1y_1)f(\ldots\mid y_2\ldots y_m\mid\ldots)$$
$$-\sigma_F(x,y_2)f(\ldots\mid y_1y_3\ldots y_m\mid\ldots)$$
$$+\sigma_F(x,y_3)f(\ldots\mid y_1y_2y_4\ldots y_m\mid\ldots)-\ldots].$$

The system of equations for $f(x\ldots|y\ldots|z\ldots)$ arising from the second equation (4.9) is

$$(-1)^n\sqrt{m+1} \tag{4.11}$$
$$\times D(-y)f(x_1\ldots x_n\mid yy_1\ldots y_m\mid z_1\ldots z_k)$$
$$+\frac{1}{\sqrt{n}}[\{D(-y)\sigma_F(x_1,y)$$
$$-\delta^4(x_1-y)\}f(x_2\ldots x_n\mid y_1\ldots y_m\mid z_1\ldots z_k)]_x$$
$$=g\gamma_5(y)\{\sqrt{m+1}\sqrt{k+1}\,(-1)^n$$
$$\times f(x_1\ldots,x_n\mid yy_1\ldots y_m\mid yz_1\ldots z_k)$$
$$+(-1)^n[d_F(y,z_1)$$
$$f(x_1\ldots x_n\mid yy_1\ldots y_m\mid z_2\ldots z_k)]_z$$
$$+n^{-1/2}[\sigma_F(x_1,y)$$
$$\times f(x_2\ldots x_n\mid y_1\ldots y_m\mid yz_1\ldots z_k)]_x$$
$$+(nk)^{-1/2}[d_F(y,z_1)$$
$$\times f(x_2\ldots x_n\mid y_1\ldots y_m\mid z_2\ldots z_k)\sigma_F(x_1-y)]_{x,z}\}.$$

The system of equations for $f(x\ldots|y\ldots|z\ldots)$ equivalent to the third equation (4.9) is

$$i(\Box_z-\mu^2) \tag{4.12}$$
$$\times f(x_1\ldots x_n\mid y_1\ldots y_m\mid zz_1\ldots z_k)$$
$$\times\sqrt{k+1}+\frac{1}{\sqrt{k}}[\{i(\Box_z-\mu^2)d_F(z,z_1)$$
$$-\delta^4(z-z_1)\}f(x_1\ldots x_n\mid y_1\ldots y_m\mid z_2\ldots z_k)]_z$$
$$=g\{\sqrt{m+1}\sqrt{n+1}\,\gamma_5(z)$$
$$\times f(zx_1\ldots x_n\mid zy_1\ldots y_m\mid z_1\ldots z_k)(-1)^n$$
$$-(-1)^n(nm)^{-1/2}[\sigma_F(x_1,z)\gamma_5(z)\sigma_F(z,y_1)$$
$$\times f(x_2\ldots x_n\mid y_2\ldots y_m\mid z_1\ldots z_k)]_{x,y}$$
$$+[\sigma_F(x_1,z)\gamma_5(z)$$
$$\times f(z_2x_2\ldots x_n\mid y_1\ldots y_m\mid z_1\ldots z_k)]^*_x$$
$$+[\gamma_5(z)\sigma_F(z,y_1)$$
$$\times f(x_1\ldots x_n\mid zy_2\ldots y_m\mid z_1\ldots z_k)]^*_y\},$$

where $[\ \ ]^*_x$ and $[\ \ ]^*_y$ mean that the expression in the first bracket must be antisymmetrized in the coordinates ($zx_1\ldots x_n$), and the expression in the second bracket must be antisymmetrized in ($zy_1\ldots y_m$).

Equations (4.10), (4.11) and (4.12) have a very clumsy appearance. They can be simplified a little by suitable choice of the functions $\sigma_F(x,y)$ and $d_F(x,y)$. Compared with the functions $T(x\ldots|y\ldots|z\ldots)$ for which equations were derived by Zimmerman[9] and others[10], the functions $f(x\ldots|y\ldots|z\ldots)$ have the advantage of referring to an orthogonal system of states. A detailed investigation of Eqs. (4.10)-(4.12) will be published separately.

5. THE ENERGY-MOMENTUM VECTOR

In the usual three-dimensional treatment of quantum field theory, the expression for the energy-momentum vector is derived from the Lagrangian, and the identification of the energy-momentum vector as a displacement operator is a consequence of the canonical commutation relations. In our four-dimensional treatment of quantum field theory there is no variation principle, no canonical commutation relations, and no equations of motion of the field-operators. We therefore define the energy-momentum vector $P_\mu (P_4 = iP_0)$ as a quantity possessing the properties of a displacement operator,

$$-i[P_\mu, a(x)] = \partial a(x)/\partial x_\mu; \tag{5.1}$$

$$-i[P_\mu, a^+(x)] = \partial a^+(x)/\partial x_\mu;$$

$$-i[P_\mu, b(x)] = \partial b(x)/\partial x_\mu;$$

$$-i[P_\mu, c^+(x)] = \partial c^+(x)/\partial x_\mu,$$

etc., and with components commuting with one another, $[P_\mu, P_\nu] = 0$. The vacuum state Ω_0 must satisfy the condition

$$P_\mu \Omega_0 = 0. \tag{5.2}$$

We determine the operator P_μ by considering the variation $\delta_{(\mu)}\Omega$ produced in any state-vector (4.2) by variations of the operators a^+, b^+, c^+ as a result of an infinitesimal displacement of coordinates $x_\mu \to x_\mu + \delta x_\mu^0$. The displacement δx_μ^0 is independent of the point x. The variation $\delta_{(\mu)}\Omega$ will be equal to

$$\delta_{(\mu)}\Omega = -iP_\mu \delta x_\mu^0 \Omega.$$

The energy-momentum tensor which satisfies the conditions (5.1) and (5.2) is, by virtue of Eq. (4.2),

$$P_\mu = -i \int \Big\{a^+(x)\frac{\partial a(x)}{\partial x_\mu} + b^+(x)\frac{\partial b(x)}{\partial x_\mu} + c^+(x)\frac{\partial c(x)}{\partial x_\mu}\Big\} d^4x. \tag{5.3}$$

In momentum-space this becomes

$$P_\mu = \int \{a^+(q)\,a(q) + b^+(q)\,b(q) + c^+(q)\,c(q)\}\, q_\mu d^4q.$$

Here q_0 is an independent variable.

It is important to distinguish the variation $\delta_{(\mu)}\Omega$ from the total variation $\delta_\mu\Omega$ connected with the displacement $x_\mu \to x_\mu + \delta x_\mu^0$. The total variation $\delta_\mu\Omega$ is zero, because the variation $\delta_\mu W$ of the action is zero then the coordinates are displaced, by virtue of the conservation law for energy and momentum [see Eq. (2.3)]. To obtain $\delta_\mu\Omega$ explicitly, one must first consider the state-vector (4.2) and the action (2.4), with each four-dimensional integration extending not over the whole space-time but over a volume bounded by two hypersurfaces σ_1 and σ_2. The limiting process which extends the integration to the whole space-time is only to be performed afterwards. Then $\delta_\mu\Omega$ and $\delta_\mu W$ refer to a rigid displacement of σ_1 and σ_2 through the vector δx_μ. The equation $\delta_\mu\Omega = 0$ means that the derivatives of the integrand of Eq. (3.2) with respect to every coordinate x_μ, y_μ, z_μ are zero. Therefore, the effect of the operator P_μ on the state-vector Ω is to differentiate the amplitudes $f(x\ldots|y\ldots|z\ldots)$. If $P_\mu\Omega = \Omega'$, and if $f'(x\ldots|y\ldots|z\ldots)$ are the amplitudes of the functional Ω', then

$$f'(x_1\ldots x_n \mid y_1\ldots y_m \mid z_1\ldots z_k) = -i\Big[\sum_n \frac{\partial}{\partial x_\mu} + \sum_m \frac{\partial}{\partial y} + \sum_k \frac{\partial}{\partial z_\mu}\Big] \times f(x_1\ldots x_n \mid y_1\ldots y_m \mid z_1\ldots z_k). \tag{5.4}$$

The meaning of P_μ as an energy-momentum vector can easily be verified by considering the example of noninteracting fields. In the special case of a free nucleon field, for example, we find by using the equation of motion (4.7)

$$i\int a^+(x)\frac{\partial a(x)}{\partial x_0} d^4x\Omega = \int a^+(x)\,[\alpha \mathbf{p} + \gamma_4 m]\, a(x)\,\Omega.$$

The operator Q, representing the total charge of the nucleon field, is obtained, just as in the usual three-dimensional treatment, by considering an infinitesimal phase transformation of the causal operators. The result is

$$Q = \int [a^+(x)\,a(x) - b^+(x)\,b(x)]\, d^4x. \tag{5.5}$$

CONCLUSION

The theory of the operators χ, $\bar{\chi}$ and Φ is a consistent space-time formulation of the quantum theory of fields, closely connected with the usual formulation of field theory with external sources. But there is an important difference between the roles played by the external sources in the two theories. In the usual three-dimensional theory, the external sources $\eta(x)$, $\bar{\eta}(x)$, $J(x)$ are auxiliary quantities, external to the theory itself. In the four-dimensional treatment, these quantities are derived from the fundamental operators $\chi(x)$, $\bar{\chi}(x)$, $\Phi(x)$ and from the creation and absorption operators a, b, c, a^+, b^+, c^+. This may be considered an advantage of the four-dimensional theory.

One may expect that the mathematical formalism of the operators χ, $\bar{\chi}$ and Φ will be suitable for constructing field theories without using representations which refer to "bare" particles. Lehmann, Zimmermann and Symanzik[11] attempted to construct a theory of this kind. Starting from general conditions of invariance, causality and asymptotic behavior of field operators at $t = \pm\infty$, they obtained a variety of equations connecting matrix elements of T-products. In the space-time treatment, these equations can be derived very easily by using the representation in which the state-vector is a Fock functional. As an example we give the derivation of the reduction formula which occupies a central position in the work of Lehmann *et al.*[11]. By definition,

$$T(x\ldots|y\ldots|z\ldots) \tag{6.1}$$

$$= (\Omega_0, \chi(x)\ldots\bar{\chi}(y)\ldots\Phi(z)\ldots\Omega(g))$$

$$= (\Omega_0, \chi(x)\ldots\bar{\chi}(y)\ldots\Phi(z)\ldots S\Omega(0)),$$

where $\Omega(0)$ is a functional of the noninteracting fields, and S is the scattering matrix [see Eq. (3.12)], the detailed form of which is irrelevant. The important point is that S commutes with χ, $\bar{\chi}$ and Φ. We assume for simplicity that $\Omega(0)$ refers to a state of one nucleon with wave-function $f_1^0(x')$ and one meson with wave-function $f_3^0(z')$. Then Eqs. (4.2) and (4.9) give

$$\Omega(0) = \int f_1^0(x') f_3^0(z') a^+(x') c^+(z') d^4x' d^4z' \, \Omega_0 \tag{6.2}$$

$$= \int D(x')\chi(x') \times i(\Box_{z'} - \mu^2)\Phi(z')\Omega_0 f_1^0(x') f_3^0(z') d^4x' d^4z'$$

and

$$T(x\ldots|y\ldots|z\ldots) \tag{6.3}$$

$$= \int D(x') i(\Box_{z'} - \mu^2)(\Omega_0, \chi(x) \ldots\bar{\chi}(y)\ldots\Phi(z)\ldots\chi(x') \times \Phi(z') S\Omega_0) f_1^0(x') f_3^0(z') d^4x' d^4z'.$$

This is the required reduction formula[11]. The asymptotic condition is here equivalent to the requirement that the wave-functions $f_1^0(x')$ and $f_3^0(z')$ are composed of positive-energy components only.

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