Synchrotron Oscillations in Strong Focusing Accelerators. I. Linear Theory

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The equations which describe synchrotron oscillations in strong focusing accelerators are examined, taking into account the relations between the field and the frequency. A general solution is found which describes the oscillations both in the adiabatic and in the critical region and the corresponding integrals of motion are obtained. It is shown that the motion in the critical region can be simply described by means of the "effective frequency" of the oscillations. The effect of fluctuations of the radio frequency, of the accelerating voltage and of the magnetic field is considered, along with the question of the influence of noises on the synchrotron oscillations. The transition through the critical point is studied. The computations are carried to the point of the derived formulas which determine the tolerances for the corresponding fluctuations.

1. EQUATIONS OF SYNCHROTRON OSCILLATIONS

ACCELERATED particles acquire energy under the action of a high-frequency electric field, the frequency of which is equal to, or exceeds by an integral number of times q, the frequency of circulation of the particles in the annular chamber of the accelerator. We shall call the quantity q the multiplicity of the radio frequency.

Let us examine the acceleration of particles which have a charge e. Let the maximum energy acquired per revolution of the particles be eu.

We designate as equilibrium particles those which maintain a constant phase shift relative to the accelerating electric field. Denoting the length of the trajectory of the particles by L and noting that the average field intensity (along the chamber) is equal to u/L, we find for the change in momentum of an equilibrium particle

$$dp/dt = eu\sin\phi/L, \tag{1}$$

where ϕ is the phase of acceleration of the equilibrium particle.

We characterize nonequilibrium particles by the deviation of their momentum and phase from the momentum and phase of equilibrium particles, and introduce the notation Π and φ for these deviations. Then, for small deviations,

$$d\Pi/dt = (eu\cos\phi/L)\,\varphi + (e\sin\phi/L)\,\Delta u/u, \tag{2}$$

where the term containing Δu takes into account the deviation of the amplitude of the accelerating voltage from the ideal value.

For the deviations in phase we have the obvious equality

$$d\varphi/dt = \Delta\omega_p - q\Delta\omega, \tag{3}$$

where $\Delta\omega_p$ is the radiofrequency deviation and $\Delta\omega$ is the deviation of the frequency of revolution of the particles from the ideal value.

We introduce the coefficient α which takes into account the lengthening of the trajectory which is related to the deviation of the momentum,

$$\Delta L/L = \alpha\Pi/p. \tag{4}$$

It is not difficult to convince oneself that

$$\Delta\omega/\omega = [(E_0/E)^2 - \alpha]\,(\Pi/p) + \alpha\Delta H/H, \tag{5}$$

where E is the total relativistic energy of the particles, E_0 is the rest energy, H is the magnetic field intensity and

$$\omega = (2\pi c/L)\;pc/E. \tag{6}$$

From Eqs. (2), (3), (5) and (6) we find

$$\frac{d}{dt}\left[\frac{E}{(E_0/E)^2-\alpha}\frac{d\varphi}{dt}\right] + \frac{2\pi qc^2eu\cos\phi}{L^2}\varphi = \frac{d}{dt}\left[\frac{E}{(E_0/E)^2-\alpha}\Delta\omega_p\right] - \frac{2\pi}{L^2}qc^2eu\sin\phi\frac{\Delta u}{u} - \frac{2\pi q\alpha c}{L}\frac{d}{dt}\left[\frac{(E^2-E_0^2)^{1/2}}{(E_0/E)^2-\alpha}\frac{\Delta H}{H}\right]. \tag{7}$$

In studying Eq. (7) one should keep in mind that $\Delta\omega_p$ and ΔH, generally speaking, are not independent quantities. If the frequency of the accelerating field follows behind the magnetic field, the deviations $\Delta\omega_p$ are partially related to inaccuracies in the electronics and partially caused by oscillations of the magnetic field. It is natural to examine these latter together with the term which depends on $\Delta H/H$. Let the system of coupling the frequency with the field be characterized by the delay τ, so that instead of the ideal rule

$$\omega_p = (2\pi cq/L)\, eRH\,(E_0^2 + e^2R^2H^2)^{-1/2}, \qquad (8)$$

the rule:

$$\Delta\omega_p + \tau d\Delta\omega_p/dt = (2\pi c^2 q/L)\,(pE_0^2/E^3)\,\Delta H/H \quad (9)$$

occurs. The letter R denotes the radius of curvature of an equilibriun particle in the field H.

We will be interested chiefly in harmonic oscillations of the type $\Delta H = H_w \sin wt$. From Eq. (9) we find

$$\Delta\omega_p = \frac{2\pi c^2 qpE_0^2}{LE^3}\,\frac{H_w}{H}\,\frac{\sin wt - w\tau\cos wt}{1 + w^2\tau^2}. \qquad (10)$$

Let us substitute this expression in Eq. (7) and combine it with the term which depends on $\Delta H/H$, keeping the notation $\Delta\omega_p$ for deviations which are associated only with the electronics. We further note that oscillations of the magnetic field are excited by oscillations of the voltage V which supplies the magnet. Disregarding the effective resistance of the magnet windings in comparison with their inductive reactance, we find

$$H_w/H = (\dot{H}/wH)\,V_w/V \qquad (11)$$

$$= (eu\sin\phi/Lwp)\,V_w/V.$$

Then Eq. (7) assumes the form:

$$\frac{1}{E}\left[\frac{E_0^2}{E^2} - \alpha\right]\frac{d}{dt}\left[\frac{E}{(E_0/E)^2 - \alpha}\,\frac{d\varphi}{dt}\right] + \Omega_t^2\varphi \qquad (12)$$

$$= \frac{1}{E}\left[\frac{E_0^2}{E^2} - \alpha\right]\frac{d}{dt}\left[\frac{E}{(E_0/E)^2 - \alpha}\,\Delta\omega_p\right]$$

$$+ \frac{\Omega_t^2\,\mathrm{tg}\,\phi}{w(1 + w^2\tau^2)}\,\frac{d}{dt}\Big[\sin wt$$

$$- \frac{w\tau}{(E_0/E)^2 - \alpha}\,\frac{E_0^2}{E^2}\left(\cos wt + \alpha w\tau\,\frac{E^2}{E_0^2}\sin wt\right)\Big]\frac{V_w}{V}$$

$$- \Omega_t^2\,\mathrm{tg}\,\phi\,\frac{\Delta u}{u},$$

where

$$\Omega_t^2 = (2\pi qc^2 eu\cos\phi/L^2E)\,[(E_0/E)^2 - \alpha]. \qquad (13)$$

Let us introduce in place of the time the independent variable x, defined by the equation

$$x = pc/E_0. \qquad (14)$$

We assume that the momentum (and the magnetic field) increases linearly with time, so that the right-hand side of Eq. (1) is constant. From Eq. (1) we have

$$dx/dt = ceu\sin\phi/E_0L. \qquad (15)$$

We also introduce the quantity

$$a^2 = (1-\alpha)/\alpha \approx 1/\alpha \qquad (16)$$

and the function

$$f(x) = [1 - (x/a)^2]\,(1 + x^2)^{-3/2}a^2\alpha \qquad (17)$$

$$\approx [1 - (x/a)^2]\,(1 + x^2)^{-3/2}.$$

(In strong focusing accelerators $\alpha \ll 1$.) In the new variables, Eq. (12) has the form:

$$f(x)\frac{d}{dx}\left[\frac{1}{f(x)}\,\frac{d\varphi}{dx}\right] + \Omega_x^2\varphi \qquad (18)$$

$$= \Omega_x^2\,\mathrm{tg}\,\phi\,\frac{d}{dx}\left[\frac{x}{(1+x^2)^{1/2}f(x)}\,\frac{\Delta\omega_p}{\omega_p}\right]$$

$$- \Omega_x^2\,\mathrm{tg}\,\phi\,\frac{\Delta u}{u} + \frac{\Omega_x^2\,\mathrm{tg}\,\phi}{w_x(1 + w^2\tau^2)}\,\frac{d}{dx}\Big[\sin wx$$

$$- \frac{w\tau\cos wx}{(1+x^2)^{3/2}f(x)}\Big]\frac{V_w}{V}.$$

In writing Eq. (18) it was taken into account that in practically-important cases $\alpha\Omega_0^2\tau^2 \ll 1$. If this is not so, then to $\cos wx$ the term $(E/E_0)^2 \times \alpha w\tau\sin wx$ must be added. The following notation is introduced in Eq. (18):

$$\Omega_x = \Omega_t\,dt/dx = \Omega_0\,|f(x)|^{1/2}, \qquad (19)$$

$$\Omega_0^2 = 2\pi qE_0\,\mathrm{ctg}\,\phi/eu\sin\phi.$$

Ω_0 is a dimensionless quantity numerically equal to the frequency in x, for $x = 0$.

The equation for free oscillations is obtained from Eq. (18) if the right side is equated to zero:

$$f(x)\frac{d}{dx}\left[\frac{1}{f(x)}\,\frac{d\varphi}{dx}\right] + \Omega_x^2\varphi = 0. \qquad (20)$$

Equation (18) has a regular singular point for

$x = a$, when $f(x)$ becomes zero. After the singular point $f(x)$ changes sign and the motion remains stable only in the case that Ω_x^2, proportional to $f(x)$, does not change sign; for this it is necessary to change the phase of acceleration from ϕ to $\pi - \phi$.

In the transition through the critical point it is possible to define positive deviation of φ in different ways. It is natural to define it so that during this transition φ be continuous. The momentum Π in the critical point likewise remains continuous. These two conditions determine the transition through the critical point, where $d\varphi/dx$ changes sign.

2. FREE SYNCHROTRON OSCILLATIONS

We introduce new variables

$$v = \varphi\Omega_x^{-1/2},\quad \psi = \int_a^x \Omega_x dx\ ; \tag{21}$$

ψ has the obvious sense of the phase of oscillation. In these variables Eq. (20) acquires the form:

$$(d^2v/d\psi^2) + v\,[1 + f_x''/4f^2\Omega_0^2 - 7\,(f_x')^2/16\,|f|^3\Omega_0^2] = 0. \tag{22}$$

In cases of practical interest, Ω_0 is a very large quantity; $\Omega_0 \gg 1$, so that the terms which are added to unity in Eq. (22) are essential only in the vicinity of the point $x = a$, where $f(x)$ becomes zero. In this region they can be expanded in a series, from which we retain only the principal term. Although such a representation becomes inadequate even for $|x - a| \approx 0.4\,a$, the entire correction to unity becomes so small here that it does not play any role. In the vicinity of the point a

$$f(x) \approx 2a^{-3}\,|1 - x/a|. \tag{23}$$

Eq. (22) assumes the form

$$(d^2v/d\psi^2) + [1 - 7/36\psi^2]\,v = 0. \tag{24}$$

The solution of Eq. (24) is:

$$v(\psi) = \psi^{1/2}\,[A_2J_{2/3}(\psi) + A_1J_{-2/3}(\psi)], \tag{25}$$

where $J_{2/3}$ and $J_{-2/3}$ are Bessel functions. Reverting to the variable φ we find:

$$\varphi = (\Omega_x/\Omega_0)^{1/2}\psi^{1/2}\,[C_1J_{-2/3}(\psi) + C_2J_{2/3}(\psi)]. \tag{26}$$

The signs in front of C_1 and C_2 do not depend on which side of the critical point we are on. This follows from the continuity of phase and momentum in the critical point. We note that Bodenstedt[2], who was studying the transition through the critical point on a mechanical model, arrived at an incorrect conclusion concerning a change in sign of C_2 (see Fig. 10 in Ref. 2), which is explained by the properties of his model. For large arguments (the adiabatic region) one can use the asymptotic representation of the Bessel functions:

$$\varphi = (2/\pi)^{1/2}(\Omega_x/\Omega_0)^{1/2}\,[C_1\cos(\psi + \pi/12) + C_2\sin(\psi - \pi/12)]. \tag{27}$$

For a sufficiently small argument ($|x - a| \ll 0.4a$), one can express ψ by means of Eqs. (19), (21) and (23) through the deviation from the critical point $\xi = |x - a|$:

$$\varphi = 2^{1/4}a^{-1}\,\Omega_{\text{eff}}^{3/4}\,\xi\,[C_1J_{-2/3}(\Omega_{\text{eff}}^{1/2}\xi^{3/2}) + C_2J_{2/3}(\Omega_{\text{eff}}^{1/2}\xi^{3/2})], \tag{28}$$

$$\Omega_{\text{eff}} = 2\cdot 3^{-2/3}\Omega_0^{2/3}a^{-4/3} \approx \Omega_0^{2/3}a^{-4/3}. \tag{29}$$

Equations of the type of Eq. (28) we obtained by Kolomenskii and Sabsovich[4] and by Johnson[5].

To facilitate the transition from one region to the other, the constants in Eqs. (26)-(28) were chosen so that C_1 and C_2 agree in these equations and thus are integrals of motion.

Let us examine in greater detail Eq. (27) which describes the phase oscillations in the adiabatic region. The instantaneous frequency of the synchrotron oscillations in x is equal to $d\psi/dx = \Omega_x$. The amplitude of the oscillations decreases on approach to the critical point and increases on going away from it, varying as $(\Omega/\Omega_0)^{1/2} = f^{1/4}$. It reaches a maximum of $0.6\,a^{-3/4}(C_1^2 + C_2^2 - C_1C_2)$ when $x = (3a^2 + 2)^{1/2}$.

Equations (12) and (18) are obtained from Eqs. (2), (3) and (5) through elimination of the momentum Π. With the same success it would be possible to eliminate the phase from them and to obtain the equation for oscillations in momentum. Instead of this we indicate the conversion to momentum directly in the solution. Converting to the variable x in Eq. (2) we obtain

$$\Pi = \frac{E_0}{c}\,\text{ctg}\,\phi \int_{x_0}^{x} \varphi\,dx + \frac{E_0}{c}\int_{x_0}^{x} \frac{\Delta u}{u}\,dx. \tag{30}$$

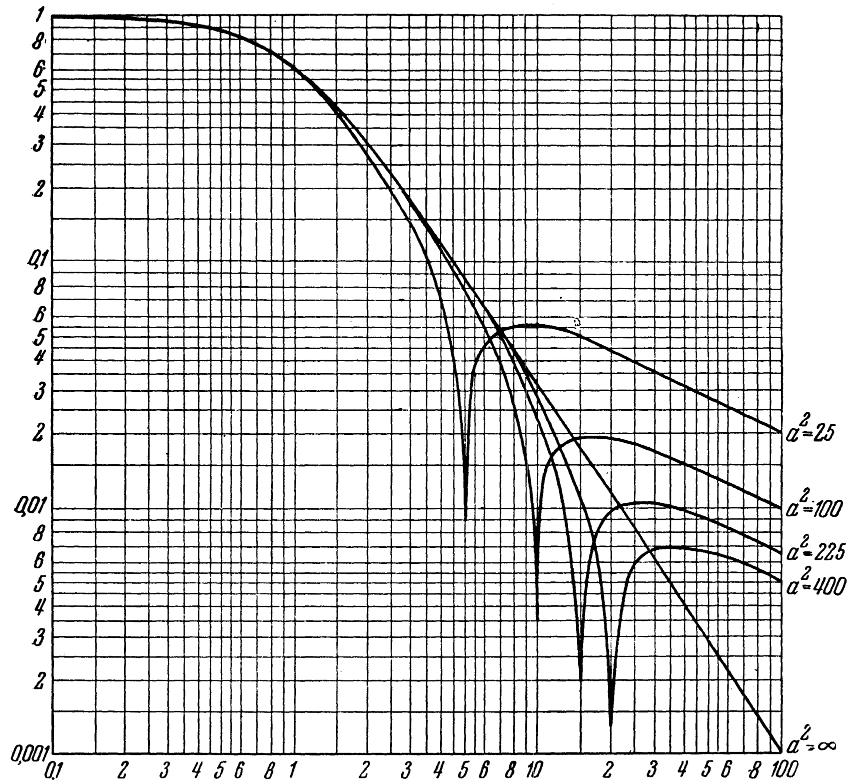

FIG. 1. Graph of the function $\sqrt{f(x)}$.

It follows from Eqs. (27) and (30) that the amplitudes of the oscillations in momentum and phase in the adiabatic region are connected by the relation

$$A_\Pi / A_\varphi = E_0 \operatorname{ctg} \phi / c\Omega_x, \qquad (31)$$

such that in this region

$$\Pi(x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{E_0 \operatorname{ctg} \phi}{c\Omega_0} \left(\frac{\Omega_0}{\Omega_x}\right)^{1/2} \left[C_1 \sin\left(\psi + \frac{\pi}{12}\right) - C_2 \cos\left(\psi - \frac{\pi}{12}\right)\right]. \qquad (32)$$

The function Ω_x/Ω_0, which enters in Eqs. (27) and (32) is depicted in Fig. 1. For the critical region we obtain:

$$\Pi(\xi) = \left(\frac{2}{3}\right)^{1/2} \frac{E_0 \operatorname{ctg} \phi}{c\Omega_0^{1/2}} \xi^{1/2} \left[C_1 J_{1/3}(\Omega_{\text{eff}}^{3/2} \xi^{3/2}) - C_2 J_{-1/3}(\Omega_{\text{eff}}^{3/2} \xi^{3/2})\right]. \qquad (33)$$

From Eq. (32) it is seen that the amplitude of the oscillations in momentum increases on approaching the critical region. Further on, close to $x = (3a^2 + 2)^{1/2}$, the amplitude falls, then rises again roughly speaking as $x^{1/4}$. We note that for the point $x = a$, there follows from Eq. (33):

$$\Pi(a) \approx 0.8 E_0 C_2 \operatorname{ctg} \phi / c (\Omega_0 \Omega_{\text{eff}})^{1/2}. \qquad (34)$$

Comparison of Eqs. (32) and (34) shows that the oscillations in the critical region have the "effective frequency" Ω_{eff}, specified by Eq. (29). Comparison of Eqs. (27) and (28) leads to practically the same value for Ω_{eff}. This effective frequency enters in almost all tolerances in the critical region.

Here, the calculations lead to the following results, which are unexpected at first glance. In the calculation of tolerances connected with forced oscillations in the critical region, one can assume that, in the region indicated, regular harmonic oscillations occur with constant amplitude and constant frequency numerically equal to the "effective frequency." Although the real picture is essentially more complicated, such a calculation leads to correct answers (within an accuracy of the order of 20%).

Of interest for a beam of particles is the value of the mean square deviation in phase and momentum which is especially essential for an examination of noise modulations (see Sec. 6).

We write Eq. (27) in the form

$$\varphi = a_1 \cos\psi + b_1 \sin\psi, \qquad (35)$$

$$a_1 = \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\Omega}{\Omega_0}\right)^{1/2}\left[C_1 \cos\frac{\pi}{12} - C_2 \sin\frac{\pi}{12}\right], \qquad (36)$$
$$b_1 = \left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\Omega}{\Omega_0}\right)^{1/2}\left[C_2 \cos\frac{\pi}{12} - C_1 \sin\frac{\pi}{12}\right].$$

For a beam of particles one can write

$$a_1 = A\cos\chi; \quad b_1 = A\sin\chi, \qquad (37)$$

where A and χ are an arbitrary amplitude and phase. Averaging over A and χ we find:

$$\overline{\varphi^2} = {}^1/_2\overline{A^2} = {}^1/_2(\overline{a_1^2} + \overline{b_1^2}) \qquad (38)$$
$$= (\Omega/\pi\Omega_0)(\overline{C_1^2} + \overline{C_2^2} - \overline{C_1C_2}).$$

From Eq. (36) it is easy to obtain $\overline{C_1^2} = \overline{C_2^2} = \overline{2C_1C_2}$, such that

$$\overline{\varphi^2} = (3\Omega_x/2\pi\Omega_0)\overline{C_2^2}. \qquad (39)$$

Comparing the mean square deviations of the momentum in the critical point and in the adiabatic region, we find

$$\overline{\Pi_{cr}^2}/\overline{\Pi_{ad}^2} \approx 1.2\,(\Omega_x/\Omega_{eff}). \qquad (40)$$

3. ELECTROTECHNICAL AND RADIOTECHNICAL TOLERANCES. (NONRESONANCE CASE)

In estimating tolerances, oscillations of the parameters which occur with a frequency close to the instantaneous frequency of the synchrotron oscillations are especially effective. It is natural to examine such perturbations separately. Faster oscillations present no danger, since they are rapidly averaged out. Let us consider abrupt changes in ω_p (a step function) and slow variations (in comparison with the frequency of the free oscillations) in $\Delta\omega_p$, Δu and ΔV.

a) Tolerances associated with jumps in ω_p.

In order to find the swing of the oscillations associated with jumps in ω_p, we integrate Eq. (18) over the period of a jump. Considering that before the jump $\varphi = \varphi' = 0$, we get

$$\varphi' = \Omega_0^2 x(1 + x^2)^{-1/2}\Delta\omega_p \operatorname{tg}\phi/\omega_p. \qquad (41)$$

Thus, a jump in ω_p gives rise (in the adiabatic region) to the appearance of oscillations with amplitude

$$A_\varphi = (\Omega_0^2/\Omega_x)\,x(1 + x^2)^{-1/2}\operatorname{tg}\phi\cdot\Delta\omega_p/\omega_p. \qquad (42)$$

In Eq. (42) the variable x pertains to the moment of the jump. Furthermore, this amplitude will vary according to the general formula, i.e., as $f^{1/4}$ The effect of the jumps in ω_p becomes all the stronger on approaching the critical region. It is not difficult to find an estimate in the immediate vicinity of the critical point, where the Bessel functions are approximated closely by the first term of a series expansion. It is just as easy in this case to convince onself that

$$\Delta C_2 \approx 0{,}6\cdot a^2 \operatorname{tg}\phi\,\Omega_0^{3/2}\Omega_{eff}^{-1}\xi^{-1}\Delta\omega_p/\omega_p, \qquad (43)$$

so that the tolerance in the magnitude of the jump becomes very inflexible. One should have in mind, however, that in this case changes in ω_p, which are rapid in comparison with changes in $1/\xi$, should qualify as jumps, so that the rigid restrictions apply only to extremely sharp changes in ω_p, which are not very substantial in practice. By means of Eq. (31), we obtain the amplitude of the oscillations in momentum excited by a jump in ω_p:

$$A_\Pi = (E_0/c)(\Omega_0/\Omega_x)^2 x(1 + x^2)^{-1/2}\Delta\omega_p/\omega_p. \qquad (44)$$

b) Tolerances associated with slow variation of $\Delta\omega_p$, Δu, ΔV.

The effect of smooth (in comparison with the free oscillations) changes in $\Delta\omega_p$, $\Delta u/u$, $\Delta V/V$ is easy to take into account, considering the right-hand side of Eq. (18) constant and finding the shift of the equilibrium point of the oscillations.

$$\varphi = \operatorname{tg}\phi\frac{d}{dx}\left[\frac{x}{(1 + x^2)^{1/2}}\frac{\Omega_0^2}{\Omega_x^2}\frac{\Delta\omega_p}{\omega_p}\right] - \operatorname{tg}\phi\frac{\Delta u}{u} \qquad (45)$$
$$+ \operatorname{tg}\phi\frac{1}{w(1 + w^2\tau^2)}\frac{d}{dx}\left[\sin wx - \frac{w\tau\cos wx}{(1 + x^2)^{3/2}}\frac{\Omega_0^2}{\Omega_x^2}\right]\frac{V_w}{V}.$$

For the maximum deviation of the momentum we find:

$$\Pi = \frac{E_0}{c}\cdot\frac{x}{(1 + x^2)^{1/2}}\frac{\Omega_0^2}{\Omega_x^2}\frac{\Delta\omega_p}{\omega_p} - \frac{E_0}{c}\frac{w}{\Omega_x^2}\frac{u_w}{u} \qquad (46)$$
$$+ \frac{E_0}{c}\frac{1}{w(1 + w^2\tau^2)}\left[\sin wx - \frac{w\tau\cos wx}{(1 + x^2)^{3/2}}\frac{\Omega_0^2}{\Omega_x^2}\right]\frac{V_w}{V}.$$

Strictly speaking, Eqs. (45) and (46) are applied only for the adiabatic region. As was pointed out above, however, in the critical region one can simply replace Ω_x by Ω_{eff}, which is corroborated by accurate calculations. Then, denoting the allowable deviation in phase and in momentum by $\Delta\varphi$ and $\Delta\Pi$, we find for the critical region:

$$\Delta\omega_p / \omega_p = (\Delta\Pi / p) [eu \sin\phi / 2\pi q E_0 \operatorname{ctg}\phi]^{1/3} [E_0 / E_{cr}]^{5/3}, \quad (47)$$

$$\Delta\omega_p / \omega_p = (\Delta\varphi / \operatorname{tg}\phi) [eu \sin\phi / 2\pi q E_0 \operatorname{ctg}\phi]^{2/3} [E_0 / E_{cr}]^{4/3}. \quad (48)$$

4. THE TRANSITION THROUGH THE CRITICAL POINT

As was pointed out above, the transition through the critical point requires a change in the phase of the acceleration from ϕ to $\pi - \phi$. This switchover cannot be realized precisely at the moment when an equilibrium particle reaches the critical point; there must always be some discrepancies, which we shall characterize by an error τ_1 in the switchover time. During this time Ω_x^2 in Eq. (20) is negative, which corresponds to defocusing of the particles. We shall assume, for definiteness, that the phase of the acceleration is switched over later than it ordinarily should be. (By virtue of the symmetry of the equations about the critical point, the answer does not depend on this assumption.) Then the equation of the phase oscillations in this region takes the form:

$$\varphi = (2^{1/4}/a)\,\Omega_{eff}^{3/4}\,\xi\,[C_1 i^{2/3} J_{-2/3}(i\Omega_{eff}^{3/2}\,\xi^{3/2}) + C_2 i^{-2/3} J_{2/3}(i\Omega_{eff}^{3/2}\,\xi^{3/2})], \quad (49)$$

where C_1 and C_2 have those values which they had up to the critical point. In the moment $\Delta\xi$ the solution Eq. (49) goes over to the solution Eq. (28) with different constants. Having denoted the corrections to the corresponding coefficients by ΔC_1 and ΔC_2, we find for small values of $\Delta\xi$:

$$\Delta C_2/C_1 \approx 2\Omega_{eff}\,\Delta\xi, \qquad \Delta C_1/C_1 \approx -0.8\cdot\Omega_{eff}^3\,(\Delta\xi)^3. \quad (50)$$

It is natural to require that $\Delta C_2/C_1$ and $\Delta C_1/C_1$ be small, let us say about 0.1. Then

$$\Delta x = 5\cdot 10^{-2}\cdot\Omega_{eff}^{-1} \quad (51)$$

and for the error in time (in seconds)

$$\tau_1 = 5\cdot 10^{-2}\,[eu \sin\phi / 2\pi q E_0 \operatorname{ctg}\phi]^{1/3} \times (E_{cr}/E_0)^{4/3} E_0 L / ceu \sin\phi. \quad (52)$$

Let us now estimate the disturbances which originate from the phase of the accelerating voltage not switching over instantaneously. We assume that the accelerating voltage is at first switched off and is switched on after the time τ_2, already in the second phase. During the time τ_2, the particles travel by inertia, not being accelerated. It would appear natural to assume, in this case, that the coordinate x does not vary. The calculations turn out simpler, however, if we assume x to vary according to the old rule. Hereupon, the continuous increase of the magnetic field and the change of frequency of the accelerating voltage will be properly taken into account. We assume for definiteness that the switching off of the accelerating voltage occurred after the transition through the critical point. The equation of motion can be obtained by Eq. (18) if we take $\Delta u/u = -1$ and cancel the term $\Omega_x^2\varphi$.

Then

$$\frac{d}{dx}\left[\frac{1}{f(x)}\frac{d\varphi}{dx}\right] = \Omega_0^2 \operatorname{tg}\phi. \quad (53)$$

Integrating Eq. (53), having used the representation of Eq. (23) for $f(x)$, we find:

$$\varphi = \varphi_0 + {}^1/_2\varphi_0'(\xi^2 - \xi_0^2)\,\xi_0^{-1} + 2\operatorname{tg}\phi\,\Omega_0^2 a^{-4}\,[({}^1/_3\xi - {}^1/_2\xi_0)\,\xi^2 + {}^1/_6\xi_0^3], \quad (54)$$

$$\varphi' = \xi\xi_0^{-1}\varphi_0' + 2\operatorname{tg}\phi\,\Omega_0^2 a^{-4}\,[\xi - \xi_0]\,\xi,$$

where ξ_0 is the moment of switching off the accelerating voltage and ξ is the moment of switching it on. At the point ξ this solution must be connected with the solution Eq. (28), which also determines the changes in the constants C_1 and C_2. For small values of $\Delta\xi = \Delta x$, neglecting cubic terms in comparison with linear terms, we obtain:

$$\Delta C_1 \approx 0, \quad \Delta C_2 \approx \Omega_{eff}\,C_1\Delta x + 1\ 3\cdot\Omega_0^{5/4} a^{-2/3}\operatorname{tg}\phi\,\Delta x. \quad (55)$$

Knowing ΔC_1 and ΔC_2, it is easy to determine the increase in amplitude of the oscillations at the most dangerous point $x = (3a^2 + 2)^{1/2}$.

Inasmuch as a deviation of the momentum is effective precisely in the critical region, where the natural oscillations reach a maximum, it is useful to calculate, by means of Eq. (55), the shift in the momentum common for all particles: $\Delta\Pi/_p = \Delta x/a$, which gives

$$\tau_2 = (\Delta\Pi / p)(E_{cr} / E_0) E_0 L / ceu \sin\phi. \quad (56)$$

Kolomenskii and Sabsovich[4] and apparently, Johnson[5], also dealt with questions related to the transition through the critcial point.

5. RESONANCE

The frequency of the generator, the magnitude of the accelerating voltage and the voltage on the magnet can oscillate with a frequency equal to the instantaneous frequency of the phase oscillations. In this case a resonance arises, which leads to a strong wave of oscillations.

Accordingly, we shall examine resonance harmonics of the right-hand part. By virtue of the incoherence of the perturbations, one can examine the terms which enter into the right-hand part separately. In the adiabatic approximation we write Eq. (18) in the form

$$(d^2\varphi / dx^2) + \Omega_x^2\varphi = d\cos(wx + \alpha), \quad (57)$$

where α is the phase of the perturbation. The solution of Eq. (57) without the right-hand part can be represented in the form

$$\varphi = \varphi_1 e^{i\psi} + \overset{*}{\varphi}_1 e^{-i\psi}. \quad (58)$$

We impose on φ_1 and φ_1^* the supplementary condition:

$$\varphi_1' e^{i\psi} + \overset{*}{\varphi}_1{}' e^{-i\psi} = 0. \quad (59)$$

Having used Eqs. (58) and (59), we obtain for the principal part of φ_1

$$\varphi_1 = \frac{d}{4i\Omega_x}\int_{x_0}^{x} e^{i(wx-\psi+\alpha)}dx. \quad (60)$$

Expanding the frequency Ω_x in a series at the resonance point, we obtain

$$\Omega_x = w + \Omega'_{res}(x - x_{res}) + \ldots \quad (61)$$

Then

$$\varphi_1 = \frac{de^{i\beta}}{4\Omega_x}\int_{x_0-x_{pe3}}^{x-x_{pe3}} \exp\{-i\Omega'_{pe3}z^2/2\}\, dz. \quad (62)$$

Taking $x - x_{res} = \infty$, $x_0 - x_{res} = -\infty$ and integrating Eq. (62) we obtain for φ:

$$\varphi = 2\,\mathrm{Re}\,\varphi_1 e^{i\psi} = (d/\Omega_x)\,[2\pi/\Omega'_{pe3}]^{1/2}\cos(\psi + \gamma). \quad (63)$$

Averaging over the arbitrary phase γ, we find for the square of the phase amplitude

$$\overline{\varphi^2} = \pi d^2 / 2\Omega_{res}^2\Omega'_{res} = \pi d^2 / \Omega_{res}\Omega_0^2 f'_{res}. \quad (64)$$

Comparing Eqs. (57) and (18) and taking into account the change in the amplitude of the oscillations with frequency, we find finally:

$$\overline{\Delta\varphi^2} = \pi\,\mathrm{tg}^2\phi\,\frac{\Omega_x}{|f'_{res}|}\left[\frac{\Omega_0^2x^2}{1+x^2}\right]_{res}\left(\frac{\omega_\Omega}{\omega_{res}}\right)^2, \quad (65)$$

$$\overline{\Delta\varphi^2} = \pi\,\mathrm{tg}^2\phi\,\frac{\Omega_x}{|f'_{res}|}\,\frac{\Omega_{res}^2}{\Omega_0^2}\left(\frac{u_\Omega}{u}\right)^2,$$

$$\overline{\Delta\varphi^2} = \pi\,\mathrm{tg}^2\phi\,\frac{\Omega_x}{|f'_{res}|}\,\frac{\Omega_{res}^2}{\Omega_0^2(1+\Omega_{res}^2\tau^2)^2} \times\left[1 + \frac{\Omega_0^4\tau^2}{(1+x^2)_{res}^3\Omega_{res}^2}\right]\left(\frac{V_\Omega}{V}\right)^2$$

Here, ω_Ω, u_Ω, V_Ω are the amplitudes of the perturbations with frequency Ω;

$$\Delta\omega = \sum_0^\infty \omega_\Omega\cos(\Omega x + \alpha).$$

This result is found to be in agreement with the result of Blachman[1]. Eqs. (65) give the increase of the mean square amplitude of the phase oscillations at any moment, if by Ω_x we understand the instantaneous frequency of the oscillations and by Ω_{res} the resonance frequency, whose effect we are studying. Equations (65) do not have any singularities in the neighborhood of the critical point, which is natural, since on approaching the critical point the resonance is passed all the more rapidly and does not have time to set synchrotron oscillations going to any appreciable extent. Therefore, the critical region does not require special consideration. One must have in mind that the resonance frequencies are included between Ω_0 and Ω_{eff}. At the point where the rate of change of frequency becomes zero $[x = (3a^2 + 2)^{1/2}]$, the next term in the expansion must be taken into account. In this case:

$$\overline{\Delta\varphi^2} = 4{,}4\,\mathrm{tg}^2\phi\,\Omega_0^{10/3}a^{7/3}(\omega_\Omega / \omega)^2, \quad (66)$$

$$\overline{\Delta\varphi^2} = 1{,}7\,\mathrm{tg}^2\phi\,\Omega_0^{4/3}a^{-2/3}(u_\Omega/u)^2,$$

$$\overline{\Delta\varphi^2} = 1{,}7\,\mathrm{tg}^2\phi\,\Omega_0^{4/3}a^{-2/3}(V_\Omega / V)^2.$$

The oscillations in momentum which reach a maximum in the critical point merit special examination. Variation of the momentum is expressed by Eq. (34), taking into account Eqs. (39) and (65).

6. NOISE EXCITATION OF SYNCHROTRON OSCILLATIONS

The problem of noise excitation of ocsillations without account of adiabatic damping was solved by Blachman, who examined Eq. (12) in the adiabatic approximation, i.e., in the form

$$(d\varphi^2/dt^2)+\Omega_t^2\varphi = F(t), \tag{67}$$

where $F(t)$ is an arbitrary noise perturbation with spectral intensity Φ (calculated for a band-width of 1 cycle). According To Ref. 1, the mean square deviation of the phase amplitude is equal to

$$\overline{\varphi^2} = {}^1/_2\int_{t_0}^{t}(\Phi/\Omega_t^2)\,dt. \tag{68}$$

Comparing Eq. (12) with Eq. (67) and transforming to the variable x, we obtainfor the noise modulation of the frequency with spectral intensity ν (cycles^2/cycle)

$$\overline{\varphi^2} = (2\pi^2E_0L/ceu\sin\phi)\int_{x_0}^{x}\nu\,dx. \tag{69}$$

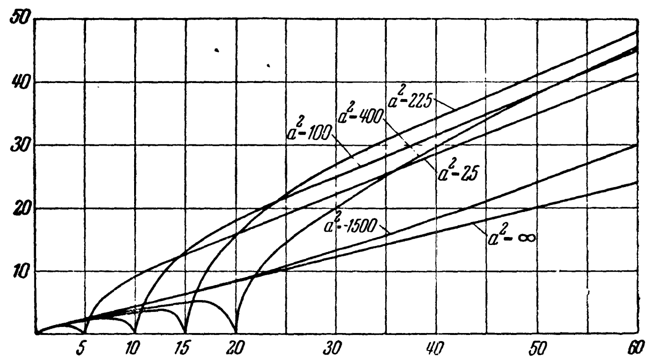

FIG. 2. Graph of the function $f^{1/2}(x)\int_0^x f^{-1/2}(x)\,dx$.

For the noise modulation of the amplitude of the accelerating voltage $\Delta u/u$ with spectral intensity η (cycle^{-1})

$$\overline{\varphi^2} = {}^1/_2\,\mathrm{tg}^2\phi\,(ceu\sin\phi/E_0L)\int_{x_0}^{x}\eta\Omega_x^2\,dx \tag{70}$$

and for the noise modulation of the voltage on the magnet $\Delta V/V$ with spectral intensity μ (cycle^{-1})

$$\overline{\varphi^2} = {}^1/_2\,\mathrm{tg}^2\phi\,\frac{ceu\sin\phi}{E_0L}\times\int_{x_0}^{x}\left[\Omega_x^2+\frac{\Omega_0^4\tau^2}{(1+x^2)^3}\right]\frac{\mu\,dx}{(1+\Omega^2\tau^2)^2}. \tag{71}$$

We now find the expression for $\overline{\varphi^2}$ accounting for the adiabatic change of amplitude of the oscillations. From the relation Eq. (27) for the mean square amplitude we obtain the equation

$$d\overline{\varphi^2}/dx = \overline{\varphi^2}f'/2f. \tag{72}$$

On the other hand, differentiating Eq. (69), we obtain for the increase in amplitude on account of the noise modulation of the frequency

$$d\overline{\varphi^2}/dx = 2\pi^2(E_0L/ceu\sin\phi)\,\nu. \tag{73}$$

On account of the independence of the processes of adiabatic contraction and of noise excitation, there follows from Eqs. (72) and (73):

$$f^{1/2}(x)\frac{d}{dx}[f^{-1/2}(x)\overline{\varphi^2}] = 2\pi^2\frac{E_0L}{ceu\sin\varphi}\nu. \tag{74}$$

Integrating Eq. (74), we obtain:

$$\overline{\varphi_\nu^2} = 2\pi^2\nu\frac{E_0L}{ceu\sin\phi}f^{1/2}(x)\int_{x_0}^{x}f^{-1/2}(x)\,dx. \tag{75}$$

Analogously, for the noise modulation of the amplitude of the accelerating voltage

$$\overline{\varphi_\eta^2} = \frac{1}{2}\,\mathrm{tg}^2\phi\,\eta\frac{ceu\sin\phi}{E_0L}\Omega_0^2f^{1/2}(x)\int_{x_0}^{x}f^{1/2}(x)\,dx \tag{76}$$

and for the noise modulation of the voltage on the magnet

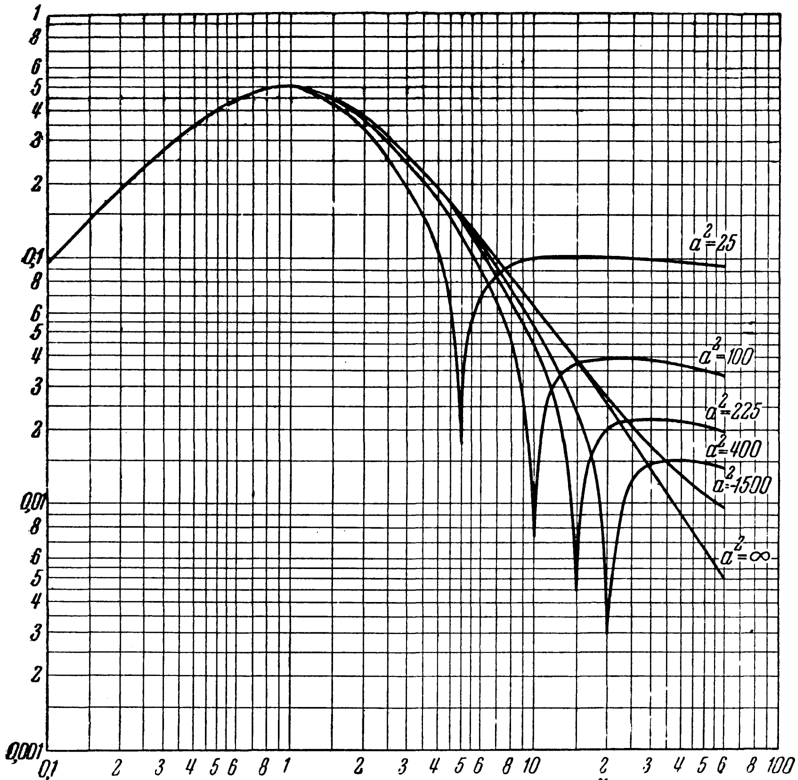

FIG. 3. Graph of the function $f^{1/2}(x)\int_0^x f^{1/2}(x)\,dx$.

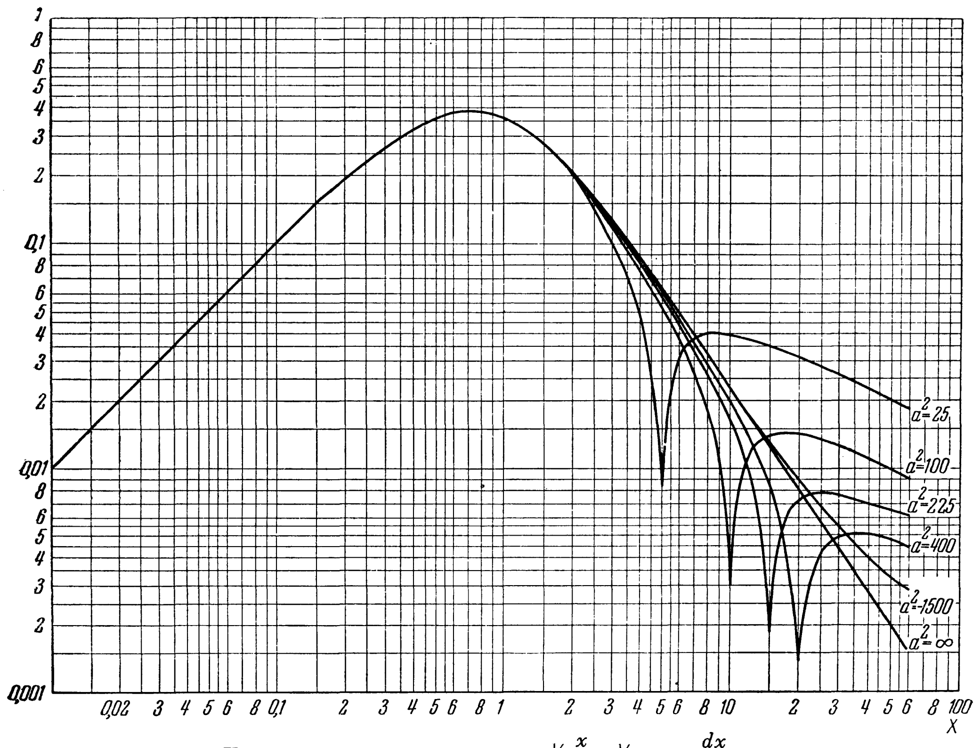

FIG. 4. Graph of the function $f^{1/2}\int_0^x f^{-1/2}(x)\frac{dx}{(1+x^2)^3}$.

$$\overline{\varphi_\mu^2} = {}^1/_2\, \mathrm{tg}^2\phi\mu \frac{ceu \sin\phi}{E_0 L} \Omega_0^2 f^{1/2}(x) \int\limits_{x_0}^{x} \left[f^{1/2}(x) + \frac{\Omega_0^2 \tau^2}{(1+x^2)^3} f^{-1/2}(x) \right] \frac{dx}{(1+\Omega^2\tau^2)^2}. \quad (77)$$

In Eqs. (75) through (77), it is assumed that the values of the spectral intensities ν, η and μ do not depend on the frequency. These formulas determine $\overline{\varphi^2}$ in the adiabatic region. Inasmuch as the effect of noises decreases on approaching the critical region, they can be used during the entire cycle of acceleration.

The mean square deviation of the momentum in the critical region is of interest. It can be calculated by Eqs. (34) and (39) on substituting for $\overline{\varphi^2}$ from Eqs. (75) through (77).

In conclusion, we present graphs of the functions which enter in Eqs. (75) through (77). These functions are evaluated for several positive- and for one negative-value of a^2. Negative values of a^2, characteristic for strong focusing accelerators without a critical energy, were proposed by Vladimirskii and Tarasov[3].

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A Comparison of the Fermi-Landau Theory with Some Experimental Data on Cosmic Rays

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The energy spectra of the secondary particles originating in nuclear collisions of high-energy particles ($10^{12} - 10^{18}$ ev) have been calculated on the basis of the theory of L. D. Landau. A calculation of the altitude dependence of the radioactive particles in the atmosphere, as well as of the number of high-energy μ-mesons at sea level, has been carried out on the basis of the spectra obtained. The results of the calculations are compared with experimental data.

UP to the present time there have been few direct experimental data concerning collisions of super-high energy nucleons ($> 10^{12}$ ev) with nucleons or light nuclei. There exists practically no knowledge of the distribution of energy between the secondary particles of various types formed as a result of such collisions. Hence, in spite of the considerable time since the publication of the Landau theory[1], there have been, up to the present, almost no papers devoted to a comparison of the results of this theory with experimental data. However, although direct experimental data relative to the energy distribution of secondary particles in the energy region $E > 10^{12}$ ev are lacking, we nevertheless have indirect experimental data relative to the formation of secondary particles of super-high energy. For example, in the works of Ryzhkova and Sarycheva[2], and also Kaplon *et al.*[3], the coefficient of absorption is measured in the atmosphere of radioactive particles with energy $\gtrsim 10^{12}$ ev. Moreover, there exist data relative to the number of μ-mesons[4] penetrating a depth of the earth of as much as 3 km of water equivalent, i.e., possessing energies of the order of 3×10^{12} ev. As the calculations show, these data are very sensitive to the mechanism of the elementary act