Theory of Nonlocal Interaction

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The possibility of a Hamiltonian structure in the theory of nonlocal interaction is considered. By way of example, the one-dimensional oscillator is considered by Pauli's method. The initial integro-differential equations of motion and the canonical system of equations of motion obtained by Pauli's method are not equivalent. The procedure of Hayashi is shown to be internally inconsistent in any given order. Thus, conclusions about the essentially non-Hamiltonian character of nonlocal interactions are confirmed.

1. THE theory of nonlocal fields has been intensively discussed in recent years* in connection with the investigation of possible ways to remove the fundamental difficulties of field theory, which are reflected in the apparatus of the present theory in the presence of divergent quantities having no physical meaning, and also in the impossibility of consistently carrying through a renormalization program, and in the divergence of experimental and theoretical data in mesodynamics, etc. At present, we still know comparatively little about the laws of motion in small ($r \lesssim 10^{-13}$ cm, $\tau \lesssim 10^{-23}$ sec) space-time regions. Therefore, a theoretical study of possible deviations in small space-time regions from the requirements and principles, justified through phenomena taking place in macroscopic space-time regions, is completely in order and even unavoidable. At the present time the theory of nonlocal interactions should, in our view, be considered as a phenomenological study of the possibility of introducing signals which have velocities greater than light in small space-time regions into the framework of linear field theory, as a first rough phenomenological exploration into the region of possible structures of elementary particles.

In this connection, the investigation of problems concerning the Hamiltonian structure of theories of nonlocal interactions is important. By 1945-1948, it had been shown in the works of Blokhintsev and Markov[1, 4-7] that, in the theory with form-factors, the state of the physical system given on the hyper surface σ does not uniquely determine the state on the hyper surface $\sigma + \delta\sigma$ as a result of the presence of signals with velocity greater than light; in microscopic space-time regions; in other words, that the theory does not have a Hamiltonian structure and that it is necessary to formulate supplementary conditions** to pick out the solution with a definite physical meaning.

However, following this, work was published[5,6] in which it was asserted that a Hamiltonian formulation of the theory is possible. In addition, it was asserted that the conditions of integrability are satisfied, not only on a space-like hyper surface, but also in finite time-like regions, and that they are conserved in the limit of the transition to the theory with local interactions.

2. Pauli[5] set out to construct a theory with nonlocal interaction in Hamiltonian form, using the connection, which is well known in analytical dynamics, of the Hamiltonian system of the equations of motion with bilinear differential forms obtained from the equations of motion in the Lagrangian variables[8]. For example, for the case of the one-dimensional oscillator with nonlocal interaction, from the equation of motion

$$\ddot{q}(t) + \omega^2 \int_{-\infty}^{+\infty} K(t - t_1)\, q(t_1)\, dt_1 = 0 \qquad (1)$$

it is easy to obtain the bilinear differential form

$$I(t;\ D;\ \delta) = (Dq(t);\ \delta\dot{q}(t)) + \frac{\omega^2}{2} \int_{-\infty}^{+\infty} \varepsilon(t - t_2)\, K(t_2 - t_1)\, (Dq(t_2);\ \delta q(t_1))\, dt_1 dt_2, \qquad (2)$$

satisfying the condition

$$(d/dt)\, I(t;\ D;\ \delta) = 0. \qquad (3)$$

* A detailed bibliography is given in the reviews, Refs. 1 and 2, and also in Ref. 3.

** We do not consider the theory with a dynamically deformable form-factor.

Here

$$(Dq(t);\ \delta\dot{q}(t)) \equiv [Dq(t)\,\delta\dot{q}(t) - \delta q(t)\,D\dot{q}(t)];$$

$$\varepsilon(t) = \begin{cases} +1 & \text{for} \quad t > 0 \\ -1 & \text{for} \quad t < 0. \end{cases}$$

It is essential to note that in (2) the independent variations of the dynamical variables are taken at different times, which prevents the direct applications of theorems of analytical dynamics. In order to overcome this difficulty, Pauli proposed, starting from the equation of motion (1), to express $q(t_1)$ and $q(t_2)$ through the independent dynamical variables q and q, taken at time t, where

$$q(t_j) = q(t)\,Q_1(t_j - t) + \dot{q}(t)\,Q_2(t_j - t), \tag{4}$$

$$Q_1(t_j - t) = \cos k^*(t_j - t);$$

$$Q_2(t_j - t) = (1/k^*)\sin k^*(t_j - t)$$

and k^* is the root of the equation*

$$k^2 - \omega^2 \mathscr{K}(t) = 0;\ \mathscr{K}(k) = \int_{-\infty}^{+\infty} K(t)\,e^{-itk}dk. \tag{5}$$

The differential form (2), when expressed in terms of the independent variations $Dq(t)$, $\delta q(t)$ and $dq(t)$, also satisfies the condition

$$dI(t;\ D;\ \delta) + \delta I(t;\ d;\ D) + DI(t;\ \delta;\ d) = 0$$

and it can be considered as a bilinear covariant of some Pfaffian form[5], which can be reduced to canonical form by the corresponding transformation of dynamical variables[9].

The argument given above is essentially based on the possibility of solving the equation of motion (1) which, however, requires supplementary conditions to be given either in the whole interval $[-\infty;\ +\infty]$ or, in some cases, at the infinite points* $t = \pm\infty$. As a consequence, the initial equation of motion (1) and the canonical system of equations of motion found by the method of Pauli are not equivalent. In other words, the method of Pauli is equivalent to the replacement of integro-differential equations of motion by differential equations for which the problem of Cauchy can be formulated (i.e., which do not require supplementary conditions over the infinite interval) and the solutions of which coincide with those of the initial integro-differential equations of motion obtained from the variational principle.

Since the canonical system of equations of Pauli is constructed with the aid of a solution of the initial integro-differential equation (1) found previously, that system of equations is essentially superfluous and cannot give anything new over and above that given by Eq. (1).

3. In Hayashi's work[6] the method of Umezawa and Takahashi[7], which had previously been successfully applied to the theory with local interaction, was employed to write down the system of equations of motion for nonlocal interaction in Hamiltonian form.

The system of equations of motion of Bloch and Møller[11] for interaction of a scalar neutral field $A(x)$ and a scalar charged field $\psi(x)$ can be written in the form

$$\frac{\delta A_0[x;\ \sigma]}{\delta\sigma(x')} = -g\int \Delta_\mu(x - x')\,\psi^*(x_1)\,\psi(x_3)\,F(x_1x'x_3)\,d^4(x_1x_3) - \frac{\delta}{\delta\sigma(x')}\sum_{k=1}^{\infty} g^k a_k[x;\ \sigma]; \tag{6}$$

$$\frac{\delta\psi_0[x;\ \sigma]}{\delta\sigma(x')} = -g\int \Delta_m(x - x')\,A(x_2)\,\psi(x_3)\,F(x'x_2x_3)\,d^4(x_2x_3) - \frac{\delta}{\delta\sigma(x')}\sum_{k=1}^{\infty} g^k\varphi_k[x;\ \sigma], \tag{7}$$

$$A(x) = \sum_{k=0}^{\infty} g^k A_k[x;\ \sigma];\qquad \psi(x) = \sum_{k=0}^{\infty} g^k\psi_k[x;\ \sigma]. \tag{8}$$

If the equations (6) and (7) are rewritten in the form

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* It is assumed that the form-factor $K(t)$ is such that (5) has only two roots $k = \pm k^*$, differing by sign. Otherwise, the argument would have to be changed because in that case there will be a larger number of independent dynamical variables.

* Supplementary conditions must be given in the interval $[-\infty;\ +\infty]$ even in the simpler case of the solution of the homogeneous integral equation[10]

$$q(t) = \int_{-\infty}^{+\infty} K(t - t_1)\,q(t_1)\,dt_1.$$

$$\delta A_0[x;\ \sigma]/\delta\sigma(x') = i[H(x'/\sigma);\ A_0[x;\ \sigma]]; \quad (9)$$

$$\delta\psi_0[x;\ \sigma]/\delta\sigma(x') = i[H(x'/\sigma);\ \psi_0[x;\ \sigma]], \quad (10)$$

then, comparing the right-hand sides of Eqs. (6) and (9), (7) and (10), and choosing in a corresponding fashion expressions for the supplementary fields $a_k[x;\sigma]$ and $\varphi_k[x,\sigma]$, it is possible to find a formal expression* for the functional $H[x;\sigma]$, also in the form of a series in powers of g:

$$H[x;\ \sigma] = \sum_{k=1}^{\infty} g^k H_k[x;\ \sigma]^5. \quad (11)$$

The equations (9), (10) are integrable only if

$$\left[\frac{\delta}{\delta\sigma(x'')};\ \frac{\delta}{\delta\sigma(x')}\right] A_0[x;\ \sigma] = 0; \quad (12)$$

$$\left[\frac{\delta}{\delta\sigma(x'')};\ \frac{\delta}{\delta\sigma(x')}\right] \psi_0[x;\sigma] = 0,$$

or, taking into account the expansion, Eq. (11),

$$[\delta/\delta\sigma(x'');\ \delta/\delta\sigma(x')]\, A_0[x;\ \sigma]$$
$$= i\sum_{k=1}^{\infty} g^k\left\{\left[H_k(x'/\sigma);\ \frac{\delta A_0[x;\ \sigma]}{\delta\sigma(x'')}\right] - \left[H_k(x''/\sigma);\ \frac{\delta A_0[x;\sigma]}{\delta\sigma(x')}\right] + \left[\frac{\partial H_k(x'/\sigma)}{\partial\sigma(x'')};\ A_0[x;\ \sigma]\right] - \left[\frac{\partial H_k(x''/\sigma)}{\partial\sigma(x')};\ A_0[x;\ \sigma]\right]\right\} \text{etc.}$$

where the symbol $\partial/\partial\sigma(x)$ denotes differentiation on the hyper surface σ, and clearly enters into the expression for the Hamiltonian $H_k[x;\sigma]$.

Employing the expression for $H_k[x;\sigma]$ obtained by Hayashi[6], it can be shown that

$$\sum_{k=1}^{\infty} g^k\left\{\left[H_k(x'/\sigma);\ \frac{\delta A_0[x;\ \sigma]}{\delta\sigma(x'')}\right] - \left[H_k(x''/\sigma);\ \frac{\delta A_0[x;\ \sigma]}{\delta\sigma(x')}\right]\right\}$$
$$= i[[H_1(x'/\sigma);\ H_1(x''/\sigma)];\ A_0[x;\ \sigma]]\, g^2 + O(g^3)$$
$$= i\int F(x_1x_2x_3)\, F(x_1'x_2'x_3') \sum_{i,j=1}^{3} a_i a_j [\delta(x'-x_i)\,\delta(x''-x_j')$$
$$- \delta(x'-x_j')\,\delta(x''-x_i)]\ \Delta_m(x_3'-x_1)\,\psi_0^*[x_1';\ \sigma]\ \psi_0[x_3,\ \sigma]\{\Delta_\mu(x_2-x)\, A_0[x_2';\sigma]$$
$$+ \Delta_\mu(x_2'-x)\, A_0[x_2;\ \sigma]\}\, d^4(x_1\ldots x_3') + O(g^3);$$

$$\left[\frac{\partial H_2[x';\ \sigma]}{\partial\sigma(x'')};\ A_0[x;\ \sigma]\right] = -\frac{i}{2}\int F(x_1x_2x_3)\, F(x_1'x_2'x_3')$$
$$\times \sum_{i;j=1}^{3} a_i a_j \delta(x'-x_i)\,\delta(x''-x_j')\{\Delta_\mu(x_2-x)\, A_0[x_2';\ \sigma]$$
$$+ \Delta_\mu(x_2'-x)\, A_0[x_2;\ \sigma]\}\{\Delta_m(x_1-x_1')\,\psi_0[x_3';\ \sigma]\,\psi_0[x_3;\ \sigma]$$
$$+ \Delta_m(x_3-x_3')\,\psi_0^*[x_1;\ \sigma]\,\psi_0^*[x_1';\ \sigma]\}\, d^4(x_1\ldots x_3');$$

$$\left[\frac{\delta}{\delta\sigma(x'')};\ \frac{\delta}{\delta\sigma(x')}\right] A_0[x;\ \sigma] = -g^2\int d^4(x_1\ldots x_3')\, F(x_1x_2x_3)\, F(x_1'x_2'x_3')$$
$$\times \sum_{i;j=1}^{3} a_i a_j [\delta(x'-x_i)\,\delta(x''-x_j') - \delta(x'-x_j')\,\delta(x''-x_i)]$$
$$\times \{\Delta_\mu(x_2-x)\, A_0[x_2';\ \sigma] + \Delta_\mu(x_2'-x)\, A_0[x_2;\ \sigma]\}$$
$$\times \{\Delta_m(x_3-x_1)\,\psi_0^*[x_1';\sigma]\,\psi_0[x_3;\sigma] - {}^1\!/_2\Delta_m(x_1-x_1')\,\psi_0[x_3';\sigma]\,\psi_0[x_3;\sigma]$$
$$- {}^1\!/_2\Delta_m(x_3-x_3')\,\psi_0^*[x_1;\ \sigma]\,\psi_0^*[x_1';\ \sigma]\} + O(g^3).$$

An analogous expression can be obtained also for the field $\psi_0[x;\sigma]$. In the theory of nonlocal interaction these expressions do not go to zero, and consequently, the conditions (12) are not satisfied. Thus, the procedure of Hayashi is internally inconsistent in an arbitrary kth order ($k \neq \infty$), and the equations of motion do not have a Hamiltonian structure.

However, from consideration of Eqs. (6) and (7), it follows that the condition of integrability (12)

* In the particular case of the first approximation ($k=1$) we obtain

$$H_1[x';\ \sigma] = -\int \psi_0^*[x_1;\ \sigma]\, A_0[x_2;\ \sigma]\,\psi_0[x_3;\ \sigma]\, F(x_1x_2x_3) \sum_{i=1}^{3} a_i\delta(x'-x_i), \quad (11')$$

where $a_1 = a_3$; $a_1 + a_2 + a_3 = 1$. It is easy to convince oneself of the validity of this expression by the direct substitution of (11′) in the equations (9) and (10). After substitutions, these equations coincide with (6) and (7) to order g^2. The hyper surface σ clearly enters into $H_k[x;\sigma]$ in subsequent approximations ($k \geq 2$).

will be fulfilled in the limit of $k \neq \infty$, if for arbitrary* $k > 1$

$$\left[\frac{\delta}{\delta\sigma(x'')}; \frac{\delta}{\delta\sigma(x')}\right] a_k[x; \sigma] = 0; \qquad (13)$$

$$\left[\frac{\delta}{\delta\sigma(x'')}; \frac{\delta}{\delta\sigma(x')}\right] \varphi_k[x; \sigma] = 0,$$

and the series (8) and (11) converge.

In this case the many-time equations of motion for the fields $A_0[x, \sigma]$ and $\psi_0[x; \sigma]$, and also the corresponding equation of Tomonaga-Schwinger in the interaction representation would have solutions for signals with velocity faster than light. These equations would also have single-valued solutions in finite time-like regions for transitions to the limit $\lambda \to 0$.

The investigation of the convergence of the series (8) and (11) presents considerable difficulty because of the complexity of the procedure of determining the quantities $H_k[x; \sigma]$; $a_k[x; \sigma]$ and $\varphi_k[x; \sigma]$. However, the very rapid increase in the number of terms in the expressions for $H_k[x; \sigma]$; $a_k[x; \sigma]$ and $\varphi_k[x; \sigma]$ as $k \to \infty$ gives no grounds for expecting that the series (8) and (11) will be convergent.

At this point there is an essential difference from the theory with local interaction, where--ignoring the divergence of the expansion in powers of the coupling constant g--after carrying out the procedure of renormalization, it is possible to use several of the initial terms in these expansions; this gives, in many cases, a sufficiently good asymptotic representation of the quantities considered, as is shown by comparison with experiment. In the theory with nonlocal interaction, it is impossible to stop at a finite number of terms, since the condition (12) is not fulfilled in this case.

* Since the fields $a_k[x; \sigma]$ and $\varphi_k[x; \sigma]$ in Ref. 6 can still be expanded in series in powers of the constant g, it is clear that the fulfillment of Eq. (12) to order g^{k^*} does not follow from the validity of Eq. (13) for some value of $k = k^* > 1$.

It is also important to note that the theory of Hayashi and the method of Pauli are both essentially tially based on the possibility of solving the integro-differential equations of motion of Bloch and Møller, which is impossible without the use of supplementary conditions over the finite interval $x \epsilon [-\infty + \infty]$.

Thus, in both the Heisenberg representation and the interaction representation[3] it is not possible to construct a theory with nonlocal interaction which has a Hamiltonian structure.

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