Expansion of Single Particle Wave Functions in Functions of The Relative Motion of the Nucleons

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For small energies of relative motion, identical nucleons interact with each other only when their relative orbital momentum is zero. Therefore, it can be expected that a nucleon in the nucleus does not interact with all of the nucleons, but only with those three which are in an S-state relative to it. These considerations lead to a picture which is in many respects similar to the α-particle model of the nucleus. The wave functions of the shell model are investigated from this point of view and a number of common properties of the shell and α-particle models of the nucleus are shown. Several regularities in the binding energies of light nuclei are discussed.

USUALLY the state of the nucleons in the nucleus is described by giving their angular momenta relative to the center of gravity of the nucleus (shell model). In such a case, the question of what angular momenta the nucleons in the nucleus have relative to each other is not considered. However, just this question is of particular interest for an understanding of the energetics of the nucleus and for an explanation of the structure of its state. The point is that the nucleons in the nucleus have a comparatively small kinetic energy (the mean energy is of the order 10-20 mev) and, as is known from experiments on nucleon-nucleon scattering, at these energies the nucleons interact only if they are in S-states relative to each other. Therefore, we can conclude that a nucleon in the nucleus does not interact with all nucleons, but only with those which are in S-states relative to it, and thus, for an evaluation of the energy of a given state of the nucleus and for an explanation of its structure, it is necessary to know the distribution of the angular momenta of relative motion of the nucleons.

The basic properties of the nucleus are clearly best described by the shell model, and therefore it is of interest to ask what the distribution of angular momenta of relative motion is in the shell-model wave functions. The elucidation of this question is the main aim of this article.

1. TWO NUCLEONS IN THE $1p$ SHELL

The simplest cases to study are the Li^6 and He^6 nuclei, where in addition to the full shell $(1s)^4$ there are two nucleons in states with $l = 1$ relative to the center of gravity of the nucleus. In the LS-coupling scheme, where the total orbital angular momentum L and spin S of the two nucleons are given, the wave functions of these nucleons break up into the product of the radial wave function $R(r_1)R(r_2)$, the angular wave function Λ, the spin function χ and isotopic spin function Ω:

$$\Psi(LST) = R(r_1)\,R(r_2)\,\Lambda(\theta_1, \varphi_1; \theta_2, \varphi_2)\,\chi(S)\,\Omega(T). \quad (1)$$

Here the angular wave function has the form

$$L = 0; M_L = 0; \quad (2)$$

$$\Lambda = \frac{1}{\sqrt{3}}[Y_{11}(\theta_1, \varphi_1)\,Y_{1-1}(\theta_2, \varphi_2) + Y_{1-1}(\theta_1, \varphi_1)\,Y_{11}(\theta_2, \varphi_2) - Y_{10}(\theta_1, \varphi_1)\,Y_{10}(\theta_2, \varphi_2)];$$

$$L = 1; M_L = 1; \Lambda = \frac{1}{\sqrt{2}}[Y_{11}(\theta_1, \varphi_1)\,Y_{10}(\theta_2, \varphi_2) - Y_{10}(\theta_1, \varphi_1)\,Y_{11}(\theta_2, \varphi_2)]$$

$$L = 2;\quad M^L = 2;\quad \Lambda = Y_{11}(\theta_1, \varphi_1)\,Y_{11}(\theta_2, \varphi_2),$$

where Y_{lm} is the spherical harmonic, and r_i, θ_i, φ_i are the spherical coordinates of the ith nucleon.

We introduce new coordinates: the radius vector of the center of gravity of the two nucleons $\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ and the radius vector of the distance between the nucleons $\mathbf{r} = \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2)$. We obtain the desired expansion of the wave function in states with given relative angular momentum of the nucleons by replacing $\mathbf{r}_1$ and $\mathbf{r}_2$ in the wave function (1) by their expression in terms of $\mathbf{R}$ and $\mathbf{r}$ and expanding the resulting expression in angular functions of Θ, Φ and ϑ, φ (spherical angles of the vectors $\mathbf{R}$ and $\mathbf{r}$, respectively). The expansion of the angular part of the wave function can be carried out at once, using the relations

$$r_1 \cos\theta_1 = R\cos\Theta + r\cos\vartheta; \quad (3)$$

$$r_1 \sin\theta_1 e^{\pm i\varphi_1} = R\sin\Theta e^{\pm i\Phi} + r\sin\vartheta e^{\pm i\varphi},$$

$$r_2 \cos\theta_2 = R\cos\Theta - r\cos\vartheta;$$

$$r_2 \sin\theta_2 e^{+i\varphi_2} = R\sin\Theta e^{\pm i\Phi} - r\sin\vartheta e^{\pm i\varphi},$$

which follow from the definition of the vectors **R** and **r**.

From this it is found that the angular wave functions of the *S*-, *P*- and *D*-states of two nucleons are proportional to

$$\Lambda(L=0) \sim Y_{00}(\Theta, \Phi)\, Y_{00}(\vartheta, \varphi)/r_1 r_2; \quad (4)$$

$$\Lambda(L=1,\ M_L=1) \sim [Y_{11}(\Theta, \Phi)\, Y_{10}(\vartheta, \varphi) - Y_{10}(\Theta, \Phi)\, Y_{11}(\vartheta, \varphi)]/r_1 r_2;$$

$$\Lambda(L=2,\ M_L=2) \sim [R^2 Y_{22}(\Theta, \Phi)\, Y_{00}(\vartheta, \varphi) - r^2 Y_{00}(\Theta, \Phi)\, Y_{22}(\vartheta, \varphi)]/r_1 r_2.$$

It is still necessary to find the expansion of the radial part of the wave function. It is easy to see that this expansion has the following form:

$$R(r_1)\,R(r_2) = \sum_{lm} B_l(r, R)\, Y_{lm}(\Theta, \Phi)\, Y_{l-m}(\vartheta, \varphi)\,(-)^m. \quad (5)$$

Since the form of the function $R(r_i)$ is unknown, it is impossible to calculate the coefficients B_l. However, this is not necessary, since for any reasonable form of this function, the first term of this expansion is most important, so that approximately $R(r_1)R(r_2) \approx B_0(r, R)$, where B_0 is a function of the absolute values of the quantities **r** and **R**. The justification of this approximation is clear from the fact that r_1 and r_2, which are respectively equal to $(r^2 + R^2 \pm 2\mathbf{rR})^{1/2}$, depend strongly on the directions of the vectors **R** and **r** only when $r \approx R$ and $\mathbf{Rr} \approx rR$, which corresponds to a comparatively small region in the space of the variables *r* and *R*.

Collecting the result , we see that in states with $L = 0$ the nucleons are in *S*-states relative to each other and their center of gravity is in an *S*-state relative to the center of gravity of the nucleus; in the state with $L = 1$ the nucleons have a relative angular momentum $l = 1$ and their center of gravity moves relative to the center of gravity of the nucleus with unit angular momentum and, finally, that in a state with $L = 2$ the nucleons spend about half the time in an *S*-state relative to each other and about half the time in a *D*-state; their center of gravity also is in either an *S*- or a *D*-state relative to the center of gravity of the nucleus with approximately equal weights.

It is already possible to draw the conclusion confirmed by experiment[1] that the *S*-state of two nucleons ($L = 0$) is energetically most favorable, than the *D*-state ($L = 2$) and, finally, the state with $L = 1$ will have the highest energy. In fact, in the framework of the shell theory, the energies of these states differ only by the interaction energy of the two nucleons considered, and since the nucleons interact strongly only if they are in an *S*-state relative to each other, the states which are most favorable energetically will be those in which the nucleons spend the most time in *S*-states relative to each other.

We note one curious feature of our results. It is well known that the ground state of Li^6 is a $^3S_1$-state ($L = 0$, $S = 1$). As we have just seen, in this state two external nucleons in the *p*-shell have a relative angular momentum equal to zero and therefore, the ground state of Li^6 can be considered as a sum of an α-particle (closed shell $(1s)^4$] and a deuteron (two nucleons in the *p*-shell in an *S*-state relative to each other). It should be kept in mind, however, that this analogy is true only as long as we are concerned with the angular wave function of Li^6; the dependence of the wave function on the distance between the two external nucleons can be completely different from that in the deuteron.

2. GENERAL CASE OF *n* NUCLEONS IN THE SAME SHELL

In the case where more than two nucleons are outside the closed shell, the expansion is, in principle, carried out in the same way as in the case of two nucleons considered above. The calculation here, however, is extremely cumbersome; therefore, we limit ourselves to several general statements.

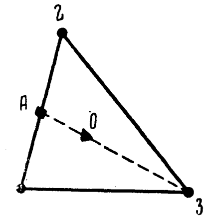

FIG. 1.

It is easy to show that the Pauli principle prohibits more than four nucleons from being in *S*-states relative to each other. (When we speak of several nucleons being in S-states relative to each other, we understand by that a state in which each nucleon of that group is in an S-state relative to any other nucleon of the group.) In fact, it is sufficient to note that nucleons which are in S-states relative to each other are also in S-states relative to the center of gravity*, and obviously, more than four nucleons cannot be in an S-state relative to the same point.

Thus, the maximum number of nucleons, which are at the same time in relative S-states, is equal to four. However, this number is an upper limit and cannot be attained in several states. The analysis shows that the criterion here is the scheme of Jahn of the given state.

In order to understand how this comes about, we consider a concrete example of five nucleons which are in a p-shell (this corresponds to the nuclei B^9 and Be^9). If four nucleons are in S-states relative to each other, then each of these is in an S-state relative to the general center of gravity. Consequently, in the coordinate system connected with their center of gravity, all of them are in the same spatial state and therefore, the spatial part of their wave functions in this system should be symmetrical in all four particles. But the symmetry property of wave functions does not change in the transition from one system of coordinates to another and thus we come to the conclusion that four nucleons, which are in the same shell, can be in S-states relative to each other in case the spatial part of the wave function is symmetrical in these nucleons. But the symmetry properties of the wave functions are determined by the scheme of Jahn, who shows how many particles of a given shell are in symmetrical states, or, in other words, how many particles can be simultaneously in one of the spatial states (that is, have the same value of the magnetic quantum number). Thus, for example, in the case of five particles in a p-shell, the states can be described by the following schemes of Jahn: [41] [32], [311], [221]. In states described by the scheme [41], the spatial part of the wave function is symmetrical in four nucleons and, consequently, the four nucleons can be in the same spatial state, because in these states the Pauli principle allows four nucleons to be simultaneously in the four S-states relative to each other. In those states, described by the scheme [32], the four nucleons cannot have the same values of the magnetic quantum number, and therefore they cannot be simultaneously in S-states relative to each other. As one sees from the scheme, in this case only three nucleons can be in one spatial state, and two others, in the other. Thus, the maximum number of nucleons simultaneously in S-states relative to each other is here equal to three. Here the remaining two nucleons can have the same magnetic quantum number and, consequently, be in S-states relative to each other.

Thus, we come to the following conclusion. In order for four nucleons in a given state to be simultaneously in S-states relative to each other, it is necessary that in the scheme of Jahn $[f_1, f_2, \ldots f_r]$, corresponding to this state, at least one number f_i must be equal to 4. If the Jahn scheme contains several numbers equal to 4: $f_1 = f_2 = \ldots = f_k = 4$, then in this state several groups of nucleons (four in each group) can exist in S-states relative to each other. The condition that in a given state three or two nucleons can be in relative S-states is formulated in a completely analogous fashion.

The conditions formulated above are necessary, but not sufficient. Direct calculations carried out for several states of three or four particles in the p-shell show, however, that in all those cases, when according to these conditions four or three nucleons can be simultaneously in S-states relative to each other, terms actually occur in the expansion of the wave function describing such groups of nucleons. The fact that, together with this term, the expansion gives also other terms corresponding, for example, to two nucleons in S-states and a third in a p-state relative to them, etc., is connected

* The proof of this statement is based on the following lemma: if two nucleons are in an S-state relative to the same point, then their center of gravity is in an S-state relative to this point. This lemma is proved directly by expanding the single-particle wave functions in functions of the relative motion of the nucleons. The corresponding calculation differs from that given in Sec. 1 only in that there nucleons in a p-shell were considered, whereas it is necessary to consider two nucleons in an s-shell.

We consider the case of three nucleons (Fig. 1). Let the 1st and 2nd nucleon be in an S-state relative to the 3rd. Then, according to our lemma, the center of gravity of the 1st and 2nd nucleons A is also in an S-state relative to the 3rd nucleon and, consequently, the 3rd nucleon and the center of gravity of the 1st and 2nd nucleons is in an S-state relative to the general center of gravity of all those nucleons O. Beginning with the calculation of the 1st and 2nd nucleon, it is possible to show in the same way that they are in an S-state relative to the general center of gravity.

The generalization to the case of a larger number of particles is obvious.

with the specific form of the wave functions of the shell model, which describe a system of noninteracting and, consequently, uncorrelated particles.

3. CONCLUSIONS AND COMPARISON WITH EXPERIMENT

We enumerate the basic theses of our article:

1. Nucleons in the nucleus interact strongly only with those nucleons which are in S-states relative to them. The number of nucleons simultaneously in S-states relative to each other cannot exceed four.

2. Insofar as nucleons interact strongly only in S-states, one can expect that in the energetically most favorable states of the nucleus (the ground and first excited states) the nucleons will be strongly correlated with each other so that the largest number of nucleons can be in S-states relative to each other. Here, in analogy with the deuteron, with tritium and the α-particle, consisting respectively of two, three and four nucleons in relative S-states, one can expect the mutual binding energy of the nucleons in S-states to grow with the number of nucleons n faster than the number of bonds between nucleons $n(n-1)/2$. Therefore, the grouping of four correlated nucleons should be especially favorable energetically.

3. The existence of such groups of nucleons (four in each) in the nucleus is definitely indicated by the experimental data. We speak here of the great aggregate of data which is usually drawn upon as a basis for the α-particle model (see Ref. 1 for the literature). One should also bear in mind the following experimental facts:

a) From the analysis of the masses of light nuclei it is possible to find out at which excitation energies the first level with different isotopic spins T appears[3] in a nucleus with atomic number A. Here it turns out that there is a striking difference between the spectra of nuclei with $A = 4n$ and $A = 4n + 2$; in nuclei with $A = 4n$ the energy differences are small between first levels with $T = 1$ and $T = 2$, and between those with $T = 3$ and $T = 4$, and large between first levels with $T = 1$ and $T = 0$, and between those with $T = 2$ and $T = 3$ (Fig. 2a). In nuclei with $A = 4n + 2$ the former differences $2-1$ and $4-3$ are, on the other hand, large and the latter differences $1-0$ and $3-2$ are small (Fig. 2b). In addition, it is of interest that with nuclei of the same type ($A = 4n$ or $A = 4n + 2$) the general form of the spectrum changes only slightly in going from nucleus to nucleus and that all spectra are periodic in T with the period $\Delta T = 2$ (the differences $1-0$ or $3-2$; and $2-1$ or $4-3$ are almost the same for nuclei of the same type). The general picture of these spectra is as if the total isotopic spin of the nucleus were divided into the isotopic spins of separate groups of nucleons (with four in each group, and with nuclei of the type $A = 4n + 2$, in addition a group of two nucleons), which are successively excited with increasing energy of the nucleus;

b) If we construct a cruve of the dependence of the nucleon binding energy as a function of mass number in nuclei with a minimum neutron surplus $(N-Z)$ for a given A, then we obtain a saw-toothed curve with maxima at $A = 4n$ and minima at $A = 4n + 1$ (Fig. 3). The picture is, especially for small A, very similar to that which would be obtained if separate groups (four in each) of strongly interacting nucleons existed in the nucleus, these groups weakly interacting with the other nucleons in the nucleus.

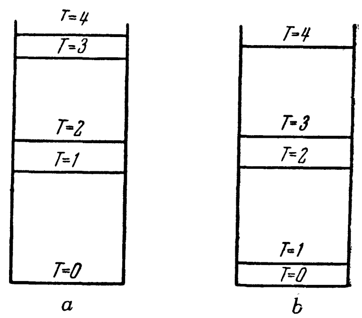

FIG. 2.

4. The shell-model wave functions contain a considerable proportion of states corresponding to the presence of one or several groups of nucleons in relative S-states. The scheme of Jahn for a given state is the criterion for how many particles in a given single-particle state are in S-states relative to each other, and for which groups they can be split up into (with two, three or four nucleons in a group). For example, in a given single-particle state described by the scheme [432] there are three groups of nucleons which are in relative S-states: groups of four, three and two particles, respectively. Similarly, in a state with the scheme [441] there exist only two groups with four nucleons in each.

Beginning from this correspondence between the Jahn scheme and the structure of the state, and remembering that the nucleons interact only in S-states, one can see why in the shell theory the states which are energetically most favorable are those states which, in the Jahn scheme, have the maximum possible number of fours, threes and twos (such states are often called states of maximum symmetry in the literature). For example, the state with the scheme [411] is energetically more favorable than the state with the scheme [321], and the latter is, in turn, more favorable than the state with the scheme [222].

5. We note, finally, the connection between the classification outlined above of states according to the number of nucleons in S-states relative to each other, and the classification according to the seniority quantum number[2], defined as $v = n - 2x$, where n is the total number of particles in the shell and x is the number of pairs of nucleons (each nucleon is counted once) having total orbital momentum $L = 0$. As we saw from the example of the Li^6 nucleus, every such nucleon pair is in an S-state relative to the other. Therefore, the classification of states according to the value of v is a classification according to the number of pairs of nucleons (each nucleon counted only once!) which are in S-states relative to the other.

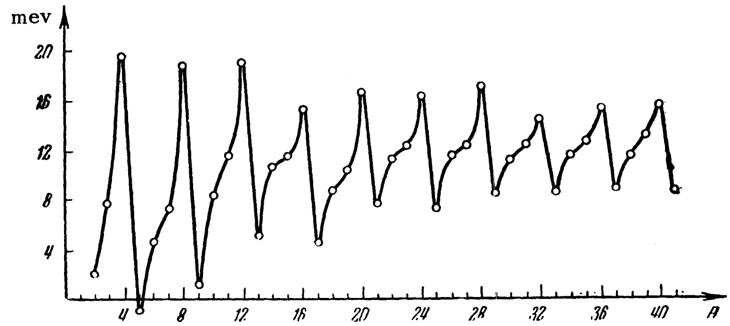

FIG. 3.

The main shortcoming of such a classification consists in the fact that each nucleon is counted only once and therefore this description of the state is not complete. In fact, if the nucleons interact only in S-states, then the total number of nucleon pairs in S-states is important for the determination of the energy of the state, and for this it may be necessary to count the same nucleon several times (if more than two nucleons are in an S-state relative to each other).

In conclusion, I would like to use the opportunity to thank Ia. A. Smorodinskii for numerous discussions and constant interest in this work.

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