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Reflection of Waves from an Isotropic Inhomogeneous Layer

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An approximate method is constructed for the calculation of the reflection coefficient of a plane electromagnetic wave from an isotropically inhomogeneous layer for media whose dielectric constant and conductivity depend on a single spatial coordinate. This method can be also applied to the solution of analogous acoustical problems.

FOR the solution of a series of applied problems, it is necessary to be able to calculate the reflection coefficient of waves from different types of stratified structures. An exact solution of the problem can be obtained only for certain individual regularities of the layers[1-6]. In the other cases, we have to apply various approximate methods[7,8], which have definite limits of applicability. Chief practical interest attaches to stratified media, for which the optical constants can undergo discontinuous changes inside the layer and on its boundaries. For such cases, it is appropriate to make use of the approximate method of calculation proposed in the present research.

1. METHOD OF SOLUTION

Let us consider an infinite-laminar inhomogeneous strip of thickness d, on which a plane electromagnetic wave is incident from the left at an angle θ_1 with the normal. In this case, the wave is partially transmitted through the layer and falls on the second boundary of the layer at an angle θ_2. Let us place the origin at the front boundary of the layer. We draw the OZ axis perpendicular to this boundary, to the right, inside the layer and the OX axis parallel to the boundary in the plane of incidence of the oncoming wave.

The propagation constant of the electromagnetic waves in the layer depends on the coordinate z according to the law

$$k(z) = (\omega/c)[n(z) + i\varkappa(z)] = (\omega/c)\sqrt{\varepsilon^*(z)}, \quad (1)$$

$$n(z) = \left[\frac{\varepsilon(z)}{2} + \sqrt{\frac{\varepsilon^2(z)}{4} + \frac{4\pi^2\sigma^2(z)}{\omega^2}}\right]^{1/2},$$

$$\varkappa(z) = \left[-\frac{\varepsilon(z)}{2} + \sqrt{\frac{\varepsilon^2(z)}{4} + \frac{4\pi^2\sigma^2(z)}{\omega^2}}\right]^{1/2}. \quad (2)$$

The dielectric constant $\epsilon(z)$ and the conductivity $\sigma(z)$ change along the layer according to some law. To the left of the layer is a semi-infinite homogeneous medium with a propagation constant $k_1 = \text{const}$, in which the incident and reflected waves are propagated. To the right of the layer is a semi-infinite homogeneous medium with propagation constant $k_2 = \text{const}$, in which the transmitted wave is propagated. All three media are presumed nonferromagnetic, and their magnetic permeability is taken to be unity. Computation of the reflection coefficient of the wave from the layer is carried out separately for waves polarized in the plane of incidence and perpendicular to the plane of incidence.

As a first case we consider s-waves, whose electric vapor is perpendicular to the plane of incidence. The amplitude $E = E_y$ satisfies the equation

$$(\partial^2 E_y/\partial x^2) + (\partial^2 E_y/\partial z^2) + k^2(z) E_y = 0. \quad (3)$$

The solution of Eq. (3) inside the layer is sought in the form (see Refs. 9 and 10)

$$E_y = \exp[ik(z)\sin\theta(z)x] \times \left\{E^+ \exp\left[i\frac{\omega}{c}\int_0^z f_s(z)\,dz\right] + E^- \exp\left[-i\frac{\omega}{c}\int_0^z F_s(z)\,dz\right]\right\}, \quad (4)$$

where $\theta(z)$ is the angle variable, defined by the conditions

$$k(z)\sin\theta(z) = k_1 \sin\theta_1 = \text{const}, \quad (5)$$

and $f_s(z)$ and $F_s(z)$ are unknown functions which satisfy conjugate equations of first order:

$$f_s^2 - \varepsilon^*\cos^2\theta - i\frac{c}{\omega}\frac{df_s}{dz} = 0, \quad (6)$$

$$F_s^2 - \varepsilon^*\cos^2\theta + i\frac{c}{\omega}\frac{dF_s}{dz} = 0.$$

Solving these equations with consideration of the boundary conditions for the incident, reflected and transmitted waves, we can then determine the reflection coefficient from the layer.

For an approximate solution of Eq. (6), we expand the functions f_s and F_s in power series. For layers whose thickness is many times greater than the wavelength ($d \gg c/\omega$), we look for the solution in the form of the series

$$f_s = \sum_{m=0}^{\infty} (c/\omega)^m f_{s,m}(z), \qquad F_s = \sum_{m=0}^{\infty} (c/\omega)^m F_{s,m}(z). \tag{7}$$

Substituting (7) in (6), we find the following expressions for the coefficients of the series:

$$f_{s,0} = \sqrt{\varepsilon^*(z)} \cos\theta(z) = F_{s,0}, \tag{8}$$

$$f_{s,1} = \frac{i}{2} \frac{d \ln(\sqrt{\varepsilon^*} \cos\theta)}{dz} = -F_{s,1},$$

$$f_{s,2} = \frac{1}{4\sqrt{\varepsilon^*} \cos\theta} \left\{ \frac{1}{2} \left[\frac{d \ln(\sqrt{\varepsilon^*} \cos\theta)}{dz} \right]^2 - \frac{d^2 \ln(\sqrt{\varepsilon^*} \cos\theta)}{dz^2} \right\} = F_{s,2}$$

etc. The coefficients $f_{s,0}$ and $F_{s,0}$ correspond to the usual approximation of geometric optics.

For layers that are thin in comparison with the wavelength ($d \ll c/\omega$), the solution is sought in the form of the series

$$f_s = \sum_{m=0}^{\infty} (\omega/c)^m f_{s,m}(z), \qquad F_s = \sum_{m=0}^{\infty} (\omega/c)^m F_{s,m}(z). \tag{9}$$

The coefficients of these series are computed by successive integration:

$$f_{s,0} = \text{const}, \qquad F_{s,0} = \text{const}, \tag{10}$$

$$f_{s,1} = i \int_0^z [\varepsilon^* \cos^2\theta - f_{s,0}^2]\, dz,$$

$$F_{s,1} = -i \int_0^z [\varepsilon^* \cos^2\theta - F_{s,0}]\, dz,$$

$$f_{s,2} = 2 f_{s,0} \int_0^z \int_0^{z'} [\varepsilon^* \cos^2\theta - f_{s,0}^2]\, dz\, dz',$$

$$F_{s,2} = 2 F_{s,0} \int_0^z \int_0^{z'} [\varepsilon^* \cos^2\theta - F_{s,0}^2]\, dz\, dz'$$

etc. Making use of Eqs. (8) and (10), we can compute the functions f_s and F_s in any approximation, after which it is easy to find the amplitude reflection coefficient.

In the second case of the p-wave, whose electric vector lies in the place of incidence, it is more appropriate to solve the equation for the magnetic vector

$$\frac{\partial^2 H_y}{\partial x^2} + \frac{d^2 H_y}{dz^2} + \frac{1}{k^2(z)} \frac{dk^2(z)}{dz} \frac{\partial H_y}{\partial z} + k^2(z) H_y = 0. \tag{11}$$

Applying the analogous substitution

$$H_y = \exp[ik(z)\sin\theta(z)\, x] \times \left\{ H^+ \exp\left[i \frac{\omega}{c} \int_0^z f_p(z)\, dz \right] + H^- \exp\left[-i \frac{\omega}{c} \int_0^z F_p(z)\, dz \right] \right\} \tag{12}$$

and expanding the functions $f_p(z)$ and $F_p(z)$ in power series in c/ω or ω/c, we get: for "thick" layers,

$$f_{p,0} = \sqrt{\varepsilon^*(z)} \cos\theta(z) = F_{p,0}, \tag{13}$$

$$f_{p,1} = \frac{i}{2} \frac{d}{dz} \ln \frac{\cos\theta(z)}{\sqrt{\varepsilon^*(z)}} = -F_{p,1},$$

$$f_{p,2} = \frac{f_{p,1}}{2 f_{p,0}} \left[i \frac{d}{dz} \ln \frac{f_{p,1}}{\varepsilon^*} - f_{p,1} \right];$$

$$F_{p,2} = \frac{F_{p,1}}{2 F_{p,0}} \left[-i \frac{d}{dz} \ln \frac{F_{p,1}}{\varepsilon^*} - F_{p,1} \right],$$

and for "thin" layers,

$$f_{p,0} = \varepsilon^*(z) = F_{p,0}, \tag{14}$$

$$f_{p,1} = i\varepsilon^*(z) \int_0^z [\cos^2\theta(z) - \varepsilon^*(z)]\, dz = -F_{p,1},$$

$$f_{p,2} = 2\varepsilon^* \int_0^z \varepsilon^*(z') \int_0^{z'} [\cos^2\theta(z) - \varepsilon^*(z)]\, dz\, dz' = F_{p,2}.$$

For the analogous acoustical problem, the pressure in the layer satisfies the equation

$$\frac{d^2p}{dz^2} - \frac{d\ln\rho}{dz}\frac{dp}{dz} + \frac{\omega^2}{c_3^2} n^2 p = 0, \quad (15)$$

where the density $\rho(z)$, the sound velocity $c_s(z)$, and the relative index of refraction of the medium $n(z)$ are given as functions of the coordinate z, perpendicular to the surface of the layer.

Introducing the similar substitution

$$p = p^+ \exp\left[i\frac{\omega}{c_3}\int_0^z \varphi(z)\,dz\right] + p^- \exp\left[-i\frac{\omega}{c_3}\int_0^z \varphi(z)\,dz\right], \quad (16)$$

we can, in the same way as for electromagnetic waves, derive the differential equation for the function $\varphi(z)$, expand it in a power series in ω/c_s or c_s/ω, and find the reflection coefficient from the entire surface.

In practice, we can limit ourselves to 2-3 first terms of the corresponding series. In the intermediate region ($d \approx \lambda$), we can make use of interpolated expressions for R which consist of calculated limiting formulas.

2. EXAMPLE

For a layer law

$$n(z) = -2d/z \quad (17)$$

with boundary conditions: $n = n_1 = 1$ for $z \leq -2d$, and $n = n_2$ for $z \geq -d$, the problem has an exact solution. Calculation of the energy reflection coefficient for the normally incident electromagnetic waves yields the exact expression

$$R(x) = \frac{\sin^2[\sqrt{4x^2 - 1/4}\ln 2]}{16x^2 - \cos^2[\sqrt{4x^2 - 1/4}\ln 2]}, \quad (18)$$

$$x = \frac{\omega}{c}d = 2\pi\frac{d}{\lambda_0}.$$

Calculation according to our approximate formulas yields, with accuracy up to terms of second order of smallness,

$$R_0(x) = (1 - 0{,}56x^2)/(9 - 0.56x^2) \quad (19)$$

$$\text{for } x \ll 1,$$

$$R_\infty(x) = \frac{\sin^2[2x\ln 2 - \ln 2/16x]}{16x^2 - \cos^2[2x\ln 2 - \ln 2/16x]} \quad (20)$$

$$\text{for } x \gg 1.$$

Numerical computation according to these equations shows that over the whole range of values of $\omega d/c$, the divergence between the exact and the approximate formulas for $\sqrt{R(x)}$ lies only in the third decimal place. Moreover, both expansions (in powers of x and of $1/x$) give neighboring values even for $x \approx 1$.

In those cases when our expansion in powers of x and $1/x$ diverge strongly in the region $x \approx 1$, we must compute these series with higher order approximations for the region $x \approx 1$. We then use interpolation formulas of the type

$$R(x) \approx [R_0(x) + xR_\infty(x)]/(1 + x). \quad (21)$$

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