Three Body Problem for Short Range Forces. I. Scattering of Low Energy Neutrons by Deuterons

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An exact solution is obtained for the three body problem in the limiting case of a vanishingly small radius of action of the forces. In this case, the Schrödinger equation for the system of three particles reduces, for motion with a definite momentum, to an integral equation for a function of a single variable. The solution is used for the calculation of the neutron-deuteron scattering cross section. In the limiting case of zero energy of the neutrons, the theory gives the values $a_{3/2} = 0.51 \times 10^{-12}$ cm, $a_{1/2} = 0.30 \times 10^{-12}$cm for the scattering amplitudes.

THE problem of the motion of two nucleons at low energy E has a solution in the form of a series in powers of the small parameters r_0/λ and αr_0, where r_0 is the radius of action of the nuclear forces, $\lambda = \hbar/\sqrt{ME}$ is the wavelength, and $1/\alpha$ is the radius of the bound state or scattering length.

In the zeroth approximation, which corresponds to $r_0 \to 0$, the wave function of the two-nucleon system can be constructed beyond the range of action of the forces if we put upon it the boundary condition

$$\left\{\frac{d}{d\rho} \ln(\rho\Psi)\right\}_{\rho=0} = -\alpha \tag{1}$$

and consider that the potential in the Schrödinger equation has the form:

$$U^{(0)}(\rho) = -(4\pi\hbar^2/M)\,\rho\delta(\rho), \tag{2}$$

i.e., that it vanishes everywhere for $\rho \neq 0$. In this approximation, which corresponds to the Bethe-Peierls theory of the deuteron[1], the properties of the system do not depend upon the details of the path of the actual potential function $U(\rho)$ and are determined by the value of only a single parameter α.

In the next approximation the wave function can be determined in an expansion in powers of r_0 (outside the range of action of the forces) if the potential in the Schrödinger equation remains in the form (2) as before, but the boundary condition is made more precise:

$$\left[\frac{d}{d\rho} \ln(\rho\Psi)\right]_{\rho=0} = -\alpha + {}^1/_2(\alpha^2 + \lambda^{-2})\, r_0 + T r_0^3 \lambda^{-4} + \ldots, \tag{1a}$$

where α, r_0 and T are parameters which characterize the actual potential $U(\rho)$.

A similar expansion in powers of r_0 can be carried out for the problem of the motion of three nucleons at small values of the energy E, when the characteristic dimensions of the system, which are defined by the length $\bar{\lambda} = \hbar/\sqrt{ME}$, greatly exceed the radius of force action, r_0. We consider below the zeroth approximation of this expansion, which corresponds to $r_0 \to 0$ (i.e., the Bethe-Peierls theory in the case of two nucleons), in application to the problem of the scattering of neutrons at low energy $E_n < 20$ mev by deuterons.

The problem of the bound state of three nucleons (the nuclei H^3 and He^3) is not considered here, since, in the approximation $r_0 \to 0$, it has no solution; for $r_0 \to 0$, the binding energy becomes infinite[2]. In this case, a more accurate analysis is required, with account being taken of terms of higher order in the expansion in r_0.

In the approximation $r_0 \to 0$ (where the effective depth $U_0 \sim \hbar^2/Mr_0^2$ of the potential well is infinite) the wave function $\Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ of the system of three nucleons satisfies the boundary condition (1), just as in the case of two nucleons, for $\rho_{ik} = |\mathbf{r}_i - \mathbf{r}_k| \to 0$ $(i, k = 1, 2, 3)$. Here the boundary condition can be constructed approximately (with accuracy up to terms of order $r_0/\bar{\lambda}$ and αr_0) outside the range of force action for a potential of the form (2) in the Schrödinger equation. As is shown below, the problem here reduces to the solution of an integral equation for a function which depends on the three variables, or, if we are dealing with a state of definite momentum, on a single variable.

For simplicity, the analysis is first carried out without consideration of the isotopic spin of the nucleons for the three particle system.

1. CASE OF THREE IDENTICAL PARTICLES. THE INTEGRAL EQUATION.

We shall describe the position of the particles in the center-of-mass system:

$$\rho_{23} = \mathbf{r}_2 - \mathbf{r}_3; \quad \rho_1 = \mathbf{r}_1 - {}^1/_2(\mathbf{r}_2 + \mathbf{r}_3),$$

where, in analogy to ρ_{23}, ρ_1, we can also introduce two pairs of vectors ρ_{12}, ρ_3 and ρ_{31}, ρ_2, the furnishing of which (along with the first pair) determines the position of the particles. Between these three systems of vectors we have the following relations

$$\begin{aligned}\rho_{23} &= -{}^1/_2\,\rho_{12} - \rho_3 = -{}^1/_2\,\rho_{31} + \rho_2,\\ \rho_1 &= {}^3/_4\rho_{12} - {}^1/_2\,\rho_3 = -{}^3/_4\,\rho_{31} - {}^1/_2\rho_2.\end{aligned} \tag{3}$$

The wave function of the system is symmetric relative to a permutation of particles; in particular,

$$\Psi(\rho_{23}, \rho_1) = \Psi(\rho_{12}, \rho_3) = \Psi(\rho_{13}, \rho_2); \tag{3a}$$

$$\Psi(\rho_{23}, \rho_1) = \Psi(-\rho_{23}, \rho_1)$$

etc., and satisfies the Schrödinger equation

$$\begin{aligned}\{-(\hbar^2/M)(\nabla^2_{\rho 23} + {}^3/_4\nabla^2_{\rho 1}) - E\}\,\Psi(\rho_{23}, \rho_1)\\ = -\{U(\rho_{23}) + U(\rho_{12}) + U(\rho_{31})\}\,\Psi(\rho_{23}, \rho_1).\end{aligned} \tag{4}$$

For what follows, it is useful to transform this equation into integral form. For this purpose, we introduce the Green's function

$$G_E(\rho_{23}, \rho_1) = \int e^{ik\rho_1}\,(e^{-\gamma_k \rho_{23}}/\rho_{23})\,d\mathbf{k}\,/\,(2\pi)^3, \tag{5}$$

where $\gamma_k = \sqrt{3k^2/4 - ME/\hbar^2}$ if $3k^2/4 - ME/\hbar^2 > 0$ and $\gamma_k = -i\sqrt{ME/\hbar^2 - 3k^2/4}$ in the opposite case*; it satisfies the condition

$$\left(\nabla^2_{\rho_{23}} + \frac{3}{4}\nabla^2_{\rho_1} + \frac{ME}{\hbar^2}\right) G_E(\rho_{23}, \rho_1) = -4\pi\delta(\rho_{23})\,\delta(\rho_1).$$

In spite of the external asymmetry of the description, the function G_E is symmetric relative to a permutation of coordinates $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$; actually, it is not difficult to prove that Eq. (5) for G_E represents an expansion in the Fourier integral of a function that is symmetric in the coordinates of the nucleons:

$$G_E(\rho_{23}, \rho_1) = \frac{2i}{3\sqrt{3}\pi}\frac{ME}{\hbar^2 R^2} H_2^{(1)}\left(\sqrt{\frac{ME}{\hbar^2}}\,R\right)$$

$$= -\frac{8}{3\sqrt{3}\pi^2}\left(\frac{ME}{\hbar^2}\right)\int_0^{2\infty} \exp = \left\{i\tau\frac{ME}{\hbar^2}R^2 + \frac{i}{4\tau}\right\}\tau\,d\tau,$$

where $H_2^{(1)}$ is an Hankel function and $R^2 = \rho_{23}^2 + 4/3\rho_1^2 \equiv 4/3(r_1^2 + r_2^2 + r_3^2 - \mathbf{r}_1\cdot\mathbf{r}_2 - \mathbf{r}_1\cdot\mathbf{r}_3 - \mathbf{r}_2\cdot\mathbf{r}_3)$, and where it must be kept in mind that $\sqrt{ME/\hbar^2} = +i\sqrt{M(-E)/\hbar^2}$, if E is negative (in order that G_E decrease and not increase as $R \to \infty$; this corresponds to the definition of γ_k given above).

By means of the function G_E, the Schrödinger equation (4) can be written in the form*

$$\Psi(\rho_{23}, \rho_1) = \int G_E(\rho_{23} - \rho'_{23}; \rho_1 - \rho'_1)\,\{u(\rho'_{23}) + u(\rho'_{12}) + u(\rho'_{31})\} \times \Psi(\rho'_{23}, \rho'_1)\,d\rho'_{23}\,d\rho'_1. \tag{6}$$

Here we have introduced the notation $u(\rho) = -(M/4\pi\hbar^2)U(\rho)$ where, for $r_0 \to 0$, $u(\rho) \to \delta\rho(\rho)$.

Substituting the function (5) in (6), and transforming to the Fourier representation of the wave function,

$$F(\rho_{23}, \mathbf{k}) = \int e^{-i\mathbf{k}\cdot\rho_1}\Psi(\rho_{23}, \rho_1)\,d\rho_1,$$

we get [taking into account the symmetry of Ψ relative to r_1, r_2, r_3 and making use of Eq. (3)], after

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* In order that G_E have the form of a diverging wave at large ρ_{23}.

* It would be necessary to add the plane wave

$$\Psi_0 = \exp(i\mathbf{k}_0\rho_1 + i\mathbf{f}_0\rho_{23}),\ f_0^2 + 3k_0^2/4 = ME/\hbar^2,$$

to the right-hand side of Eq. (6) if we were interested in the state $\Psi(\rho_{23}, \rho_1)$, which corresponds to the motion of all three nucleons to infinity with definite momenta (such a state can arise, for example, in the photoeffect on H^3 or He^3). In the problem of the scattering of a particle on the bound state of two others (or in the problem of the bound state of all three particles) the right side of (6) ought not to contain the term Ψ_0.

some transformations*,

$$F(\rho, \mathbf{k}) \tag{7}$$

$$= \int \frac{\exp\{-\gamma_k |\rho - \rho'|\}}{|\rho - \rho'|} u(\rho') F(\rho', \mathbf{k}) d\rho' + 8\pi \int \frac{d\mathbf{k}'}{(2\pi)^3}$$

$$\times \frac{\cos(\mathbf{k}' + \mathbf{k}/2, \rho) \int \cos(\mathbf{k} + \mathbf{k}'/2, \rho') u(\rho') F(\rho', \mathbf{k}') d\rho'}{k^2 + k'^2 + \mathbf{k}\mathbf{k}' - ME/\hbar^2 - i\tau}$$

or

$$\{\nabla_\rho^2 - 4\pi u(\rho) + ME_k/\hbar^2\} F(\rho, \mathbf{k}) \tag{8}$$

$$= -8\pi \int \frac{d\mathbf{k}'}{(2\pi)^3} \cos(\mathbf{k}' + \mathbf{k}/2, \rho) \times \int \cos(\mathbf{k} + \mathbf{k}'/2, \rho') u(\rho') F(\rho', \mathbf{k}') d\rho'.$$

Here $E_k = E - 3\hbar^2 k^2/4M$; for brevity, the indices on ρ_{23} are omitted everywhere; the quantity $\tau > 0, \tau \to 0$ furnishes the rule in (7) for detouring the poles; (8) is obtained from (7) by application of the operator $\nabla_\rho^2 - \gamma_k^2 = \nabla_\rho^2 + ME_k/\hbar^2$. The form of Eqs. (7) and (8) of the Schrödinger equation is suitable for consideration of the case in which $u(\rho)$ is the short range potential. It follows from Eq. (7) that the function $F(\rho, \mathbf{k})$ is defined for arbitrary ρ if its values are known in the region of action of the forces, $\rho \leq r_0$. In this region, we can limit consideration, with accuracy up to terms of order $(r_0/\lambda\!\!\!^-)^3$, to the spherically symmetric part of F:

* The first term in Eq. (7) follows trivially from the term in (6) which contains the factor $u(\rho'_{23})$; the Fourier component of the following term in (6), which contains $u(\rho'_{12})$ is transformed, by means of (3) and (3a), into the form

$$\int \exp\left[i\mathbf{k}\left(\frac{3}{4}\rho'_{12} - \frac{1}{2}\rho'_3\right)\right] \frac{\exp[-\gamma_k |\rho_{23} + {}^1/_2\rho'_{12} + \rho'_3|]}{|\rho_{23} + {}^1/_2\rho'_{12} + \rho'_3|} \times u(\rho'_{12}) \Psi(\rho'_{12}, \rho'_3) d\rho'_{12} d\rho'_3 =$$

$$= \int \frac{d\mathbf{k}\, 4\pi \exp[-i(\mathbf{k}' + \mathbf{k}/2, \rho_{23})]}{(2\pi)^3 (\kappa^2 + k'^2 + \mathbf{k}\mathbf{k}' - ME/\hbar^2 - i\tau)} \times \int \exp[-i(\mathbf{k} + \frac{1}{2}\mathbf{k}', \rho'_{12})] u(\rho'_{12}) F(\rho'_{12}, \mathbf{k}') d\rho'_{12}.$$

Along with the Fourier component of the third term in Eq. (6), this expression gives the second component in Eq. (7).

$$F_0(\rho, \mathbf{k}) = \int F(\rho, \mathbf{k}) d\Omega_\rho / 4\pi$$

[in order to establish this, we can substitute in Eq. (7)

$$F(\rho_1, \mathbf{k}) = \sum_{l=0}^{\infty} (2l+1) F_l(\rho, \mathbf{k}) P_l(\rho\mathbf{k}/\rho k)$$

and estimate the contribution from terms with $l \neq 0$].

But in the region $\rho \leq r_0$, we can neglect the right-hand side of Eq. (8). Actually, substituting unity for the cosines in Eq. (8), we get for it the estimate

$$\sim \int (2\pi)^{-3} d\mathbf{k} \int u(\rho') F_0(\rho', \mathbf{k}') d\rho' \sim r_0 \int (2\pi)^{-3} d\mathbf{k}' F_0(r_0, \mathbf{k}') \sim$$

$$\sim r_0^{-2} (r_0/\lambda\!\!\!^-)^3 F_0(r_0, \mathbf{k}).$$

[In obtaining this estimate, it has been taken into account that $\int u(\rho') d\rho' \sim r^0$ and also that k and k' are everywhere of order α (or $1/\lambda\!\!\!^-$, if $\lambda\!\!\!^{-\,-1} > \alpha$); for large values of k, the function $F_0(\rho, \mathbf{k})$ vanishes or oscillates.]

Comparing this estimate with any of the terms on the left-hand side of Eq. (8),

$$\nabla_\rho^2 F \sim u(\rho) F \sim r_0^{-2} F \quad (\rho \lesssim r_0),$$

$$(ME_k/\hbar^2) F \sim r_0^{-2} (r_0/\lambda\!\!\!^-)^2 F,$$

we note that the right-hand side of Eq. (8) differs by the factor $(r_0/\lambda\!\!\!^-) < 1$ from the smallest term $(ME_k/\hbar^2)F$ on the left-hand side.

Thus, in the zeroth approximation, we do not have to take the right-hand side of Eq. (8) into consideration. But this means that the function $F_0(\rho, \mathbf{k})$, which is regular for $\rho = 0$, satisfies the same equation

$$[d^2/d\rho^2 - 4\pi u(\rho) + ME_k/\hbar^2] [\rho F_0(\rho, \mathbf{k})] = 0 \tag{9}$$

in the region $\rho \lesssim r_0$ as the function of the system of two particles, i.e., for $\rho = r_0$, it has the same value (1) of the logarithmic derivative as the wave function for the two-particle system. Physically, neglect of the right-hand side of Eq. (8) [i.e., use of Eq. (1)] corresponds to neglect of random penetrations of the third particle into the region of interaction of the other two.

Therefore, in accord with Eq. (1), we have, in the approximation $r_0 \to 0$,

$$\left\{\frac{d}{d\rho}[\rho F_0(\rho,\mathbf{k})]\right\}_{\rho=0} = -\alpha[\rho F_0(\rho,\mathbf{k})]_{\rho=0}. \qquad (10)$$

In this approximation, we have, by Eqs. (2) and (7),

$$F(\rho,\mathbf{k}) = \frac{\exp(-\gamma_k\rho)}{\rho}\chi(\mathbf{k}) \qquad (11)$$

$$+ 8\pi\int\frac{d\mathbf{k}'}{(2\pi)^3}\frac{\chi(\mathbf{k}')\cos(\mathbf{k}'+\mathbf{k}/2,\rho)}{k^2+k'^2+\mathbf{k}\mathbf{k}'-(ME/\hbar^2)-i\tau},$$

$$\chi(\mathbf{k}) = \lim_{\rho\to 0}\{\rho F(\rho,\mathbf{k})\} = \lim_{\rho\to 0}\{\rho F_0(\rho,\mathbf{k})\}.$$

Substituting Eq. (11) in Eq. (10), we get an integral equation for $\chi(\mathbf{k})$:

$$(\alpha-\gamma_k)\chi(\mathbf{k}) \qquad (12)$$

$$+ 8\pi\int\frac{d\mathbf{k}'}{(2\pi)^3}\frac{\chi(\mathbf{k}')}{k^2+k'^2+\mathbf{k}\mathbf{k}'-(ME/\hbar^2)-i\tau} = 0.$$

The solution of this equation determines the wave function of the system, in accord with Eq. (11). For states with a definite quantity of momentum, Eq. (12) reduces to an equation for a function which depends on one independent variable, which can be solved numerically.

The idea for this consideration of the three body problem was supplied by L. D. Landau.

2. CASE OF THREE IDENTICAL PARTICLES. THE SCATTERING OF A PARTICLE BY THE BOUND STATE OF THE OTHER TWO.

If $\alpha > 0$, then there exists a bound state of two particles with binding energy $\epsilon = \hbar^2\alpha^2/M$. This state is described, beyond the region of action of the forces, by the wave function

$$\varphi_0(\rho) = \sqrt{\alpha/2\pi}\exp(-\alpha\rho)/\rho.$$

Let us consider the problem of the scattering of a third particle by such a system.

Let $\hbar\,\mathbf{k}_0$ be the momentum relative to the motion of the incident particle. Then

$$E = \hbar^2 k_0^2/2\mu - \hbar^2\alpha^2/M; \quad \mu = {}^2/_3 M,$$

$$\text{i.e.,}\quad ME/\hbar^2 = -[\alpha^2 - 3k_0^2/4].$$

The wave function of the system of three particles Ψ, for large ρ_1 (or ρ_2 or ρ_3), ought to undergo transition to the product of the function φ_0 and the sum of a plane and a diverging wave. This requirement will be satisfied if Eq. (11) has a solution of the form

$$\chi(\mathbf{k}) = \sqrt{\frac{\alpha}{2\pi}}\left\{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}_0) + \frac{4\pi a(\mathbf{k},\mathbf{k}_0)}{k^2-k_0^2-i\tau}\right\}, \qquad (13)$$

where the amplitude of elastic scattering is

$$a(\vartheta) = \lim_{k^2\to k_0^2}\frac{k^2-k_0^2}{4\pi}\int\varphi_0^*(\rho)F(\rho,\mathbf{k})\,d\rho;$$

for such a choice, $\chi(\mathbf{k})$ is seen equal to

$$a(\vartheta) = [a(\mathbf{k},\mathbf{k}_0)]_{k=k_0};$$

ϑ is the angle between $\mathbf{k}$ and $\mathbf{k}_0$ [the second term in Eq. (11) vanishes after multiplication by $k^2-k_0^2$, if $k^2\to k_0^2$].

In a similar way, the inelastic scattering amplitude is determined by the equation

$$a(\mathbf{k}_f,\mathbf{f};\mathbf{k}_0 d) = \left\{\frac{k^2-k_f^2}{4\pi}\int\varphi_\mathbf{f}^*(\rho)F(\rho,\mathbf{k})\,d\rho\right\}_{k\to k_f}$$

Here, in the approximation $r_0\to 0$,

$$\varphi_\mathbf{f} = e^{if\rho} + a_f e^{if\rho}/\rho, \quad a_f = -1/(\alpha+if)$$

is the wave function of the final state of two particles with relative momentum $\mathbf{f}$:

$$^3/_4 k_f^2 + f^2 = {}^3/_4 k_0^2 - \alpha^2;$$

$$k_f = \sqrt{k_0^2 - {}^4/_3(\alpha^2+f^2)},$$

k_f is the momentum of the particle after inelastic scattering. This amplitude can also be expressed by $a(\mathbf{k},\mathbf{k}_0)$, with the aid of Eqs. (11) and (12):

$$a(\mathbf{k}_f,\mathbf{f},\mathbf{k}_0 d) = \sqrt{8\pi a}\left\{\frac{a(\mathbf{k}_f,\mathbf{k}_0)}{(f+i\alpha)^2} + \frac{a(\mathbf{f}-{}^1/_2\mathbf{k}_f,\mathbf{k}_0)}{{}^3/_4[(\mathbf{f}-{}^1/_2\mathbf{k}_f)^2-\mathbf{k}_0^2]} + \frac{a(-\mathbf{f}-{}^1/_2\mathbf{k}_f,\mathbf{k}_0)}{{}^3/_4[(\mathbf{f}+{}^1/_2\mathbf{k}_f)^2-k_0^2]}\right\}.$$

Substituting (13) in (12), we get the inhomogeneous equation

$$\frac{{}^3/_8\, a(\mathbf{k},\mathbf{k}_0)}{\alpha+\sqrt{3k^2/4-ME/\hbar^2}} \qquad (14)$$

$$= \frac{1}{k^2+k_0^2+\mathbf{k}\mathbf{k}_0-(ME/\hbar^2)-i\tau}$$

$$+ 4\pi\int\frac{a(\mathbf{k}',\mathbf{k}_0)}{(k^2+k'^2+\mathbf{k}\mathbf{k}'-(ME/\hbar^2)-i\tau)(k'^2-k_0^2-i\tau)} \times\frac{d\mathbf{k}'}{(2\pi)^3},$$

whose solution, in accord with Eqs. (13) and (11), determines (in zeroth approximation in r_0) the wave function of the system and all the scattering amplitudes. This equation has a much simpler form for $k_0 \to 0$, $ME/\hbar^2 \to -\alpha^2$, when the scattering is spherically symmetric.

If we introduce the dimensionless quantity

$$a(\mathbf{k}, 0) = a(k, \hat{0}) = \alpha^{-1}\tilde{a}(x), \quad x = k/\alpha,$$

we get

$$\frac{{}^3/_8\, \tilde{a}(x)}{\sqrt{3x^2/4+1}+1} = \frac{1}{1+x^2} \tag{15}$$

$$+\frac{2}{\pi}\int_0^\infty \frac{1}{2xx'} \ln\frac{1+x^2+x'^2+xx'}{1+x^2+x'^2-xx'}\, \tilde{a}(x')\, dx',$$

where the scattering amplitude of particles with zero energy is determined by the value $\tilde{a}$ (0):

$$a_0 = a(0, 0) = \alpha^{-1}\tilde{a}(0).$$

The function $\tilde{a}(x)$ is plotted in Fig. 1. This function was obtained by numerical integration of Eq. (15). According to Fig. 1,

$$\tilde{a}(0) = \left(\frac{8}{3}\right)\left[1+\frac{4}{\pi}\int_0^\infty \frac{\tilde{a}(x')\,dx'}{1+x'^2}\right] \approx 2.5.$$

For $k_0 \neq 0$, we can expand the function $a(\mathbf{k}, \mathbf{k}_0)$ in a series of Legendre polynomials:

$$a(\mathbf{k}, \mathbf{k}_0) = \sum_{l=0}^\infty (2l+1)\, a_l(k, k_0)\, P_l(\cos\vartheta). \tag{16}$$

The equations for $a_l(k, k_0)$ follow from (14), in analogy to Eq. (15). These can be solved numerically (this is done in the Appendix at the end of this paper). In contrast to Eq. (15), the kernel of these equations is complex*, so that the quantities $a_l(k, k_0)$ are complex for $k_0 \neq 0$.

It should be noted that Eq. (12) formally has the form of a three-dimensional Schrödinger equation. Actually, introducing the function

$$\psi(\rho_1) \tag{17}$$

$$= \int\left[\frac{4\pi}{\alpha+\sqrt{3k^2/4 - ME/\hbar^2}}\right]^{1/2} \chi(\mathbf{k})\, e^{i\mathbf{k}\rho_1}\, d\mathbf{k}/(2\pi)^3,$$

which, for $\rho_1 \to \infty$ has the form [in accord with Eq. (13)]:

$$\psi(\rho_1) \sim e^{i\mathbf{k}_0\rho_1} + a(\mathbf{k}_0', \mathbf{k}_0)\, e^{ik_0\rho_1}/\rho_1,$$

$\mathbf{k}' = k_0\rho_1/\rho_1$, i.e., it describes the elastic scattering of a particle on the bound state of the other two; we note that Eq. (12) is equivalent to the Schrödinger equation for $\psi(\rho_1)$ in the usual form:

$$[-(\hbar^2/2\mu)\nabla^2_{\rho 1} + \hat{V}]\,\psi(\rho_1) = (\hbar^2 k_0^2/2\mu)\,\psi(\rho_1),$$

where $\mu = 2M/3$ is the reduced mass. Here $\hat{V}$ is the interaction operator (of one particle with the bound state of the two others), defined by the matrix elements

$$\int e^{i\mathbf{k}\rho_1}\hat{V}e^{-i\mathbf{k}\rho_1}\, d\rho_1 \tag{18}$$

$$= -\frac{8\pi\hbar^2}{M}\frac{[(\alpha+\gamma_k)(\alpha+\gamma_{k'})]^{1/2}}{(K^2/4)+3q^2-(ME/\hbar^2)-i\tau} = V(\mathbf{K}, \mathbf{q}),$$

$$\mathbf{k} = \mathbf{q}+\mathbf{K}/2, \ \ \mathbf{k}' = \mathbf{q}-\mathbf{K}/2, \ \ \hat{V}\,\psi(\rho_1)$$

$$= \int V(\rho_1, \rho_1')\,\psi(\rho_1')\, d\rho_1',$$

$$V(\rho_1, \rho_1') = \iint \exp\{i\mathbf{K}(\rho_1+\rho_1')/2$$

$$+ i\mathbf{q}(\rho_1-\rho_1')\}\, V(\mathbf{K}, \mathbf{q})(2\pi)^{-6}\, d\mathbf{K}\, d\mathbf{q}.$$

The operator $\hat{V}$ is Hermitian only when the energy $\hbar^2 k_0^2/2\mu$ of the particle is less than the energy ϵ binding the other two, i.e., under the condition

$$E = \hbar^2 k_0^2/2\mu - \varepsilon < 0;$$

in the converse case (when inelastic scattering is impossible) the potential (18) is complex*.

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* For $k_0 \neq 0$, the quantity

$$k'^2[k'^2-k_0^2-i\tau]^{-1} \equiv P\frac{k'^2}{k'^2-k_0^2} + i\pi k'^2\delta(k'^2-k_0^2)$$

in Eq. (14) is complex (for $k_0 = 0$, it reduces to unity).

* For its crude estimate, we set $V(\mathbf{K}, \mathbf{q}) \approx V(\mathbf{K}, 0)$. This gives $V(\rho_1) \approx V(\rho_1)\psi(\rho_1)$ where, for $V(\rho_1)$, we get the estimate $V(\rho_1) \approx -(16\epsilon/\alpha\rho_1)\, e^{-2\alpha\rho_1}$ for $k_0 = 0$ and $V(\rho_1) \sim \exp(i\rho_1/\lambda\!\!\!^{-})$ for $3k_0^2/4 \gg \alpha^2$. For $k_0 = 0$, the potential falls off with increase in ρ_1 as the density of the bound state $|\varphi_0|^2$.

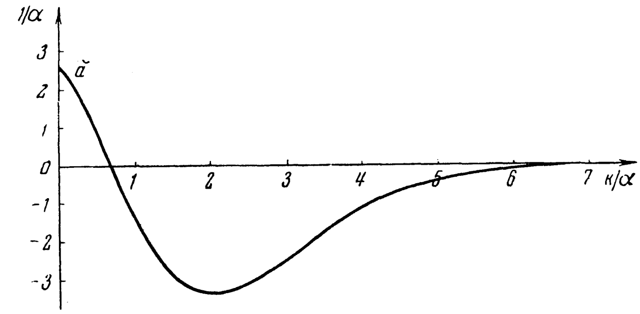

FIG. 1.

3. CONSIDERATION OF SPIN AND ISOTOPIC SPIN

The previous analysis can be generalized to the case of the motion of three nucleons, in which spin and isotopic spin are taken into account. We assume that central forces (we do not consider non-pair forces, triple forces and also tensor forces) act between each pair of nucleons. The potential for such forces,

$$\hat{U}(\rho) = U_1(\rho) + (\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\, U_2(\rho) \qquad (19)$$

$$+ (\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\, U_3(\rho) + (\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\, U_4(\rho)$$

corresponds to the hypothesis of isotopic invariance of nuclear forces; here U_1, U_2, U_3 and U_4 are certain functions of ρ which vanish for $\rho > r_0$, and σ and τ are the operators of spin and isotopic spin, respectively. In this case, the total spin S, the isotopic spin T, and their projections on the OZ axis---S_z and T_z---will be constants of the motion.

Let $\alpha_i = \alpha(\xi_i)$ and $\beta_i = \beta(\xi_i)$ be the spin wave functions of the ith nucleon (ξ_i is the spin variable) corresponding to the spin projections $S_z = 1/2$ and $S_z = -1/2$. The wave functions of the system of three nucleons can be constructed from α_i and β_i by making use of the well-known values of the coefficients of the Clebsch-Gordan series; for $S_z = 1/2$, we get two functions in the case $S = 1/2$:

$$\chi_1 = \frac{1}{\sqrt{2}}(\alpha_2\beta_3 - \beta_2\alpha_3)\,\alpha_1,$$

$$\chi_2 = \frac{1}{\sqrt{3}}\alpha_1\frac{1}{\sqrt{2}}(\alpha_2\beta_3 + \beta_2\alpha_3) - \sqrt{\frac{2}{3}}\,\beta_1\alpha_2\alpha_3,$$

and one in the case $S = 3/2$:

$$\chi_3 = \sqrt{\frac{2}{3}}\alpha_1\frac{1}{\sqrt{2}}(\alpha_2\beta_3 + \beta_2\alpha_3) + \frac{1}{\sqrt{3}}\beta_1\alpha_2\alpha_3.$$

Of these, χ_3 is symmetric relative to a permutation $\hat{P}^{(\sigma)}_{ik}$ of the spin variables of an arbitrary pair of nucleons: $\hat{P}^{(\sigma)}_{ik}\chi_3 \equiv \chi_3$, while χ_1 and χ_2 transform lineraly into each other upon permutations: χ_1 is antisymmetric and χ_2 is symmetric relative to the permutation $\hat{P}^{(\sigma)}_{23}$ so that the permutation $\hat{P}^{(\sigma)}_{23}$ corresponds to the matrix $M_{23} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ of a linear transformation of the functions χ_1 and χ_2; similarly, it is not difficult to prove that the matrices

$$M_{13} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \text{ and } M_{12} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}.$$

correspond to the permutations $\hat{P}^{(\sigma)}_{13}$ and $\hat{P}^{(\sigma)}_{12}$. We note that the functions χ_1 and χ_2 and χ_3 can be obtained as a result of application of the operators

$$\hat{P}^{(\sigma)}_1 = \frac{1}{\sqrt{2}}(\hat{P}^{(\sigma)}_{13} - \hat{P}^{(\sigma)}_{12}), \qquad (20)$$

$$\hat{P}^{(\sigma)}_2 = \frac{1}{\sqrt{6}}(\hat{P}^{(\sigma)}_{13} + \hat{P}^{(\sigma)}_{12}) - \sqrt{\frac{2}{3}}\,\hat{P}^{(\sigma)}_{23},$$

and the symmetrization operator

$$\hat{P}^{(\sigma)}_3 = \frac{1}{\sqrt{3}}(\hat{P}^{(\sigma)}_{13} + \hat{P}^{(\sigma)}_{12} + \hat{P}^{(\sigma)}_{23})$$

to the function $\beta_1\alpha_2\alpha_3$, which is symmetric in the spins of the 2nd and 3rd nucleons. Similar functions of isotopic spin for $T_z = 1/2$ (this value of T_z corresponds to a system of two neutrons and one proton) are denoted by ϑ_1 and ϑ_2 for $T = 1/2$ and ϑ_3 for $T = 3/2$.

The spin and isotopic spin wave functions Φ are obtained by multiplication of the functions χ and ϑ;

we have, evidently,

for $T = 3/2$, $S = 3/2$, the function $\tilde{\Phi}_c = \chi_3 \vartheta_3$;
for $T = 3/2$, $S = 1/2$, two functions $\tilde{\Phi}'_1 = \chi_1 \vartheta_3$ and $\tilde{\Phi}'_2 = \chi_2 \vartheta_3$;
for $T = 1/2$, $S = 3/2$, two functions $\Phi'_1 = \chi_3 \vartheta_1$ and $\Phi'_2 = \chi_3 \vartheta_2$;
for $T = 1/2$, $S = 1/2$, four functions $\chi_1\vartheta_1$, $\chi_2\vartheta_2$, $\chi_2 \vartheta_1$ and $\chi_1 \vartheta_2$.

In the latter case, instead of the four functions listed, it is appropriate to choose (as the basic system) linear combinations of them, which possess definite properties relative to permutations. Such combinations are[4]:

$$\Phi_a = \frac{1}{\sqrt{2}}(\chi_1\vartheta_2 - \chi_2\vartheta_1); \quad \Phi_1 = \frac{1}{\sqrt{2}}(\chi_1\vartheta_2 + \chi_2\vartheta_1);$$
$$\Phi_c = \frac{1}{\sqrt{2}}(\chi_1\vartheta_1 + \chi_2\vartheta_2); \quad \Phi_2 = \frac{1}{\sqrt{2}}(\chi_1\vartheta_1 - \chi_2\vartheta_2)$$

[the factor $1/\sqrt{2}$ is introduced for normalization: $(\Phi^+, \Phi) = 1$]. As is not difficult to show, Φ_a is antisymmetric relative to simultaneous permutation of the spin and isotopic spin of an arbitrary pair of nucleons, Φ_c is completely symmetric, while the pair of functions Φ_1 and Φ_2 transform into each other in the same fashion as the functions χ_1 and χ_2 (i.e., with the help of the matrices M_{23}, M_{13} and M_{12}). Evidently, the pairs Φ'_1, Φ'_2 and $\tilde{\Phi}'_1$, $\tilde{\Phi}'_2$ possess this same property of transformation.

It should be noted that if Ψ_1 and Ψ_2 (functions of the coordinates of the nucleons), which transform into one another upon a permutation $\hat{P}^{(r)}_{ik}$ of the coordinates $\mathbf{r}_i$ and $\mathbf{r}_k$ ($i, k = 1, 2, 3$; $i \neq k$), in the same way as χ_1 and χ_2 transform under a permutation of the corresponding spins, then we can construct (in analogy to Φ_a, constructed from χ and ϑ) the function

$$\Psi_1\Phi_2 - \Psi_2\Phi_1,$$

which will be antisymmetric relative to the permutation $\hat{P}^{(\sigma)}_{ik}\hat{P}^{(\tau)}_{ik}\hat{P}^{(r)}_{ik}$ of all coordinates of any pair of nucleons. Taking this into account, we write the antisymmetric function of a system of three nucleons in the form:

$$\begin{aligned}
T &= 3/2, & S &= 3/2: & \Psi_{3/2,\,3/2} &= \tilde{\Psi}_a\tilde{\Phi}_c, & (21)\\
T &= 3/2, & S &= 1/2: & \Psi_{3/2,\,1/2} &= \tilde{\Psi}'_1\tilde{\Phi}'_1 - \tilde{\Psi}'_2\tilde{\Phi}'_1, &\\
T &= 1/2, & S &= 3/2: & \Psi_{1/2,\,3/2} &= \Psi'_1\Phi'_2 - \Psi'_2\Phi'_1 &\\
T &= 1/2, & S &= 1/2: & \sqrt{2}\,\Psi_{1/2,\,1/2} &= \Psi_a\Phi_c - \Psi_c\Phi_a &\\
& & & & &\quad + \Psi_1\Phi_2 - \Psi_2\Phi_1. &
\end{aligned}$$

All the functions $\Psi_{T,S}$ will be antisymmetric relative to a permutation $\hat{P}^{(\sigma)}_{ik}\hat{P}^{(\tau)}_{ik}\hat{P}^{(r)}_{ik}$ of all coordinates of any pair of nucleons if, upon permutation $\hat{P}^{(r)}_{ik}$ of the coordinates $\mathbf{r}_i$, $\mathbf{r}_k$, the functions $\tilde{\Psi}_a$ and Ψ_a will be antisymmetric, Ψ_c will be symmetric and the pairs Ψ_1 and Ψ_2; Ψ'_1 and Ψ'_2; $\tilde{\Psi}'_1$ and $\tilde{\Psi}'_2$ will transform linearly one into the other, just as χ_1 and χ_2 transform upon the corresponding permutation of spins (the factor $\sqrt{2}$ for $\Psi_{1/2,1/2}$ is written for normalization purposes).

The isotopic spin of the deuteron is equal to zero, and that of the neutron is one-half. Therefore, keeping in mind the problem of deuteron-neutron scattering, it suffices to limit ourselves to a consideration of the latter two of the states described above, i.e., the states with $T = 1/2$. We find the coordinate part of Ψ_c, Ψ_a, Ψ_1, Ψ_2 and Ψ'_1, Ψ'_2 corresponding to the wave functions.

The wave function is a solution of the Schrödinger equation which can be written in the form (6) if by $U(\rho)$ is understood the potential (19).

In the approximation $r_0 \to 0$, in the first term on the right-hand side of (6),

$$\int (G_E(\rho_{23} - \rho'_{23}; \rho_1 - \rho'_1) \times \frac{\{-MU(\rho'_{23})\}}{4\pi\hbar^2}\Psi(\rho'_{23}, \rho'_1)\,d\rho'_{23}\,d\rho'_1,$$

only the region $\rho'_{23} \leq r_0 \to 0$ will be essential for the integration; therefore, the dependence of this term on the coordinates ρ_{23} and ρ_1 is determined by the relation

$$f_{23} = f(\rho_{23}, \rho_1) = \int G_E(\rho_{23}, \rho_1 - \rho) f(\rho)\,d\rho, \quad (22)$$

where

$$f(\rho) = \lim_{\rho_{23}\to 0}\{\rho_{23}\Psi(\rho_{23}, \rho)\}$$

is some unknown function. Similarly, the dependence of the coordinates of the following two terms in (6) is determined by the functions

$$f_{13} = f(\rho_{13}, \rho_2) \text{ and } f_{12} = f(\rho_{12}, \rho_3).$$

In other words, according to Eq. (6), the coordinate part of the wave function is represented by some linear combination of terms of the form f_{23}, f_{13} and f_{12}. In principle, we could, by substituting the potential (19) in Eq. (6) and multiplying successively both parts of (6) by different functions Ψ of the spin and the isotopic spin, obtain integral equations for the coordinate parts of Ψ only, which would also define the form of the linear combination described above. However, this could also be done, not by writing out the equation for the coordinate part, but by making use only of the symmetry properties of the function Ψ. Since the function Ψ, being symmetric relative to $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, ought to be equal to the sum $f_{23} + f_{12} + f_{13}$ of the functions f_{ik} which are determined by a single function $f(\rho)$, since only this sum remains unchanged upon the permutation $\hat{P}^{(r)}_{ik}$ of the coordinates of an arbitrary pair of nucleons:

$$\Psi_c = f_{23} + f_{13} + f_{12}. \tag{23}$$

The antisymmetric Ψ_a in the coordinates of the nucleons is a function identically equal to zero, since, in accord with (5) a function of the type (22) is symmetric in $\mathbf{r}_2$ and $\mathbf{r}_3$: $f(\rho_{23},\rho_1) = f(-\rho_{23},\rho_1)$ and as a result of its antisymmetrization in $\mathbf{r}_2$ and $\mathbf{r}_3$, we obtain zero. With the help of integral equations for the coordinate parts of Ψ of the wave function we can show that this result is valid with accuracy up to terms of order $(r_0/\lambda\!\!\!^-)^3$ in comparison with unity.

The functions Ψ_1 and Ψ_2, which transform upon permutation like χ_1 and χ_2, are represented in the form of a linear combination of functions of the type (22), φ_{23}, φ_{13} and φ_{12} [i.e., $\varphi_{23} = \int G_E(\rho_{23}, \rho_1 - \rho)\,\varphi(\rho)\,d\rho$, where $\varphi(\rho)$ is an unknown function]. Introducing the operators $\hat{P}^{(r)}_1$ and $\hat{P}^{(r)}_2$ which are analogous to (20) but which permute the coordinates of the nucleons, we evidently get $\Psi_1 = \sqrt{3/2}\,\hat{P}^{(r)}_1 \varphi_{23}$; $\Psi_2 = \sqrt{3/2}\,\hat{P}^{(r)}_2 \varphi_{23}$ [just the same as χ_1 and χ_2 could be obtained as a result of the action of $\hat{P}^{(\sigma)}_1$ and $\hat{P}^{(\sigma)}_2$ on the function $\beta_1\alpha_2\alpha_3$, which is symmetric in the spins of nucleons 2 and 3; the factor $\sqrt{3/2}$ was chosen for normalizing the functions $\varphi(\rho)$]. Thus,

$$\Psi_1 = (\sqrt{3}/2)(\varphi_{12} - \varphi_{13}), \tag{24}$$

$$\Psi_2 = {}^1\!/_2(\varphi_{12} + \varphi_{13}) - \varphi_{23}.$$

In similar fashion, Ψ_1' and Ψ' are expressed by the functions φ_{ik}', where

$$\varphi_{23}' = \int G_E(\rho_{23}, \rho_1 - \rho)\,\varphi'(\rho)\,d\rho;$$

$\varphi'(\rho)$ is a third unknown function.

Just as in the case of three identical particles, it is appropriate to consider the Fourier component

$$F(\rho_{23}, \mathbf{k}) = \int e^{-i\mathbf{k}\rho_1}\Psi(\rho_{23}, \rho_1)\,d\rho_1$$

of the wave functions in the coordinate ρ_1 of any one of the three nucleons (in view of the symmetry, all are equally suitable); by Eqs. (22), (23) and (24), and using the value (5) of the Green's function G_E, we get [just as was done above for the Fourier components F obtained from the values of (7), (8)]:

$$F_a = 0, \quad F_c = \frac{e^{-\gamma_k \rho_{23}}}{\rho_{23}}\chi(\mathbf{k}) \tag{25}$$

$$+ 8\pi \int \frac{\chi(\mathbf{k}')\cos(\mathbf{k}' + \mathbf{k}/2, \rho_{23})}{k^2 + k'^2 + \mathbf{k}\mathbf{k}' - (ME/\hbar^2) - i\tau}\frac{d\mathbf{k}'}{(2\pi)^3},$$

$$F_1 = 4\pi i\sqrt{3}\int \frac{\xi(\mathbf{k}')\sin(\mathbf{k}' + \mathbf{k}/2, \rho_{23})}{k^2 + k'^2 + \mathbf{k}\mathbf{k}' - (ME/\hbar^2) - i\tau}\frac{d\mathbf{k}'}{(2\pi)^3},$$

$$F_2 = 4\pi \int \frac{\xi(\mathbf{k}')\cos(\mathbf{k}' + \mathbf{k}/2, \rho_{23})}{k^2 + k'^2 + \mathbf{k}\mathbf{k}' - (ME/\hbar^2) - i\tau}\frac{d\mathbf{k}'}{(2\pi)^3} - \frac{e^{-\gamma_k \rho_{23}}}{\rho_{23}}\xi(\mathbf{k}),$$

and F_1' and F_2' (the Fourier components of Ψ_1' and Ψ_2') have values similar to F_1 and F_2 upon replacement of $\xi(\mathbf{k})$ by $\xi'(\mathbf{k})$; here $\chi(\mathbf{k})$, $\xi(\mathbf{k})$ and $\xi'(\mathbf{k})$ are functions analogous to $\chi(\mathbf{k})$ in the case of identical particles; these are Fourier components, correspondingly, of $f(\rho)$, $\varphi(\rho)$ and $\varphi'(\rho)$.

Substituting the value of the function $F_{T,S}$ in Eq. (21), we write down the Fourier component of the wave function $F_{T,S}$ of a system of three nucleons for the case $T = 1/2$. For $S = 1/2$;

$$F_{1/2,1/2} = {}^1\!/_2 F_1(\chi_1\vartheta_1 - \chi_2\vartheta_2) \tag{26}$$

$$- {}^1\!/_2(F_c + F_2)\chi_1\vartheta_2 + {}^1\!/_2(F_c - F_2)\chi_2\vartheta_1;$$

for $S = 3/2$:

$$F_{^1/_2,\,^3/_2} = F_1'\chi_3\vartheta_2 - F_2'\chi_3\vartheta_1. \qquad (26')$$

We can find the functions $\chi(\mathbf{k})$, $\xi(\mathbf{k})$ and $\xi'(\mathbf{k})$ by imposing on the wave function the boundary condition similar to (1) or, more precisely, (1a). In order to obtain this, we note that, similar to the way in which Eq. (9) follows from the integral equation (6), there follow from (6), upon substitution of a potential in the form (19), equations for the coefficients

$$F_t = {}^1/_2(F_c - F_2) \text{ and } F_s = -{}^1/_2(F_c + F_2)$$

for the case of $\chi_2\ \vartheta_1$ and $\chi_1\ \vartheta_2$ in Eq. (26):

$$[(d^2/d\rho^2) - (M/\hbar^2)\,U_t(\rho) + (ME_k/\hbar^2)]\,\{\rho F_t(\rho, \mathbf{k})\} \approx 0, \qquad (27)$$
$$[(d^2/d\rho^2) - (M/\hbar^2)\,U_s(\rho) + (ME_k/\hbar^2)]\,\{\rho F_s(\rho, \mathbf{k})\} \approx 0.$$

Here U_t and U_s are potentials acting between the neutron and the proton in the triplet and singlet states:

$$U_t(\rho_{23}) = (\chi_2^*\vartheta_1^*\hat{U}_{23}\chi_2\vartheta_1) = U_1 + U_2 - 3U_3 - 3U_4,$$

$$U_s(\rho_{23}) = (\chi_1^*\vartheta_2^*\hat{U}_{23}\chi_1\vartheta_2) = U_1 - 3U_2 + U_3 - 3U_4.$$

Equations (27) are approximate, valid with accuracy up to terms of order $(r_0/\lambdabar)^3$ in the region of force action [since, on the right-hand side, there are actually small terms in the exact equations similar to the term on the right side of Eq. (8)]. We can also obtain an equation for the function F_2' in the coefficient for $\chi_3\ \vartheta_1$ in Eq. (26′). This equation is identical with the first of Eq. (27). The validity of Eq. (27), analogous to Eq. (9), is almost self-evident; its precise demonstration is connected with several rough calculations and is not carried out here.

It follows from Eq. (27) that the functions F_t and F_s satisfy boundary conditions of the type (10) at the point $\rho = r_0 \to 0$:

$$\left\{\frac{d}{d\rho_{23}}[\rho_{23}F_t(\rho_{23}, \mathbf{k})]\right\}_{\rho_{23}\to 0} = -\alpha_t\{\rho_{23}F_t(\rho_{23}, \mathbf{k})\}_{\rho_{23}\to 0}, \qquad (28)$$
$$\left\{\frac{d}{d\rho_{23}}[\rho_{23}F_s(\rho_{23}, \mathbf{k})]\right\}_{\rho_{23}\to 0} = -\alpha_s\{\rho_{23}F_s(\rho_{23}, \mathbf{k})\}_{\rho_{23}\to 0},$$

where α_t and α_s are reciprocals of the scattering length (in the approximation $r_0 \to 0$, the difference between the scattering length and the radius of the bound state disappears), and are constants which characterize the potentials U_s and U_t. The function $F_t' = -F_2'$ satisfies the boundary condition identical to the condition (28) for F_t.

Substituting (25) in (28), we get, for $S = 1/2$:

$$\left(\sqrt{\frac{3k^2}{4} - \frac{ME}{\hbar^2}} - \alpha_t\right)\{\chi(\mathbf{k}) + \xi(\mathbf{k})\} = \int \frac{4\pi\,\{2\chi(\mathbf{k}') - \xi(\mathbf{k}')\}}{k^2 + k'^2 + \mathbf{k}\mathbf{k}' - (ME/\hbar^2) - i\tau}\,\frac{d\mathbf{k}'}{(2\pi)^3}, \qquad (29)$$
$$\left(\sqrt{\frac{3k^2}{4} - \frac{ME}{\hbar^2}} - \alpha_s\right)\{\chi(\mathbf{k}) - \xi(\mathbf{k})\} = \int \frac{4\pi\,\{2\chi(\mathbf{k}') + \xi(\mathbf{k}')\}}{k^2 + k'^2 + \mathbf{k}\mathbf{k}' - (ME/\hbar^2) - i\tau}\,\frac{d\mathbf{k}'}{(2\pi)^3},$$

and for $S = 3/2$,

$$\left(\sqrt{\frac{3k^2}{4} - \frac{ME}{\hbar^2}} - \alpha_t\right)\xi'(\mathbf{k}) + \int \frac{4\pi\xi'(\mathbf{k}')}{k^2 + k'^2 + \mathbf{k}\mathbf{k}' - (ME/\hbar^2) - i\tau}\,\frac{d\mathbf{k}'}{(2\pi)^3} = 0.$$

The solutions of these equations determine the wave function of the system in accord with Eqs. (25) and (26). In the case $S = 1/2$, the equations can be written in a more appropriate form if we denote:

$$\chi_t(\mathbf{k}) = {}^1/_2\{\chi(\mathbf{k}) + \xi(\mathbf{k})\},$$
$$\chi_s(\mathbf{k}) = {}^1/_2\{\chi(\mathbf{k}) - \xi(\mathbf{k})\};$$

then we obtain

$$\left(\sqrt{\frac{3k^2}{4} - \frac{ME}{\hbar^2}} - \alpha_t\right)\chi_t(\mathbf{k}) = \int \frac{4\pi\,\{{}^1/_2\chi_t(\mathbf{k}') + {}^3/_2\chi_s(\mathbf{k}')\}}{k^2 + k'^2 + \mathbf{k}\mathbf{k}' - (ME/\hbar^2) - i\tau}\,\frac{d\mathbf{k}'}{(2\pi)^3}; \qquad (30)$$
$$\left(\sqrt{\frac{3k^2}{4} - \frac{ME}{\hbar^2}} - \alpha_s\right)\chi_s(\mathbf{k}) = \int \frac{4\pi\,\{{}^3/_2\chi_t(\mathbf{k}') + {}^1/_2\chi_s(\mathbf{k}')\}}{k^2 + k'^2 + \mathbf{k}\mathbf{k}' - (ME/\hbar^2) - i\tau}\,\frac{d\mathbf{k}'}{(2\pi)^3}.$$

Here, we have everywhere

$$\sqrt{\frac{3k^2}{4}-\frac{ME}{\hbar^2}} \to -i\sqrt{\frac{ME}{\hbar^2}-\frac{3k^2}{4}},$$

$$\text{if}\quad \frac{ME}{\hbar^2}-\frac{3k^2}{4}>0.$$

Let us consider the problem of the scattering of neutrons by deuterons. For large ρ_1 the wave function of the system must contain the incident wave (which is proportional to $\chi_2\ \vartheta_1$ if $S = 1/2$, and to $\chi_3\ \vartheta_1$ if $S = 3/2$) and various diverging waves. In accord with (25) and (26), this will occur if (29) and (30) have solutions of the form

$$\xi'(\mathbf{k}) = \sqrt{\frac{\alpha_t}{2\pi}}\left\{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}_0)+\frac{4\pi a_{3/2}(\mathbf{k},\mathbf{k}_0)}{k^2-k_0^2-i\tau}\right\};$$

$$\chi_t(\mathbf{k}) = \sqrt{\frac{\alpha_t}{2\pi}}\left\{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}_0)+\frac{4\pi a_{1/2}(\mathbf{k},\mathbf{k}_0)}{k^2-k_0^2-i\tau}\right\};$$

$$\chi_s(\mathbf{k}) = \sqrt{\frac{\alpha_t}{2\pi}}\,\frac{4\pi b_{1/2}(\mathbf{k},\mathbf{k}_0)}{k^2-k_0^2-i\tau} \qquad (31)$$

(the function $\chi_s(\mathbf{k})$ actually (as is evident from Eq. (30), where $ME/\hbar^2 = 3k_0^2/4 - \alpha_t^2$), has no pole for $k = k_0$, i.e., $\{b_{1/2}(\mathbf{k},\mathbf{k}_0)\}_{k=k_0} = 0$ is described in the form written above for one-dimensional notation]. Here the amplitudes of elastic scattering of neutrons are:

$$f_{3/2}(\vartheta) = \lim_{k\to k_0}\left\{\frac{k^2-k_0^2}{4\pi}(\chi_3^*\vartheta_1^*\int\varphi_0^*(\rho_{23})\,F_{1/2,\,3/2}(\rho_{23},\mathbf{k})\,d\rho_{23})\right\},$$

$$f_{1/2}(\vartheta) = \lim_{k\to k_0}\left\{\frac{k^2-k_0^2}{4\pi}(\chi_2^*\vartheta_1^*\int\varphi_0^*(\rho_{23})\,F_{1/2,\,1/2}(\rho_{23},\mathbf{k})\,d\rho_{23})\right\},$$

where $\varphi_0(\rho_{23}) = \sqrt{\alpha_t/2\pi}\,e^{-\alpha_t\rho_{23}}/\rho_{23}$ is the wave function of the deuteron, and, by Eqs. (25), (26) and (31), are equal to

$$f_{3/2}(\vartheta) = \{a_{3/2}(\mathbf{k},\mathbf{k}_0)\}_{k=k_0},$$

$$f_{1/2}(\vartheta) = \{a_{1/2}(\mathbf{k},\mathbf{k}_0)\}_{k=k_0}.$$

Substituting (31) in (29), (30), we get inhomogeneous equations which permit us to find the wave functions of the complicated spectrum:

$$S = {}^3/_2:\ \frac{(\sqrt{3k^2/4-(ME/\hbar^2)}-\alpha_t)}{k^2-k_0^2}\,a_{3/2}(\mathbf{k},\mathbf{k}_0) = \frac{-1}{k_0^2+k^2+\mathbf{k}\mathbf{k}_0-(ME/\hbar^2)-i\tau}$$
$$-\int\frac{4\pi a_{3/2}(\mathbf{k}',\mathbf{k}_0)}{(k^2+k'^2+\mathbf{k}\mathbf{k}'-(ME/\hbar^2)-i\tau)\,(k'^2-k_0^2-i\tau)}\frac{d\mathbf{k}'}{(2\pi)^3}; \qquad (32)$$

$$S = {}^1/_2\left\{\begin{aligned}
&\frac{(\sqrt{3k^2/4-(ME/\hbar^2)}-\alpha_t)}{k^2-k_0^2}\,a_{1/2}(\mathbf{k},\mathbf{k}_0) = \frac{1/2}{k_0^2+k^2+\mathbf{k}\mathbf{k}_0-(ME/\hbar^2)-i\tau}\\
&\qquad+\int\frac{4\pi\{{}^1/_2a_{1/2}(\mathbf{k}',\mathbf{k}_0)+{}^3/_2b_{1/2}(\mathbf{k}',\mathbf{k}_0)\}}{(k^2+k'^2+\mathbf{k}\mathbf{k}'-(ME/\hbar^2)-i\tau)\,(k'^2-k_0^2-i\tau)}\frac{d\mathbf{k}'}{(2\pi)^3},\\
&\frac{(\sqrt{3k^2/4-(ME/\hbar^2)}-\alpha_s)}{k^2-k_0^2}\,b_{1/2}(\mathbf{k},\mathbf{k}_0) = \frac{3/2}{k_0^2+k^2+\mathbf{k}\mathbf{k}_0-(ME/\hbar^2)-i\tau}\\
&\qquad+\int\frac{4\pi\{{}^3/_2a_{1/2}(\mathbf{k}',\mathbf{k}_0)+{}^1/_2b_{1/2}(\mathbf{k}',\mathbf{k}_0)\}}{(k^2+k'^2+\mathbf{k}\mathbf{k}'-(ME/\hbar^2)-i\tau)\,(k'^2-k_0-i\tau)}\frac{d\mathbf{k}'}{(2\pi)^3}.
\end{aligned}\right.$$

For $k_0 \to 0$, similar to the case of three identical particles, we can introduce the dimensionless quantities

$$\tilde{a}_{3/2}(x) = \alpha_t a_{3/2}(k,0),$$

$$\tilde{a}_{1/2}(x) = \alpha_t a_{1/2}(k,0),$$

$$\tilde{b}_{1/2}(x) = \alpha_t b_{1/2}(k,0); \quad x = k/\alpha_t,$$

for which it follows that

$$S = {}^3/_3: \frac{\sqrt{3x^2/4+1}-1}{x^2}\tilde{a}_{^3/_2}(x) = \frac{-1}{x^2+1} - \frac{1}{\pi}\int_0^\infty \tilde{a}_{^3/_2}(x)\ln\frac{1+x^2+x'^2+xx'}{1+x^2+x'^2-xx'}\frac{dx'}{xx'};$$

$$S = {}^1/_2 \begin{cases} \dfrac{\sqrt{3x^2/4+1}-1}{x^2}\tilde{a}_{^1/_2}(x) = \dfrac{^1/_2}{x^2+1} + \\ + \dfrac{1}{\pi}\displaystyle\int_0^\infty \{{}^1/_2\tilde{a}_{^1/_2}(x') + {}^3/_2\tilde{b}_{^1/_2}(x')\}\ln\frac{1+x^2+x'^2+xx'}{1+x^2+x'^2-xx'}\frac{dx'}{xx'}; \\ \dfrac{\sqrt{3x^2/4+1}-\dfrac{\alpha_s}{\alpha_t}}{x^2}\tilde{b}_{^1/_2}(x) = \dfrac{^3/_2}{x^2+1} + \\ + \dfrac{1}{\pi}\displaystyle\int_0^\infty \{{}^3/_2\tilde{a}_{^1/_2}(x') + {}^1/_2\tilde{b}_{^1/_2}(x')\}\ln\frac{1+x^2+x'^2+xx'}{1+x^2+x'^2-xx'}\frac{dx'}{xx'}. \end{cases}$$

These equations are integrated numerically. The functions $a_{3/2}(k,0)$, $a_{1/2}(k,0)$ and $b_{1/2}(k,0)$ are plotted in Fig. 2; the scattering amplitudes $a_{3/2}(0,0)$ and $a_{1/2}(0,0)$ of neutrons with $k_0=0$ by deuterons are seen to be equal to

$$a_{^3/_2}(0,0) = 0.51\cdot 10^{-12}\ \text{cm};$$

$$a_{^1/_2}(0,0) = 0.30\cdot 10^{-12}\ \text{cm}.$$

Here, for α_t and α_s, we have taken the following values: $1/\alpha_t = 4.32\times 10^{-13}$ cm; $1/\alpha_s = -25\times 10^{-13}$ cm. The results of the calculation depend weakly on the value of α_s, since this quantity enters in the form of a small parameter $-\alpha_s/\alpha_t = 0.173 < 1$ in the sum, with $\sqrt{3x^2/4+1} \gtrsim 1$.

For $k_0 \neq 0$, the functions $a_{3/2}$, $a_{1/2}$, $b_{1/2}$ can be expanded in a series of Legendre polynomials:

$$a_{^3/_2}(\mathbf{k},\mathbf{k}_0) = \sum_{l=0}^\infty (2l+1)\{C_l^{(1)}(k) + iC_l^{(2)}(k)\}P_l(\cos\vartheta); \quad (34)$$

$$a_{^1/_2}(\mathbf{k},\mathbf{k}_0) = \sum_{l=0}^\infty (2l+1)\{A_l^{(1)}(k) + iA_l^{(2)}(k)\}P_l(\cos\vartheta);$$

$$b_{^1/_2}(\mathbf{k},\mathbf{k}_0) = \sum_{l=0}^\infty (2l+1)\{B_l^{(1)}(k) + iB_l^{(2)}(k)\}P_l(\cos\vartheta).$$

The equations which determine the functions $C_l^{(1)}, C_l^{(2)}; A_l^{(1)}, A_l^{(2)}; B_l^{(1)}, B_l^{(2)}$ are developed in the Appendix.

For the case $k_0 \to 0$, the zeroth approximation in r_0 of the theory corresponds to neglect of terms of the order $\alpha_t r_0 \sim (1.7/4.34) \sim 0.4$, in comparison with unity; therefore, the theoretical values of $a_{3/2}$ and $\alpha_{1/2}$ are valid with accuracy up to terms of the order

$$(\alpha_t r_0)/\alpha_t \sim r_0 = 0.17\cdot 10^{-12}\ \text{cm}.$$

In agreement with the experimental data of Hurst and Alcock[5], $a_{3/2} = (0.64+0.01)\times 10^{-12}$ cm; $a_{1/2} = (0.07 \pm 0.03)\times 10^{-12}$ cm; thus, for spin 3/2, the correspondence with experiment is excellent; but the small amplitude of $\alpha_{1/2}$ as determined by the zeroth approximation of the theory is imprecise. Consequently, the first approximation of the theory in an expansion in r_0 will be considered later.

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APPENDIX

INTEGRAL EQUATIONS FOR THE CASE $k_0 \neq 0$

Below we obtain equations which determine the wave function of the three-particle system with $k_0 \neq 0$ and with arbitrary momentum.

First, let us consider the case of three identical particles. Making use of the expansion (16) and representing the kernel of the integral equation (14) in the form

$$[k^2 + k'^2 + \mathbf{kk}' - (ME/\hbar^2) - i\tau]^{-1} \qquad \text{(a)}$$
$$= [kk'(Q + \cos\gamma - i\tau)]^{-1} =$$
$$= \sum_{l=0}^{\infty} (2l+1) P_l(\cos\gamma) \frac{G_l(k, k')}{2kk'},$$
$$G_l(k, k') = \int_{-1}^{1} \frac{P_l(x)\,dx}{Q + x - i\tau}$$

($\tau \to 0$), where we have introduced the notation $Q(k, k') = (k^2 + k'^2 - ME/\hbar^2)/kk'$, $x = \cos\gamma = \mathbf{kk}'/kk'$, we obtain, after multiplication of (14) by $P_l(\cos\vartheta)$ and integration over the directions of the vector $\mathbf{k}$:

$$\frac{{}^3/_8\, a_l(k, k_0)}{\alpha + \sqrt{3k^2/4 - (ME/\hbar)^2}} = \frac{G_l(k, k_0)}{2kk_0} \qquad \text{(b)}$$
$$+ \frac{2}{\pi}\int_0^{\infty} \frac{G_l(k, k')}{k'^2 - k_0^2 - i\tau} \frac{a_l(k', k_0)}{2kk'} k'^2 dk'.$$

Taking it into account that

$$\frac{1}{Q + x - i\tau} = \mathrm{P}\frac{1}{Q + x} + i\pi\delta(Q + x)$$

(P denotes the "principal part"), we obtain

$$G_l(k, k') = G_l^{(1)}(k, k') + i\pi\theta(Q) P_l(-Q), \qquad \text{(c)}$$

where $G_l^{(1)}$ is substantially:

$$G_l^{(1)}(k, k') = \int_{-1}^{1} \mathrm{P}\frac{P_l(x)\,dx}{Q + x}$$
$$= \begin{cases} \ln\left|\frac{Q+1}{Q-1}\right|, & l = 0; \\ 2 - Q\ln\left|\frac{Q+1}{Q-1}\right|, & l = 1; \\ (3Q^2 - 1)\ln\left|\frac{Q+1}{Q-1}\right| + 6Q, & l = 2, \text{ etc.} \end{cases}$$

and by θ is meant the equation $\theta(Q) = 1$, for $|Q| < 1$ and $\theta(Q) = 0$, if $|Q| > 1$.

The kernel of the integral equation (b) is complex not only because, in view of (c), $Q_l(k, k')$ has an imaginary part (it differs from zero if $3k_0^2/4 > \alpha^2$; otherwise, $Q > 1$), but also because of the fact that the factor

$$\frac{k'^2}{k'^2 - k_0^2 - i\tau} = \mathrm{P}\frac{k'^2}{k'^2 - k_0^2} + i\pi k'^2\delta(k'^2 - k_0^2)$$

is complex. Therefore, the functions $a_l(k, k_0)$ are complex [except for the case $k_0 \to 0$, where they

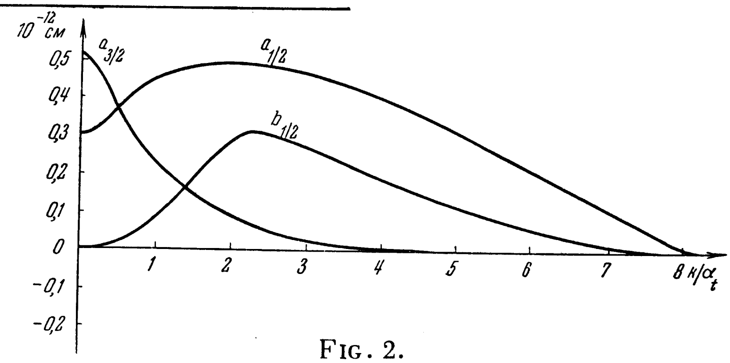

FIG. 2.

are real, by Eq. (15)]. Writing $a_l(k, k_0) = \xi_l(k) + i\eta_l(k)$, we get the following equations for ξ_l and η_l from (b) and (c):

$$\frac{(R_k - \alpha)\,\xi_l(k) + J_k\eta_l(k)}{2(k^2 - k_0^2)}$$
$$= \frac{1}{2kk_0} G_l^{(1)}(k, k_0) + \frac{2}{\pi}\int_0^{\infty} \frac{G_l^{(1)}(k, k')}{2kk'} \frac{k'^2\xi_l(k')}{(k'^2 - k_0^2)} dk'$$
$$- \frac{1}{2k} G_l^{(1)}(k, k_0)\,\eta_l(k_0)$$
$$- \int_{|J_k - k/2|}^{J_k + k/2} \frac{k'}{k} \frac{P_l(-Q)}{k'^2 - k_0}\,\eta_l(k')\,dk';$$

all the integrals are taken in the sense of the principal value (the integral with the limits from $|J_k - k/2|$ to $J_k + k/2$ arises from the imaginary part of (c): $|Q| < 1$ only if k' changes in these limits).

We treat the case of three nucleons in a similar fashion. Taking into account the expansion of Eq. (34) and denoting $C_l = C_l^{(1)} + iC_l^{(2)}$, $A_l = A_l^{(1)} + iA_l^{(2)}$, $B_l = B_l^{(1)} + iB_l^{(2)}$, we get a system of two coupled equations for $S = 3/2$:

$$\frac{(R_k-\alpha_t)\,C_l^{(1)}(k)+J_kC_l^{(2)}(k)}{k^2-k_0^2}$$

$$=-\frac{G_l^{(1)}(k,k_0)}{2kk_0}-\frac{1}{\pi}\int_0^\infty\frac{G_l^{(1)}(k,k')}{kk'}\frac{k'^2C_l^{(1)}(k')}{k'^2-k_0^2}dk'+\frac{G_l^{(1)}(k.\,k_0)}{2k}C_l^{(2)}(k_0)$$

$$+\int_{|J_k-k/2|}^{J_k+k/2}\frac{C_l^{(2)}(k')\,P_l(-Q)}{k'^2-k_0^2}\frac{k'}{k}dk';$$

$$\frac{(R_k-\alpha_t)\,C_l^{(2)}(k)-J_k\,C_l^{(1)}(k)}{k^2-k_0^2}=-\frac{1}{\pi}\int_0^\infty\frac{G_l^{(1)}(k,k')}{kk'}\frac{k'^2C_l^{(2)}(k')}{k'^2-k_0^2}dk'$$

$$-\frac{G_l^{(1)}(k,k_0)}{2k}C_l^{(1)}(k_0)-\frac{1}{\pi}\int_{|J_k-k/2|}^{|J_k+k/2|}\frac{C_l^{(1)}(k')\,P_l(-Q)}{k'^2-k_0^2}\frac{k'}{k}dk',$$

and for $S=1/2$, a system of four coupled equations:

$$\frac{(R_k-\alpha_t)\,A_l^{(1)}(k)+J_kA_l^{(2)}(k)}{k^2-k_0^2}=\frac{G_l^{(1)}(k,\,k_0)}{4kk_0}$$

$$+\frac{1}{\pi}\int_0^\infty\frac{G_l^{(1)}(k,\,k')\,k'^2\,\{{}^1/{}_2A_l^{(1)}(k')+{}^3/{}_2B_l^{(1)}(k')\}}{kk'\,(k'^2-k_0^2)}dk'-\frac{G_l^{(1)}(k,\,k_0)}{4k}A_l^{(2)}(k_0)$$

$$-\int_{|J_k-k/2|}^{J_k+k/2}\frac{P_l(-Q)\,\{{}^1/{}_2A_l^{(2)}(k')+{}^3/{}_2B_l^{(2)}(k')\}}{k'^2-k_0^2}\frac{k'}{k}dk';$$

$$\frac{(R_k-\alpha_t)\,A_l^{(2)}(k)-J_kA_l^{(1)}(k)}{k^2-k_0^2}=\frac{1}{\pi}\int_0^\infty\frac{G_l^{(1)}(k,\,k')\,k'^2\,\{{}^1/{}_2A_l^{(2)}(k')+{}^3/{}_2B_l^{(2)}(k')\}}{kk'\,(k'^2-k_0^2)}dk'$$

$$+\frac{G_l^{(1)}(k,k_0)}{4k}A_l^{(1)}(k_0)+\int_{|J_k-k/2|}^{J_k+k/2}\frac{P_l(-Q)\,\{{}^1/{}_2A_l^{(1)}(k')+{}^3/{}_2B_l^{(1)}(k')\}}{k'^2-k_0^2}\frac{k'}{k}dk';$$

$$\frac{(R_k-\alpha_s)B_l^{(1)}(k)+J_kB_l^{(2)}(k)}{k^2-k_0^2}=\frac{3G_l^{(1)}(k,k_0)}{4kk_0}$$

$$+\frac{}{\pi}\int_0^\infty\frac{G_l^{(1)}(k,k')\,k'^2\{{}^3/{}_2A_l^{(1)}(k')+{}^1/{}_2B_l^{(1)}(k')\}}{kk'\,(k'^2-k_0^2)}dk'-\frac{3G_l^{(1)}(k,\,k_0)}{4k}A_l^{(2)}(k_0)$$

$$-\int_{|J_k-k/2|}^{J_k+k/2}\frac{P_l(-Q)\,\{{}^3/{}_2A_l^{(2)}(k')+{}^1/{}_2B_l^{(2)}(k')\}}{k'^2-k_0^2}\frac{k'}{k}dk';$$

$$\frac{(R_k-\alpha_s)\,B_l^{(2)}(k)-J_kB_l^{(1)}(k)}{k^2-k_0^2}=\frac{1}{\pi}\int_0^\infty\frac{G_l^{(1)}(k,\,k')\,k'^2\,\{{}^3/{}_2A_l^{(2)}(k')+{}^1/{}_2B_l^{(2)}(k')\}}{kk'\,(k'^2-k_0^2)}dk'$$

$$+\frac{3G_l^{(1)}(k,\,k_0)}{4k}A_l^{(1)}(k_0)+\int_{|J_k-k/2|}^{J_k+k/2}\frac{P_l(-Q)\,\{{}^3/{}_2A_l^{(1)}(k')+{}^1/{}_2B_l^{(1)}(k')\}}{k'^2-k_0^2}\frac{k'}{k}dk'.$$

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Measurement of the Ionizing Power of Particles in a Bubble Chamber

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The possibility of the measurement of the ionizing power of particles in a propane bubble chamber is demonstrated. The chamber was operating in conjunction with an accelerator. The use of the method of the reduction of pressure to a controllable constant level ensured the stability of chamber operation necessary for ionization measurements. The period of sensitivity was 40 m sec. Measurements of the ionizing power of particles were carried out in the range up to eighty times minimum ionization. It was found that the track density changes with the velocity of the particle as $1/\beta^2$ for $\beta < 0.6$. For velocities close to that of light, relativistic increase in the track density is observed. The used methods on the track evaluation are described.

1. INTRODUCTION

It was shown by Glaser[1] in 1953 that ionizing particles cause an overheated liquid to boil, leaving tracks in the form of vapor bubble chains. It is possible to describe the expected dependence of the number of bubbles along the track on the charge and the velocity of the particle not entering into the detailed formation mechanism of the nuclei of vaporization under the influence of ionization. It is well known that an overheated liquid starts boiling on the vaporization nuclei, the radius of which is greater than a critical value r_{cr}.

$$r_{cr} = 2\sigma/(P_\infty - P_0), \qquad (1)$$

where σ is the surface tension coefficient of the liquid-vapor boundary, P_∞ is the pressure of the saturated vapor and P_0 is the hydrostatic pressure of the liquid. The energy losses of a particle due to the ionization of the medium can entail conditions favoring the formation of vaporization nuclei. Bubble chambers usually operate at values of σ in the range from 1–10 dyne/cm and of $(P_\infty - P_0)$ in the range from 5 to 20 atm, which correspond to $r_{cr} = 10^{-5} - 10^{6}$ cm.

It is evident that the nuclei of vaporization are formed in the region where the ionization density, over distances of the order of r_{cr} is considerably larger than the average ionization density along the particle trajectory. This condition is fulfilled by the end points of the tracks of the δ-electrons produced by the particles. Indeed, electrons of about 200 ev lose all their energy on a path shorter than 10^{-5} cm in a liquid, producing some 20 ion pairs*, while for the case of a relativistic particle, the number of ion pairs along the trajectory is not greater than 0.5 per 10^{-5} cm, and the probability of many acts of primary ionization over such a length is vanishingly small. It can be therefore assumed that the number of bubbles along the path of the particle is proportional to the number of δ-electrons of energy greater than 100–200 ev. Neglecting the binding energy of electrons in the atom, we shall obtain the following expression for the number of bubbles per unit length of track (track density):

$$g = g_0 Z^2/\beta^2, \qquad (2)$$

where g is the track density, Z is the charge of the particle, β is the particle velocity in terms of the velocity of light, and g_0 is a coefficient depending on the stopping power of the medium, on the temperature and on the "overheating" $P_\infty - P_0$.

*The given values of range and of the number of ion pairs are calculated for the case of liquid propane from data concerning the air.[2]