the constant component is equal here to 1 instead of 3, according to the theory of Davydov and Shmushkevich. Since at low temperatures the results coincide, the temperature dependence of α must also change correspondingly.

For the transverse Nernst-Ettingshausen effect:

$$\text{emf} = {}^1/_2 (n - 1) (\varkappa T / e) R\sigma H dT / dx$$

(H is the magnetic field, R is the Hall constant, σ is the electrical conductivity) one must expect an increase of the coefficient ($n - 1$) by a factor of three.

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On the Theory of the Stability of a Layer Located at a Superadiabatic Temperature Gradient in a Gravitational Field

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The stability of a layer of liquid or gas in the presence of a superadiabatic temperature gradient is investigated for cases in which the upper and lower boundaries of the layer are not fixed, and convection arising in it may spread into stable regions bordering it.

1. INTRODUCTION

IT is known that the equilibrium of a layer located in a gravitational field is stable if its entropy S increases with ehight[1]. A series of authors[2-4] have investigated the stability of a layer bounded by horizontal planes on which the temperature is given and the vertical component of the velocity $v_z = 0$. However, in a whole series of cases, the layer in which instability arises, causing an increase of convection, is bordered on one or both sides by stable layers in which the temperature gradient is less than adiabatic, but in which the motion is propagated, occupying a region considerably exceeding the region of instability. The propagation of convection beyond the limits of the unstable layer may be understood in the following way.

With a random rise of a separate convective element, a lifting force is developed proportional to the difference between the temperature ϑ of this element and that of the surrounding medium, and increases monotonically up to the upper boundary of the unstable layer. Therefore, the convective element arrives at the upper boundary with maximum acceleration. Above the boundary, the temperature difference, and consequently the acceleration, start to decrease, becoming zero at some height, at which the rising velocity is maximum. Only then does the element begin to slow down, coming to rest at a still greater height.

Similar effects also result when a convective element sinks below the boundary of the unstable region. Thus, for example, the height of the troposphere, i.e., the region of convective mixing, reaches 15 km, i.e., exceeds by a factor of three the height of the layer where the temperature gradient is superadiabatic. Such propagation of convection beyond the limits of a region of instability may occur under the most diverse circumstances, from the mixing of molten metal, or oil in a well[5], to the effects of convective mixing in the atmospheres of stars[6] and their central regions[7], and also in interstellar matter.

Up to the present time, however, this possibility has not been taken into account in spite of the fact that is consideration may substantially change a series of conclusions.

We consider two cases: 1) convection may propagate only upwards; on the lower boundary the temperature is everywhere the same, and friction is negligible, and 2) convection may propagate both upwards and downwards from the unstable layer.

We shall assume that the vertical dimensions of all regions of mixing are sufficiently small so that a) throughout its extent the relative change of temperature $\Delta T/T$, and consequently the relative changes of all other essential quantities (density, viscosity, thermal conductivity, etc.) are negligibly small, and b) the density in the absence of convection may be considered independent of height.

2. CASE OF UPWARD PROPAGATION OF CONVECTION

The system of equations of stationary convection in a linear approximation has the form[1]

$$\nabla P = -\rho\beta \mathbf{g}\vartheta + \eta\nabla^2\mathbf{v}, \ \operatorname{div}\mathbf{v} = 0,$$
$$\rho T(\mathbf{v}\nabla S) = \varkappa\nabla^2\vartheta,$$

which is easily reduced to one equation

$$\Delta^3 v_z = \frac{\rho^2\beta g T(\partial S/\partial z)}{\varkappa\eta}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v_z. \quad (1)$$

If the height h of the unstable layer is introduced as the unit of length, and if we assume that all variables, including v_z, are proportional to $\exp\{i(p_x x + p_y y)\}$, then Eq. (1) takes the form

$$(\partial^2/\partial z^2 - p^2)^3 v_z = -p^2 C v_z \quad 0 < z < 1, \quad (2)$$

$$(\partial^2/\partial z^2 - p^2)^3 v_z = p^2 C_1 v_z \quad z > 1, \quad (3)$$

where

$$C = \rho\beta g T\,|\partial S/\partial z|\,h^4/\varkappa\eta, \quad (4)$$

and in C_1 we replace $|\partial S/\partial z|$ by the corresponding value $|\partial S_1/\partial z|$ above the unstable layer. We assume that $|\partial S_1/\partial z|$ is independent of z. The difference in sign occurs since $\partial S/\partial z < 0$, and $\partial S_1/\partial z > 0$.

On the lower boundary of the unstable layer $z = 0$, neglecting friction, we have[3]

$$v_z = 0, \quad \partial^2 v_z/\partial z^2 = 0, \quad \partial^4 v_z/\partial z^4 = 0, \quad (5)$$

and on its upper boundary v_z and its derivatives up to fifth order inclusive must be continuous. At $z = \infty$,

$$v_z = 0. \quad (6)$$

The solution of Eq. (2), satisfying conditions (5) on the lower boundary of the unstable layer, has the form

$$v_z = \sum_{k=1}^{3} A_k \operatorname{sh}\mu_k z, \quad (7)$$

where A_k are arbitrary coefficients, and

$$\mu_k^2 = p^2 - (p^2C)^{1/3} e^{2\pi i(k-1)/3}. \quad (8)$$

Similarly, the solution of Eq. (3) satisfying boundary condition (6) at infinity has the form

$$v_z = \sum_{k=1}^{3} B_k e^{\nu_k z}, \quad (9)$$

$$\nu_k^2 = p^2 + (p^2 C_1)^{1/3} e^{2\pi i(k-1)/3}, \quad \operatorname{Re}\nu_k < 0. \quad (10)$$

The continuity conditions on the upper boundary of the unstable layer yield six equations. The condition that these equations have a solution is

$$\begin{Vmatrix} \sinh\mu_1 & \sinh\mu_2 & \sinh\mu_3 & 1 & 1 & 1 \\ \mu_1\cosh\mu_1 & \mu_2\cosh\mu_2 & \mu_3\cosh\mu_3 & \nu_1 & \nu_2 & \nu_3 \\ \mu_1^2\sinh\mu_1 & \mu_2^2\sinh\mu_2 & \mu_3^2\sinh\mu_3 & \nu_1^2 & \nu_2^2 & \nu_3^2 \\ \mu_1^3\cosh\mu_1 & \mu_2^3\cosh\mu_2 & \mu_3^3\cosh\mu_3 & \nu_1^3 & \nu_2^3 & \nu_3^3 \\ \mu_1^4\sinh\mu_1 & \mu_2^4\sinh\mu_2 & \mu_3^4\sinh\mu_3 & \nu_1^4 & \nu_2^4 & \nu_3^4 \\ \mu_1^5\cosh\mu_1 & \mu_2^5\cosh\mu_2 & \mu_3^5\cosh\mu_3 & \nu_1^5 & \nu_2^5 & \nu_3^5 \end{Vmatrix} = 0 \qquad (11)$$

This is the equation from which, with a fixed value of the parameter C_1 determining the degree of stability of the region lying above the unstable layer, we can determine the parameter C (i.e., the entropy gradient necessary for the realization of stationary convection) as a function of the parameter p, determining the horizontal dimensions of the convective cells. Basically, our problem consists of the determination of the minimum possible value of C, and the value of p corresponding to it.

We denote the columns of this determinant by Roman numerals and replace IV, V and VI, respectively, by the combinations

$$\mathrm{VI}' = \frac{1}{\nu_3-\nu_2}\left[\frac{\mathrm{VI}-\mathrm{IV}}{\nu_3-\nu_1} - \frac{\mathrm{V}-\mathrm{IV}}{\nu_2-\nu_1}\right],$$

$$\mathrm{V}' = \frac{\mathrm{V}-\mathrm{IV}}{\nu_2-\nu_1} - (\nu_1+\nu_2)\,\mathrm{VI}',$$

$$\mathrm{IV}' = \mathrm{IV} - \nu_1\mathrm{V}' - \nu_1^2\,\mathrm{VI}'.$$

These transformations do not change the roots of the equations having physical significance. Here the equation takes the form of a determinant (11) equal to zero, in which the three right-hand columns are replaced by the following:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma & \beta & \alpha \\ \alpha\gamma & \alpha\beta+\gamma & \alpha^2+\beta \\ (\alpha^2+\beta)\gamma & \alpha\gamma+\alpha^2\beta+\beta^2 & \alpha^3+2\alpha\beta+\gamma, \end{matrix} \qquad (12)$$

where

$$\alpha = \nu_1+\nu_2+\nu_3, \qquad (13)$$

$$\beta = -(\nu_1\nu_2+\nu_1\nu_3+\nu_2\nu_3), \quad \gamma = \nu_1\nu_2\nu_3.$$

a) *The case* $C_1 \ll 1$ (*the above-lying region near the stability limit*).

In this case, evidently the convective region will considerably exceed the dimensions of the unstable layer. On the other hand, the minimum of the parameter C must correspond to the possibility of comparable vertical and horizontal dimensions of the convective cells, and therefore it may be expected that in this case $p \ll 1$. The complete calculation shows that for minimum C, $p \sim \mu_k \sim C_1^{1/4}$.

We expand μ_k and $\cosh\mu_k$ in Eq. (11) in a series of powers of μ_k and limit ourselves to two terms of these series. If we then perform a transformation on the first three columns, completely similar to that described in the preceding section for the second three columns, and note that in view of Eq. (8), $(\mu_k^2 - p^2)^3 = -p^2C$, then the three left-hand columns of (11) take the form

$$\begin{matrix} 1 & 1/6 & 0 \\ 1 & 1/2 & 0 \\ 0 & 1 & 1/6 \\ 0 & 1 & 1/2 \\ ^1/_6\,(p^6-p^2C) & -^1/_2\,p^4 & 1+{}^1/_2\,p^2 \\ ^1/_2\,(p^6-p^2C) & -^3/_2\,p^4 & 1+{}^3/_2\,p^2. \end{matrix}$$

For future transformations we note that $\alpha \sim p$, $\beta \sim p^2$, $\gamma \sim p^3$ and we will subsequently neglect terms of order p with respect to unity.

We denote by Roman numerals I, II, ..., VI the rows of the determinant obtained, and perform two successive transformations without changing the magnitude of the determinant:

$$\mathrm{VI}' = \mathrm{VI} - \alpha\mathrm{V} - \beta\mathrm{IV}, \qquad (a)$$

$$\mathrm{V}' = \mathrm{V} - \alpha\mathrm{IV}, \ \mathrm{IV}' - \mathrm{IV};$$

$$\mathrm{IV}'' = \mathrm{IV}' - \gamma\mathrm{I} - \beta\mathrm{II} - \alpha\mathrm{III}, \quad \text{(b)}$$

$$\mathrm{V}'' = \mathrm{V}' - \gamma\mathrm{II} - \beta\,\mathrm{III}, \quad \mathrm{VI}'' = \mathrm{VI}' - \gamma\,\mathrm{III}.$$

The right lower quadrant of the determinant is transformed into a null matrix. Therefore, the equation is reduced to a determinant of the third rank, equated to zero. Expanding this determinant, we obtain in the indicated approximation

$$^1/_3\, p^2 C = \alpha\beta + \gamma. \tag{14}$$

Introducing $\xi = p/(p^2 C_1)^{1/6}$, we have, in view of Eqs. (10) and (13),

$$\begin{aligned}\alpha &= -(p^2C_1)^{1/3}\{\sqrt{\xi^2+1} + \sqrt{2(\xi^4-\xi^2+1)^{1/2}+2\xi^2-1}\},\\ \beta &= -(p^2C_1)^{2/3}\{\sqrt{\xi^2+1}\sqrt{2(\xi^4-\xi^2+1)^{1/2}+2\xi^2-1} + \sqrt{\xi^4-\xi^2+1}\},\\ \gamma &= -(p^2C_1)\sqrt{1+\xi^6}.\end{aligned} \tag{15}$$

According to Eqs. (14) and (15),

$$\begin{aligned}C &= C_1^{1/4} F(\xi),\\ F(\xi) &= 3\xi^{-3/2}\{4\sqrt{1+\xi^6} + (2\xi^2-1)\sqrt{1+\xi^2}\\ &+ (1+\xi^2+\sqrt{\xi^4-\xi^2+1})\sqrt{2(\xi^4-\xi^2+1)^{1/2}+2\xi^2-1}\}.\end{aligned} \tag{16}$$

The function $F(\xi)$ tends toward infinity for $\xi = 0$ and $\xi = \infty$, has a lower bound, and attains its minimum value $F_{min} \approx 35$ at $\xi \approx 0.7$. Therefore,

$$C_{\min} \approx 35 C_1^{1/4}, \quad p_{\min} \approx 0.6\, C_1^{1/4}. \tag{17}$$

These expressions justify the assumption used in calculating the determinant that for $C_1 \ll 1$, $\mu_k \ll 1$.

We call the total height h^* of the convective region the distance from the lower boundary of the unstable region to the point in the stable region at which the velocity v_z is $1/e$ times ths value at the lower boundary of this region, i.e.,

$$(h^* - h)/h \approx [\mathrm{Re}(-\nu_{1,2})]^{-1}, \tag{18}$$

since ν_1 and $\mathrm{Re}\,\nu_2$ are approximately equal near $p \approx p_{min}$. Keeping in mind that $C_1^{1/4} \ll 1$, we obtain from Eqs. (10) and (17)

$$h^* = C_1^{-1/4} h, \quad C_{\min} \approx 35\, h/h^*. \tag{19}$$

b) *The case $C_1 \gg 1$ (the above-lying region possesses considerable stability).*

We show that in this case $p_{min} \sim 1$, where p_{min} is the value of p corresponding to the minimum of criterion C. Therefore, $|\alpha| = |\nu_1 + \nu_2 + \nu_3| \gg 1$, $\beta \sim \alpha^2$, $\gamma \sim \alpha^3$. We will limit ourselves to the term in the determinant (11) containing α to the highest degree.

It is easily shown that the determinant A, consisting of the right lower quadrant of (11) is $\sim \alpha^9$, and all other minors have an order not exceeding α^8. Therefore, if terms of the order $1/\alpha$ are neglected, determinant (11) is reduced to the product of determinant A and an additional minor B, consisting of the elements of the upper left quadrant. Since $A \neq 0$, Eq. (11) reduces to $B = 0$. Expanding the determinant of the third rank, we obtain

$$\mu_1 \coth\mu_1(\mu_2^2 - \mu_3^2) + \mu_2\coth\mu_2(\mu_3^2 - \mu_1^2) + \mu_3\coth\mu_3(\mu_1^2 - \mu_2^2) = 0. \tag{20}$$

Let us write $\mu_1 = iz$, $\mu_2 = \mu_3^* = x - iy$, where

$$\begin{aligned}x &= {}^1/_2[2[p^4 + p^2(p^2C)^{1/3} + (p^2C)^{2/3}]^{1/2} + 2p^2 + (p^2C)^{1/3}]^{1/2},\\ y &= {}^1/_2[2[p^4 + p^2(p^2C)^{1/3} + (p^2C)^{2/3}]^{1/2} - 2p^2 - (p^2C)^{1/3}]^{1/2},\\ z &= [(p^2C)^{1/3} - p^2]^{1/2};\end{aligned} \tag{21}$$

we agree to take the root with the positive sign. Then Eq. (20) takes the form

$$-z\cot z = [(x + \sqrt{3}\,y)\sinh 2x + (\sqrt{3}\,x - y)\sin 2y]\,/\,(\cosh 2x - \cos 2y). \tag{22}$$

Since $x > 0$, $y > 0$ and, according to Eq. (21) $x > y$, the right-hand side of Eq. (22) is greater than zero. Hence, it follows that z is real, since if $\mu_1 = iz$ is real, then

$$-z\cot z = -\mu_1 \coth \mu_1 < 0.$$

Furthermore, it follows that $z > \pi/2$. But since [in view of Eq. (21)] $x > z$, we may neglect e^{-2x} compared with unity in the right-hand sideof Eq. (22). Under these conditions, we obtain

$$-z\cot z = x + \sqrt{3}y. \tag{23}$$

Substituting $(p^2C)^{1/3} = z^2 + p^2$ on the right, and solving for p^2, we find

$$p^2 = \tfrac{1}{2} z^2 [3^{-1/2} (\cot^4 z - 2\cot^2 z)^{1/2} - 1]. \tag{24}$$

This equation, together with the relation $z^2 = (p^2C)^{1/3} - p^2$, determines z and p as functions of the parameter C. We are interested in the minimum value of C at which these equations are joined, and the corresponding value of p. This means that in the (p^2, z^2) plane we must construct the curve determined by Eq. (24), and curves of $z^2 = (p^2C)^{1/3} - p^2$, and find the minimum value of C at which they intersect. This construction yields the values:

$$C_{\min} \approx 370, \quad z_{\min} \approx 2{,}7, \quad p_{\min} \approx 1.9, \quad x_{\min} \approx 3.4. \tag{25}$$

Let us note that in the usual statement of the problem, in which convection does not go beyond the limits of the unstable region, and $v_z = v_z^{\mathrm{II}} = v_z^{\mathrm{IV}} = 0$ at both boundaries,

$$C_{\min} \approx {}^{27}/_4\, \pi^4 \approx 665, \quad z_{\min} \approx \pi, \quad p_{\min} \approx \pi/\sqrt{2}. \tag{26}$$

In order to determine the total height of the region of mixing from Eq. (18), we take into account that [by Eq. (10)] with

$$(p^2C_1)^{1/3} \gg p^2, \quad \mathrm{Re}(-\nu_2) \approx \tfrac{1}{2}(p^2C_1)^{1/6}, \quad -\nu_1 \approx (p^2C_1)^{1/6},$$

the total height is determined by the least of these expressions, i.e.,

$$(h^* - h)/h = 2(p^2C_1)^{-1/6} \approx 1.6\, C_1^{-1/6}. \tag{27}$$

Thus, for $C_1^{1/6} \gg 1$, mixing occurs practically only in the unstable region. We must keep in mind that in all calculations for the case just considered we have neglected the quantity $1/\alpha \sim C_1^{-1/6}$, and consequently, the calculations are valid for $C_1^{1/6} \gg 1$.

3. CASE OF UPWARD AND DOWNWARD PROPAGATION OF CONVECTION

If convection is propagated both above and below the unstable layer, then we must add to Eqs. (2) and (3) a similar equation for $z < 0$, containing a parameter C_2 analogous to C_1. Also, the boundary conditions (5) are replaced by the condition $v_z = 0$ for $z \to -\infty$. The usual solution of this problem has the form:

$$v_z = \sum_{k=1}^{6} A_k e^{\mu_k z}, \quad 0 < z < 1, \qquad v_z = \sum_{k=1}^{3} B_k^j e^{\nu_{kj} z}, \quad j = 1 \text{ for } z > 1, \; j = 2 \text{ for } z < 0; \tag{28}$$

$$\mu_k^2 = p^2 - (p^2C)^{1/3} e^{2\pi i(k-1)/3}, \quad \nu_{kj}^2 = p^2 + (p^2C_j)^{1/3} e^{2\pi i(k-1)/3}, \quad \mathrm{Re}\,\nu_{k1} < 0, \quad \mathrm{Re}\,\nu_{k2} > 0. \tag{29}$$

The condition of continuity is equivalent to the condition that the following determinant be equal to zero:

$$\left\| \begin{array}{cccccccccccc} e^{\mu_1} & e^{\mu_2} & e^{\mu_3} & e^{\mu_4} & e^{\mu_5} & e^{\mu_6} & 1 & 1 & 1 & 0 & 0 & 0 \\ \mu_1 e^{\mu_1} & \cdot & \cdot & \cdot & \cdot & \mu_6 e^{\mu_6} & \nu_{11} & \nu_{21} & \nu_{31} & 0 & \cdot & \cdot \\ \mu_1^2 e^{\mu_1} & \cdot & \cdot & \cdot & \cdot & \mu_6^2 e^{\mu_6} & \nu_{11}^2 & \cdot & \nu_{31}^2 & 0 & \cdot & \cdot \\ \mu_1^3 e^{\mu_1} & \cdot & \cdot & \cdot & \cdot & \mu_6^3 e^{\mu_6} & \nu_{11}^3 & \cdot & \nu_{31}^3 & 0 & \cdot & \cdot \\ \mu_1^4 e^{\mu_1} & \cdot & \cdot & \cdot & \cdot & \mu_6^4 e^{\mu_6} & \nu_{11}^4 & \cdot & \nu_{31}^4 & 0 & \cdot & \cdot \\ \mu_1^5 e^{\mu_1} & \cdot & \cdot & \cdot & \cdot & \mu_6^5 e^{\mu_6} & \nu_{11}^5 & \cdot & \nu_{31}^5 & 0 & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ \mu_1 & \cdot & \cdot & \cdot & \cdot & \mu_6 & 0 & \cdot & \cdot & \nu_{12} & \nu_{22} & \nu_{23} \\ \mu_1^2 & \cdot & \cdot & \cdot & \cdot & \mu_6^2 & 0 & \cdot & \cdot & \nu_{12}^2 & \cdot & \nu_{23}^2 \\ \mu_1^3 & \cdot & \cdot & \cdot & \cdot & \mu_6^3 & 0 & \cdot & \cdot & \nu_{12}^3 & \cdot & \nu_{23}^3 \\ \mu_1^4 & \cdot & \cdot & \cdot & \cdot & \mu_6^4 & 0 & \cdot & \cdot & \nu_{12}^4 & \cdot & \nu_{23}^4 \\ \mu_1^5 & \cdot & \cdot & \cdot & \cdot & \mu_6^5 & 0 & \cdot & \cdot & \nu_{12}^5 & \cdot & \nu_{23}^5 \end{array} \right\| \qquad (30)$$

We break this determinant into four parts of six rows and columns each, and denote them by

$$\left\| \begin{array}{cc} A & B \\ C & D \end{array} \right\|. \qquad (31)$$

For the investigation of limiting cases which will be of interest to us in the future, it is convenient to subject this determinant to a transformation which results in C taking the form of a unit matrix. Determinant C is the Vandermond determinant, and can without difficulty be reduced to such a form by the use of combinations of only one of the columns. This transformation is broken into two parts:

a) We replace column II by the combination $(\mathrm{II} - \mathrm{I})/(\mu_2 - \mu_1) = \mathrm{II}_1$. Then we replace III by column III_1 which is obtained by subtracting column I from III, dividing by $\mu_3 - \mu_1$, subtracting II_1 from the new expression, and then dividing by $\mu_3 - \mu_2$. Similarly, we subtract successively from each column up to the sixth inclusive, all the preceding columns already changed by the corresponding transformations, each time dividing the difference by the corresponding difference of the form $\mu_i - \mu_k$.

b) From each (ith) column, beginning with the fifth, we subtract a linear combination of the succeeding columns, formed in the following way: the $(i+1)$th column is multiplied by $\mu_1 + \mu_2 + \ldots + \mu_i$, the $(i+2)$th column is multiplied by $\sum_{j \leq k \leq i} \mu_j \mu_k$, the $(i+3)$th column by $\sum_{j \leq k \leq l \leq i} \times \mu_j \mu_k \mu_l$, etc. Here, a column already transformed by similar preceding operations, is subtracted each time.

The elements a_{ik} of determinant A then take the form:

$$a_{k1} = {}^1/_6 (p^2C)^{-1/_3} (p^2 - p^2C) \sum_{n=1}^{6} e^{\mu_n} \mu_n^{k-3} e^{-4\pi i(n-1)/3},$$

$$a_{k2} = {}^1/_6 (p^2C)^{-1/_3} \sum_{n=1}^{6} e^{\mu_n} \mu_n^{k-2} (\mu_4^4 - 3p^2\mu_n^2 + 3p^4) e^{-4\pi i(n-1)/3},$$

$$a_{k3} = {}^1/_6 (p^2C)^{-1/_3} \sum_{n=1}^{6} e^{\mu_n} \mu_n^{k-1} (\mu_n^2 - 3p^2) e^{-4\pi i(n-1)/3},$$

$$a_{k4} = {}^1/_6 (p^2C)^{-1/_3} \sum_{n=1}^{6} e^{\mu_n} \mu_n^{k-2} (\mu_n^2 - 3p^2) e^{-4\pi i(n-1)/3},$$

$$a_{k5} = {}^1/_6 (p^2 {}_{-})^{-1/_3} \sum_{n=1}^{6} e^{\mu_n} \mu_n^{k-1} e^{-4\pi i(n-1)/3},$$

$$a_{k6} = {}^1/_6 (p^2C)^{-1/_3} \sum_{n=1}^{6} e^{\mu_n} \mu_n^{k-2} e^{-4\pi i(n-1)/3}. \qquad (32)$$

Expression (29) for μ_k was used to obtain Eq. (32).

The first three columns of determinant B and the second three columns of determinant D are transformed similarly to the transformation applied to the second three columns of determinant (11). As a result, these columns coincide with Eq. (12), except that instead of α, β, γ, we have, respectively, α_1, β_1, γ_1, α_2, β_2, γ_2, determined by the equations

$$\alpha_j = \nu_{1j} + \nu_{2j} + \nu_{3j}, \qquad (33)$$

$$\beta_j = -(\nu_{1j}\nu_{2j} + \nu_{1j}\nu_{3j} + \nu_{2j}\nu_{3j}),$$

$$\gamma_j = \nu_{1j}\nu_{2j}\nu_{3j}.$$

a) *The case in which the regions above and below the unstable layer are near the stability limit.*

As in Sec. 2a, under these conditions the μ_k corresponding to the minimum C are small, and therefore in the expression for a_{ki} we can expand e^{μ_k} in a series and limit ourselves to terms up to the second order, inclusively. Here, the expansions for all of the elements a_{ki} begin with terms of the order μ_n^4. Having performed these expansions, we apply transformations to the rows of determinants AB and C, identical to transformations a) and b) of determinant (11). After these transformations, the initial determinant is easily reduced to a determinant of the 6th order. Since under the conditions of interest to us $p \ll 1$, we neglect terms $\sim p$ with respect to unity. We take into account that $|\alpha| \sim p$, $|\beta| \sim p^2$, $|\gamma| \sim p^3$. Then the determinant of the 6th order takes the form

$$\left\| \begin{matrix} \gamma_1 & \beta_1 & \alpha_1 & -1 & -1 & -1/2 \\ 1/2\,(p^2C - p^6) & \gamma_1 & \beta_1 & \alpha_1 & -1 & -1 \\ (p^2C - p^6) & -p^6 & \gamma_1 & \beta_1 & \alpha_1 & -1 \\ \gamma_2 & \beta_2 & \alpha_2 & -1 & 0 & 0 \\ 0 & \gamma_2 & \beta_2 & \alpha_2 & -1 & 0 \\ 0 & 0 & \gamma_2 & \beta_2 & \alpha_2 & -1 \end{matrix} \right\|$$

Hence, we obtain, to the assumed approximation,

$$p^2C = F_1 / F_2; \quad F_2 = \beta^2 - \alpha\gamma + \alpha(\alpha_1\beta_2 - \alpha_2\beta_1), \qquad (34)$$

$$F_1 = \gamma^3 + \alpha(\gamma_1\alpha_2 - \gamma_2\alpha_1)$$

$$+ (\gamma_1\beta_2 - \gamma_2\beta_1)[\beta^2 - 2\alpha\gamma + \alpha(\alpha_1\beta_2 - \alpha_2\beta_1)]$$

$$+ \gamma\beta(\gamma_2\alpha_1 - \gamma_1\alpha_2),$$

where

$$\alpha = \alpha_2 - \alpha_1, \quad \beta = \beta_2 - \beta_1, \quad \gamma = \gamma_2 - \gamma_1.$$

From the explicit expressions for α_j, β_j, γ_j (33), and from Eq. (34), it follows that F_1 is a homogeneous function of the 9th degree of the quantities p, $(p^2C_1)^{1/6}$, $(p^2C_2)^{1/6}$, and F_2 is a homogeneous function of the 4th degree of the same quantities. Moreover, both functions are symmetric with respect to the parameters C_1 and C_2. We take the factor $(p^2C_2)^{9/6}$ outside the brackets in the numerator, and the factor $(p^2C_2)^{4/6}$ outside the brackets in the denominator, and introduce the symbol $\xi = p/(p^2C_2)^{1/6}$. Equation (34) then takes the form:

$$C = C_2^{3/4}\,\xi^{-1/2} J_1(\xi, C_1/C_2) / J_2(\xi, C_1/C_2). \qquad (35)$$

With a given value of the ratio C_1/C_2 and with $\xi \to \infty$

$$J_1(\xi, C_1/C_2) \sim \xi^9, \quad J_2(\xi, C_1/C_2) \sim \xi^4.$$

For $\xi \to 0$, both functions have finite positive values. Consequently, $C \to \infty$ when $\xi \to \infty \to 0$, and has a minimum at a finite ξ.

In view of the unwieldiness of the expressions for J_1 and J_2, we shall consider the case $C_1/C_2 \to 0$ and neglect $(C_1/C_2)^3$ with respect to ξ^2. This neglect will be justified by the calculations which show that C is a minimum when ξ is comparable with 1. Then

$$J_1 = (1 + \xi^2)^{1/2}[67\xi^8 - 40\xi^6 + 40\xi^4 \qquad (36)$$

$$+ 27\xi^2 + 2\xi - 1] + 3\rho(1 + 8\xi^6)$$

$$+ \eta\xi^2[27\xi^6 + 8\xi^5 + 8\xi^4$$

$$+ 27] + (1 + \xi^6)^{1/2}[64\xi^6 + 9\xi^3 - 1]$$

$$+ 3(1 + \xi^2)^{1/2}\eta\xi(1 + 8\xi^6)$$

$$+ 8\rho\eta\xi^4[6\xi^2 + 5] + 3\xi^6 + 40\xi^9;$$

$$J_2 = (1 + \xi^6)^{1/2}[(1 + \xi^2)^{1/2} \qquad (37)$$

$$+ \eta + 6\xi] + (1 + \xi^2)^{1/2}\xi(14\xi^2 - 3)$$

$$\eta\xi(3 + 3\rho + 11\xi^2) + 9\xi^2[\rho + (1 + \xi^2)^{1/2}\eta] + 18\xi^4;$$

$$\rho = (\xi^4 - \xi^2 + 1)^{1/2}, \quad \eta = (2\rho + 2\xi^2 - 1)^{1/2}.$$

The minimum C occurs at $\xi \approx 0.4$, which corresponds to $p^2_{\min} \approx 0.06\,C_2^{1/2}$. Here,

$$C_{\min} \approx 1.5 \times C_2^{3/4}. \qquad (38)$$

Replacing C_2 by C_1 in this expression, we obtain the solution of the opposite limiting case $C_2 \ll C_1$.

A solution is also easily obtained for $C_1 = C_2$. In this case, $\alpha_1 = -\alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = -\gamma_2$. Then Eq. (34) takes the form:

$$C = \gamma_2 (\gamma_2^2 + 9p^6 + 6\alpha_2\gamma_2 p^2 - 3p^4\beta_2) / 3p^4 (3p^2 - \beta_2) \tag{39}$$

or, in the previous notation,

$$C = C_2^{3/4} \xi^{-5/2} J_1'(\xi) / J_2'(\xi). \tag{40}$$

C is a minimum at $\xi \approx 0.7$, and has the value

$$C \approx 7.3\ C_2^{3/4}. \tag{41}$$

Let us now turn to physical deductions from these calculations. Together with the minimum value of the criterion C, determined in the two cases considered by Eqs. (38) and (41), and the corresponding horizontal dimensions $\sim 1/p$, we are interested in the total height of the region of mixing

$$h^* = \{1 + 1/(\mathrm{Re}(-\nu_{k1}))_{\min} + 1/(\mathrm{Re}\,\nu_{k2})_{\min}\}\,h. \tag{42}$$

For $C_1 \ll C_2$:

$$\mathrm{Re}(-\nu_{k1}) = p,\quad (\mathrm{Re}\,\nu_{k2})_{\min}$$

$$= {}^1/_2\, p\, [2(\sqrt{\xi^4 - \xi^2 + 1} + \xi^2) - 1]^{1/2},$$

using the explicit expressions (29) for ν_{kj}, and the definition $\xi = p/(p^2 C_2)^{1/6}$.

Hence, remembering that in this case $p \ll 1$, we have

$$h^*/h \approx 3/p \approx 12\, C_2^{-1/4}, \tag{43}$$

whence

$$C \approx 2.5 \times 10^3 (h/h^*)^3. \tag{44}$$

For $C_1 = C_2$,

$$h^*/h \approx 2.4/p \approx 4 C_2^{-1/4}, \tag{45}$$

whence

$$C \approx 470\,(h/h^*)^3. \tag{46}$$

Consideration of these limiting cases indicates that in the general case also, the dependence $C \sim (h/h^*)^3$ must be approximately conserved.

This result differs from the result of Sec. 2a obtained in the same approximation, but for the case in which mixing propagates only into the region lying above the unstable layer. There we found that $C \approx 35\, h/h^*$. A second essential difference is that in case (1) the total height of the region increased without limit when the stable region approached the stability limit, i.e., for $C_1 \to 0$. But in the case considered now, $h^* \to \infty$ only when $C_1 \to 0$ and $C_2 \to 0$ simultaneously. If either one of these stability criteria is finite, then h^* is also finite and is determined by this criterion.

b) *The case in which the regions lying above and below the unstable layer possess considerable stability* ($C_1 \gg 1$, $C_2 \gg 1$).

In this case α_1, α_2, β_1, β_2, γ_1, γ_2 cannot be small, since we would obtain formally the results of the preceding paragraph, from which it would follow that at $C = C_{\min}$, $p \sim C_{1,2}^{1/4}$, which would give large values of α, β, γ. Therefore, we will assume, and will confirm in the following calculation, that α, β and $\gamma \gg 1$.

We limit ourselves to the principal term of determinant (9), which as is easily shown, has the form:

$$\begin{Vmatrix} \gamma_1 & \beta_1 & \alpha_1 \\ \alpha_1\gamma_1 & \alpha_1\beta_1+\gamma_1 & \alpha_1^2+\beta_1 \\ (\alpha^2+\beta_1)\gamma_1 & \alpha_1\gamma_1+\alpha_1^2\beta_1+\beta_1^2 & \alpha_1^3+2\alpha_1\beta_1+\gamma_1 \end{Vmatrix} \cdot \tag{47}$$

$$\begin{Vmatrix} \gamma_2 & \beta_2 & \alpha_2 \\ \alpha_2\gamma_2 & \alpha_2\beta_2+\gamma_2 & \alpha_2^2+\beta_2 \\ (\alpha_2^2+\beta_2)\gamma_2 & \alpha_2\gamma_2+\alpha_2^2\beta_2+\beta_2^2 & \alpha_2^3+2\alpha_2\beta_2+\gamma_2 \end{Vmatrix} \cdot \begin{Vmatrix} a_{14} & a_{15} & a_{16} \\ a_{24} & a_{25} & a_{26} \\ a_{34} & a_{35} & a_{36} \end{Vmatrix}.$$

Since the first two factors in Eq. (47) are not equal to zero, omitting them and calculating the third determinant, we obtain:

$$2\sinh\mu_1/\mu_1 - \sinh\mu_2/\mu_2 - \sinh\mu_3/\mu_3 + i\sqrt{3}(\sinh\mu_3/\mu_3 - \sinh\mu_2/\mu_2) \tag{48}$$
$$+3(p^2C)^{1/3}\sinh\mu_1\sinh\mu_2\sinh\mu_3/\mu_1\mu_2\mu_3 - 2\sinh\mu_1\cosh\mu_1\cosh\mu_2/\mu_1 + \cosh\mu_1(\cosh\mu_2\sinh\mu_3/\mu_3$$
$$+\sinh\mu_2\cosh\mu_3/\mu_2) + i\sqrt{3}(\sinh\mu_2\cosh\mu_3/\mu_2 - \cosh\mu_2\sinh\mu_3/\mu_3)\cosh\mu_1 = 0.$$

Let

$$\mu_2 = x - iy, \quad \mu_3 = x + iy, \tag{49}$$

$$x = {}^1/_2\{2[p^4 + p^2(p^2C)^{1/3} + (p^2C)^{2/3}]^{1/2} + 2p^2 + (p^2C)^{1/3}\}, \tag{50}$$
$$y = {}^1/_2\{2[p^4 + p^2(p^2C)^{1/3} + (p^2C)^{2/3}]^{1/2} - 2p^2 - (p^2C)^{1/3}\}.$$

When $p \gtrsim 1$, $x > 1$ and therefore as a first approximation we can neglect the first three terms in Eq. (48) compared with the remaining ones. This means neglecting the quantity e^{-x} with respect to unity. Then we obtain the equation

$$\coth\mu_1 = [2\mu_2\mu_3 - 3(p^2C)^{1/3}]/2\mu_1(x - \sqrt{3}y). \tag{51}$$

We consider the domain of values of p in which $p^2 < (p^2C)^{1/3}$. We let $\mu_1 = iz$, replace $(p^2C)^{1/3}$ by $z^2 + p^2$ in the right-hand side, and introduce $\eta^2 = z^2/p^2$. Then Eq. (51) takes the form:

$$\cot z = \frac{2\sqrt{\eta^4+3\eta^2+3} - 3(\eta^2+1)}{\sqrt{2(\eta^4+3\eta^2+3)^{1/2}+\eta^2+3} - \sqrt{3}\sqrt{2(\eta^4+3\eta^2+3)^{1/2}-\eta^2-3}} = f(\eta^2). \tag{52}$$

Hence

$$z = \operatorname{arc\,cot} f(\eta^2). \tag{53}$$

On the other hand, from the definitions of η^2 and z,

$$z = C^{1/4}\eta/(1+\eta^2)^{3/4}. \tag{54}$$

The minimum value of C for which the curves (53) and (54) in the (z, η) plane intersect, and the corresponding values of z, p and x, are

$$C_{\min} \approx 106, \quad p_{\min} \approx 2{,}6, \tag{55}$$
$$z \approx 1.2, \quad x \approx 3.4.$$

We turn now to the domain of values of p, $p^2 > (p^2C)^{1/3}$, for which μ_1 is real. A numerical solution of Eq. (61) shows that for $(p^2C)^{1/3}p^2 = \eta^2 > 0.8$, the criterion C is greater than $C_{\min} = 106$, and therefore this region does not interest us. On the other hand, for $\eta^2 \ll 1$, the right-hand side of (51) tends rapidly towards 1 with decreasing η^2. At $\eta^2 \approx 0.8$, the right-hand side of (51) differs from 1 by less than 0.06, and can be calculated by the series

$$1 + (17/128)\eta^8 + \cdots$$

For $\eta^2 < 0.8$, it may appear that $C < C_{\min}$, but here, according to Eq. (51), $\coth\mu_1 \to 1$ and differs from 1 by the quantity $e^{-2\mu_1}$. But since $\mu_1 = \sqrt{1-\eta^2}\,p$, then for $\eta^2 < 0.8$, $\mu_1 \gtrsim p/2$, and therefore $e^{-2\mu_1} \sim e^{-p}$, i.e., is a magnitude which we neglected in deducing (51) from (48). This means that for $\eta^2 < 0.8$ we must set $\coth \mu_1 = 1$ in Eq. (51), but then it will have only the trivial solution $C = 0$. Therefore, we must return to Eq. (48) and investigate its behavior in this case.

For large p, $\mathrm{Re}\,\mu_2 = \mathrm{Re}\,\mu_3 = x$ is large, but $\mu_1 = p\sqrt{1-\eta^2}$ may be small when $\eta^2 \to 1$. Therefore, for $\eta^2 > 0.8$ we could also consider Eq. (28), but in the case $\eta^2 < 0.8$, we must return to Eq. (25) and consider both x and μ_1 large.

After straightforward transformations, Eq. (48) can be written in the form

$$\begin{aligned} &-\alpha \sinh\mu_1 \cosh 2x + 2\mu_2\mu_3 [\sinh(2x-\mu_1) - \sinh\mu_1 \cos 2y + 2\sinh\mu_1 - 2\sinh x \cos y] \\ &- 3(p^2C)^{1/3} [\sinh(2x-\mu_1) + \sinh\mu_1 \cos 2y - 2\sinh x \cos y] \\ &- 2\mu_1(\sqrt{3}\,x + y)[2\sinh x \sin y - \cosh\mu_1 \sin 2y] - \alpha\sinh(2x-\mu_1) \\ &+ 2\alpha \sinh x \cos y = 0, \end{aligned} \tag{56}$$

where

$$\alpha = 2\mu_2\mu_3 - 3(p^2C)^{1/3} \tag{57}$$

$$-2\mu_1(x - \sqrt{3}y) = (17/128)\,\eta^8 + \cdots$$

Clearly, the first term in Eq. (56) is dominant. If we limit ourselves only to this principal term, the equation will have a solution only for $C = 0$.

The whole of Eq. (56) will have only this solution for large p if for all the remaining terms, $C = 0$ is a root of multiplicity not less than that of the first term. Expansion in a series of powers of η^2 and $p\eta^2$ shows that $C = 0$ is a root of the same multiplicity for all the terms of the equation. We are therefore led to the conclusion that the solution (55) is unique.

Solution (55) shows that in the case considered, in which convection is propagated into the regions lying above and below the unstable layer, the minimum value of C is 2.5 times smaller than the value obtained in case 1, and is 6.5 times smaller than the value obtained in the usual case.

The total height of the region of mixing [determined by Eq. (42)] in this case is equal to

$$h^* = h\,[1 + 2/(p^2C_1)^{1/6} + 2/(p^2C_2)^{1/6}]. \tag{58}$$

The calculations of this section are valid under the conditions $C_1^{-1/6} \ll 1$, $C_2^{-1/6} \ll 1$.

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