

## The Establishment of Thermal Equilibrium between Quanta and Electrons\*

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The role of the Compton effect in the establishment of equilibrium between quanta and electrons is considered in the nonrelativistic approximation.

### 1. INTRODUCTION

**I**N order that radiation be considered as a closed system, it is appropriate to assume that it is contained in a vessel with perfectly reflecting walls. By itself, radiation in such a vessel cannot achieve thermodynamic equilibrium: the equations of electrodynamics are strictly linear and the exchange of energy between vibrations of different frequency, direction of propagation and polarization does not occur. Therefore, we must consider that there is in the container, besides the radiation, a suitably small "carbon particle" which does not appreciably disturb the field at any given moment, but which is capable of absorption and radiation of energy of all frequencies. Over a sufficiently long time, the absorption and emission of quanta by the "particle" leads to the establishment of thermodynamic equilibrium.

What will happen if we put a free electron in place of the material particle in the container? The container is taken to be large enough that we can neglect the quantization of the energy of the electron. The free electron does not absorb and emit, but only scatters quanta; therefore, the total number of quanta in the vessel does not change. What sort of equilibrium is established? If the occupation numbers  $n$  of the individual states are small in comparison with unity, then we can neglect induced transitions and assume that the probability of scattering a quantum in a certain state does not depend on the number of quanta occupying this state. In other words, we replace the factor  $1 + n$  in the transition probability by 1. Then the same distribution will be set up among the quanta as in an ideal gas with a constant number of particles; i.e., we obtain a distribution of the form  $n = e^{-h\omega/kT}$  (Wien's law). In this case the mean energy of the quantum is  $3kT$ .

Planck's distribution is established only much later, since it exists as the result of induced emission.

The physical conditions under which Wien's distribution can exist can be represented in the following way. Let the matter be momentarily in a state with very high temperature, so that all the atoms are completely ionized, but assume radiation has not yet been produced. Then the absorption and emission of quanta will occur at the expense of a "free-free" mechanism. The corresponding emission process is nothing else than the bremsstrahlung of the electrons. This process is the more probable the lower the frequency of the emitted quantum. The same applies to the probability of the reverse process, the "free-free" absorption; therefore, for sufficiently low frequencies, thermal equilibrium will be established by means of the absorption and emission of quanta. For high frequencies, the probability of Compton scattering exceeds the absorption probability. Since the quanta are scattered by the moving electrons, their frequency can also be increased in the scattering. Initially, the number of quanta is not large; they will all tend toward thermal equilibrium independently of each other (i.e., induced processes will not constitute any significant part of all the scattering processes). The Wien distribution, with mean energy  $3kT$ , is established among such quanta.

For this reason, the transition of energy from matter to radiation will be much more rapid than if it took place at the expense of bremsstrahlung of the electrons. The bremsstrahlung quanta which have frequencies  $\omega$  greater than a certain definite frequency  $\omega_0$ , will quickly and irreversibly increase their energy by a Compton mechanism, tending to go to the maximum energy  $3kT$ .

In the Sections below, the distribution function will be found for quanta which undergo the Compton process. In the Appendix we shall consider what energy the quanta extract from the electrons in a body of finite dimensions.

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2. THE KINETIC EQUATION

Let us write down the kinetic equation for the distribution function  $n$  of the quanta in an unbounded medium for the case of a single scattering. For generality, we shall at first not neglect induced processes. The equation has the following form:

$$\left(\frac{\partial n}{\partial t}\right)_C = - \int d\tau \int [n(1+n')N(\epsilon) - n'(1+n)N(\epsilon + h\omega - h\omega')]dW. \tag{1}$$

Here  $N$  is the distribution function for the free electrons,  $d\tau$  is the element of phase space of the electrons,  $dQ$  is the differential probability of transition from a given state into another, compatible with the laws of conservation of energy and momentum. The index  $C$  on  $dn/dt$  denotes that only Compton processes are considered in the equation. Statistical equilibrium among the electrons in the plasma is established very quickly, independently of the radiation; therefore, we can consider the distribution function  $N(\epsilon)$  to be Maxwellian. Then, if we replace  $n$  in Eq. (1) by the Planck distribution  $n = (e^{h\omega/kT} - 1)^{-1}$ , then the right and the left parts vanish, as they must.

We shall consider that the energy of the electrons is nonrelativistic; i.e., we assume that the inequality  $kT \ll mc^2$  holds. Then the energy transferred in each separate act is small in comparison with the energy quantum  $h\omega$ :  $\omega' - \omega \equiv \Delta \ll \omega$ .

Making use of this inequality, we expand the integrand of Eq. (1) in a power series in  $\Delta$  up to terms of the second order, inclusively. We set  $h\omega/kT \equiv x$ :

$$\left(\frac{\partial n}{\partial t}\right)_C = \left[\frac{\partial n}{\partial x} + n(1+n)\right] \frac{h}{kT} \int d\tau \int dWN(\epsilon)\Delta + \left[\frac{\partial^2 n}{\partial x^2} + 2(1+n)\frac{\partial n}{\partial x} + n(1+n)\right] \times \frac{1}{2} \left(\frac{h}{kT}\right)^2 \int d\tau \int dWN(\epsilon)\Delta^2. \tag{2}$$

The second integral on the right side of Eq. (2) is much easier to compute than the first; but it suffices to compute only one of the two integrals. The other is determined from the condition that the equation ought to guarantee conservation of the total number of quanta in the scattering.

The laws of energy and momentum conservation in the nonrelativistic approximation are written in the form:

$$(h\omega/c)\mathbf{n} + \mathbf{p} = (h\omega'/c)\mathbf{n}' + \mathbf{p}', \tag{3}$$

$$h\omega + p^2/2m = h\omega' + p'^2/2m.$$

Here  $p^2/2m = \epsilon$ ,  $\mathbf{p}$  and  $\mathbf{p}'$  are the momenta of the electron before and after the collision,  $\mathbf{n}$  and  $\mathbf{n}'$  are the directions of propagation of the quantum. Eliminating  $p'$  from these equations, we obtain an equation which determines  $\omega'$  as a function of  $\omega$ ,  $\mathbf{p}$  and the angles of scattering. In this equation, we set  $\omega = \omega' + \Delta$  and limit ourselves to terms that are linear in  $\Delta$ , at least while we determine the integral which contains  $\Delta^2$ . After simple transformations, we get

$$h\Delta = - \frac{hc\omega(\mathbf{p}, \mathbf{n} - \mathbf{n}') + (h\omega)^2(1 - \mathbf{nn}')}{mc^2[1 + (h\omega/mc^2)(1 + \mathbf{nn}') - \mathbf{pn}/mc]}. \tag{4}$$

It is valid to replace the brackets in the denominator of Eq. (4) with unity for the case of interest to us. The first term in the numerator has the order of magnitude  $(kT/mc^2)^{3/2}$  and the second term the order  $(kT/mc^2)^2$ . But we shall determine the integral of  $\Delta^2$ ; therefore, the contribution of the first term in the averaging over the angles does not vanish. Consequently, upon substitution in Eq. (2), we must compute the following integral:

$$I = (h\omega/mc)^2 \int d\tau \int dW (\mathbf{p}, \mathbf{n} - \mathbf{n}')^2 N(\epsilon). \tag{5}$$

After averaging over all directions of  $\mathbf{p}$  we get

$$I = 1/3 (h\omega/mc)^2 \int p^2 N(\epsilon) d\tau \int |\mathbf{n} - \mathbf{n}'|^2 dW. \tag{6}$$

The first integral is equal to  $2m(3kT/2) = 3mkT$ . In the second integral, we replace the Compton cross section in the nonrelativistic approximation by the Thomson cross section, which is symmetric relative to scattering over the angles  $\theta$  and  $(\pi - \theta)$ , so that  $\int \mathbf{n}\mathbf{n}' dW = 0$ . The Thomson cross section does not depend on the energy of the quantum; consequently,

$$I = (h\omega)^2 (kT/mc^2) c/l, \tag{7}$$

where  $l$  is the Compton range, which is determined by the total cross section  $(8\pi/3)(e^2/mc^2)$ .

In order that Eq. (2) guarantee the conservation of the total number of quanta, we must set

$$\int d\tau \int dWN(\varepsilon) \Delta \quad (8)$$

$$= (kT / mc^2) (c / l) h\omega (4 - h\omega / kT).$$

Equation (8) can be obtained in the following way. The kinetic equation for the quanta must look like a sort of conservation law

$$(\partial n / \partial t)_C = -x^{-2} \partial(x^2 j) / \partial x, \quad (9)$$

where  $j$  is the "flow" of quanta in frequency space. Since Eq. (2) is of second order relative to  $x$  and contains the second derivative  $d^2 n / dx^2$  linearly, without a coefficient depending on  $n$ , the current must have the form of the sum of the first derivative  $\partial n / \partial x$  and a certain function depending on  $n$ . But in the state of total equilibrium, when  $n = (e^x - 1)^{-1}$ , the flow vanishes. In this case,  $\partial n / \partial x = -n(1 + n)$ . It therefore follows that

$$j = g(x) [\partial n / \partial x + n(1 + n)], \quad (10)$$

where the function  $g(x)$  must be determined. Substituting (10) in Eq. (9), and comparing with (2), we find that  $g = -x^2$ , and the unknown function  $\Delta = x(4 - x)$ , which is equivalent to (8). Equation (8) is in qualitative agreement with Wien's law: the energy of the quantum increases so long as  $h\omega < 4kT$ .

We now introduce the dimensionless time  $y$  by the expression

$$t = (mc^2 / kT) (l / c) = y. \quad (11)$$

In these units, the kinetic equation is written in the following form:

$$\left(\frac{\partial n}{\partial y}\right)_C = \frac{1}{x^2} \frac{\partial}{\partial x} x^4 \left(\frac{\partial n}{\partial x} + n + n^2\right). \quad (12)$$

If  $n \ll 1$ , then this equation transforms into the linear relation

$$\left(\frac{\partial n}{\partial y}\right)_C = \frac{1}{x^2} \frac{\partial}{\partial x} x^4 \left(\frac{\partial n}{\partial x} + n\right). \quad (13)$$

This equation will also be solved. According to Eq. (13), it is easy to compute the time in which the energy of the quantum increases by a factor  $e$  as a result of the Compton effect. Multiplying both sides of Eq. (13) by  $x^3$  and integrating, we get

$$\frac{d}{dy} \int_0^\infty nx^3 dx = 4 \int_0^\infty nx^3 dx - \int_0^\infty nx^4 dx. \quad (14)$$

While  $h\omega$  is still small in comparison with  $kT$ , we can neglect the second integral on the right in (14). Here  $x = x_{\text{init}} e^{4y}$ . This means that the time for the energy to increase by a factor  $e$  is equal to

$$\tau_C = (mc^2 / 4kT) l / c. \quad (15)$$

### 3. BREMSSTRAHLUNG SPECTRUM

The bremsstrahlung spectrum in the general case appears rather complicated. We assume that such conditions are satisfied in which the Born approximation is applicable for the determination of the spectrum. These conditions are satisfied in the light elements for sufficiently high temperature (from several tens of kilovolts and higher). At the same time, it is taken into account that the inequality  $kT \ll mc^2$  is satisfied. We shall not take into account the scattering of electrons on electrons, with emission of quanta, since such a process makes a significant contribution only in very light materials and in each case it does not exceed the bremsstrahlung emission on nuclei, always remaining smaller than it.

In the Born approximation, and in the nonrelativistic case, the bremsstrahlung cross section is [Ref. 1, p. 183, Eq. (18)]:

$$d\sigma = \frac{3}{8} \frac{Z^2 e^6}{mhc^3} \frac{1}{\varepsilon} \ln \frac{(V\varepsilon + V\varepsilon - h\omega)^2}{h\omega} \cdot \frac{d\omega}{\omega}. \quad (16)$$

In order to go over to the total number of quanta of a given frequency that are emitted per second, we need to multiply the cross section by  $LN(\varepsilon) v d\tau$  ( $L$  is the number of nuclei per unit volume,  $v$  is the velocity of the electron), integrate over all states of the electron and refer to the number of states of a quantum of given frequency per unit volume. Thus we obtain the equation

$$\frac{1}{\tau_B^0} = 3 \sqrt{\frac{\pi^3}{8m^3 kT}} \frac{Z^3 L^2 e^6}{h\omega^3} e^{-h\omega/2kT} K_0 \left( \frac{h\omega}{2kT} \right) \quad (17)$$

$$\equiv B \frac{e^{-x/2} K_0(x/2)}{h\omega^3}.$$

Here  $K_0$  is the Macdonald function (Ref. 2, p. 206),  $\tau_B^0$  is the magnitude of the time interval which characterizes the bremsstrahlung. If we take into consideration the inverse absorption of quanta and their induced emission, then we obtain the

rate of change of the quanta as a result of bremsstrahlung and free-free absorption:

$$(\partial n / \partial t)_B = (1 / \tau_B^0) [(1 + n) - ne^{h\omega / kT}]. \quad (18)$$

The derivative vanishes if we put the Planck distribution in place of  $n$

Digressing from the Compton process, and assuming that the total change  $dn/dt$  appears on the left side of Eq. (18), we can integrate (18). If  $n(0) = 0$ , then we get

$$n = \frac{1}{e^{h\omega / kT} - 1} \left( 1 - \exp \left\{ -\frac{t}{\tau_B^0} (e^{h\omega / kT} - 1) \right\} \right). \quad (19)$$

It then follows that the relaxation time for thermal equilibrium of the quanta, for a pure bremsstrahlung mechanism, is equal to

$$\tau_B = \tau_B^0 / (e^{h\omega / kT} - 1). \quad (20)$$

The total kinetic equation, which takes into account the Compton and bremsstrahlung processes, is now written

$$\begin{aligned} \left( \frac{\partial n}{\partial t} \right)_C + \left( \frac{\partial n}{\partial t} \right)_B &= \frac{\partial n}{\partial t} \quad (21) \\ &= \frac{kT}{mc^2} \frac{c}{l} \frac{1}{x^2} \frac{\partial}{\partial x} x^4 \left( \frac{\partial n}{\partial x} + n + n^2 \right) \\ &\quad + \frac{1}{\tau_B^0} [(1 + n) - ne^x]. \end{aligned}$$

Going over again to the dimensionless time  $y$ , we get

$$\begin{aligned} \frac{\partial n}{\partial y} &= \frac{1}{x^2} \frac{\partial}{\partial x} x^4 \left( \frac{\partial n}{\partial x} + n + n^2 \right) \quad (22) \\ &\quad + \frac{4\tau_C}{\tau_B^0} [(1 + n) - ne^x]. \end{aligned}$$

The braking time  $\tau_B$  decreases with decrease in the frequency, for example, as  $\omega^2$ ; therefore, the quanta for sufficiently low frequencies will always undergo transition to statistical equilibrium in processes of emission and absorption. The quanta of high frequencies will be taken up by the Compton process and thus increase their frequency, approaching the Wien distribution. It should not be thought that frequency of the quantum increases monotonically; Eq. (12) is of second order, of the diffusion type. The approximation of quanta to incomplete equilibrium (with respect to energy but

not with respect to number) is in the fashion of Brownian motion. The limit of the frequency, for which the Compton process takes up the quanta more rapidly than it succeeds in being absorbed, is determined by the following estimate:  $\tau_C / \tau_B \sim 1$ . Substituting  $\tau_C$  and  $\tau_B$  in it, we get

$$(9\sqrt{\pi} / 64\sqrt{2}) LZ^2 m^{*1/2} c^5 h^2 e^2 \quad (23)$$

$$\times (kT)^{-3/2} e^{-x_0/2} x_0^{-3} K_0(x_0/2) (e^{x_0} - 1) \sim 1.$$

The entire process considered (i.e., the establishment of equilibrium by the Compton effect) presents interest only in the case in which  $h\omega_0 \ll kT$ , since only then is there a frequency interval,  $\omega \gg \omega_0$ , in which the Compton effect plays an essential role. For small  $x_0$ , the function  $K_0$  can be replaced by an expansion in  $x_0$  (Ref. 2, p. 96):  $K_0(x_0/2) = \ln(4/\gamma x_0)$ , where  $\ln \gamma = 0.577$ . Thus  $x_0$  is determined by the equation

$$Ax_0^{-2} \ln(2.35/x_0) \sim 1. \quad (24)$$

This equation has meaning only if  $A$  is a small number.

We now compare the transfer of energy from the electrons to the quanta by pure bremsstrahlung and by the Compton mechanism\*.

The total energy of the quanta emitted by the electrons per unit volume is

$$\begin{aligned} \left( \frac{dE}{dt} \right)_B &= \int \frac{1}{\tau_B} h\omega \frac{\omega^2 d\omega}{\pi^2 c^3} \quad (25) \\ &= \frac{BkT}{\pi^2 c^3 h} \int_0^\infty dx e^{-x/2} K_0\left(\frac{x}{2}\right) = \frac{2BkT}{\pi^2 c^3 h} \end{aligned}$$

(Ref. 2, p. 424). All quanta whose frequencies are greater than  $\omega_0$  transfer energy to the electrons by the Compton mechanism; on the average the energy of the quanta approaches  $3kT$ . Therefore, the energy transferred per unit time is

$$\left( \frac{dE}{dt} \right)_C = 3kT \int_{\omega_0}^\infty \tau_B^{-1} \omega^2 d\omega / \pi^2 c^3. \quad (26)$$

The principal contribution to the integral is provided by the low frequencies. Therefore, we can replace the function  $K_0$  under the integral by its

\* This comparison was pointed out to us by L. D. Landau.

approximate value; to avoid divergence at the upper limit, we do not integrate to infinity, but to  $\omega = kT/h$ . We get

$$\left(\frac{dE}{dt}\right)_C = 3kT \frac{B}{\pi^2 c^3 h} \int_{\omega_0}^{\hbar T} \frac{d\omega}{\omega} \ln \frac{4kT}{\gamma h \omega} \quad (27)$$

$$\sim \frac{3}{2} \frac{BkT}{\pi^2 c^3 h} \ln^2 \frac{4kT}{\gamma h \omega_0}.$$

If we do not make use of the approximate expression for  $K_0$ , then there will still stand a numerical component in front of the logarithm in Eq. (27). That is, in place of  $\ln^2(4/\gamma x_0)$ , the integral gives  $\ln^2(4/\gamma x) - 0.27$ . But we can hardly consider this to be accurate, since the frequency  $\omega_0$  is found by an estimation method.

The ratio of the energy transferred by the Compton mechanism to the energy transferred to the quanta upon emission is equal to

$$\left(\frac{dE}{dt}\right)_C : \left(\frac{dE}{dt}\right)_B = \frac{3}{4} \ln^2 \frac{4kT}{h\gamma\omega_0}, \quad (28)$$

which can reach a few score.

If we assume that quanta with frequency less than  $\omega_0$  do not undergo Compton scattering at all, we commit a certain error. There is another, and somewhat more accurate, method of computing  $dE/dt$ . For sufficiently low frequencies, the state of the quanta is stationary, and for  $x \rightarrow 0$ , it tends toward  $1/x$ , i.e., toward the limiting form of Planck's formula. Therefore,  $n$  satisfies the ordinary differential equation which is obtained from (22) if we discard  $\partial n/\partial y$  in it. Moreover, we should also discard  $n$  in comparison with  $n^2$ , and consequently regard  $x$  as a small quantity in the second component of the right side. This gives

$$\frac{d}{dx} x^3 \left( \frac{dn}{dx} + n^2 \right) + \frac{4A}{x} (1 - nx) \ln \frac{2.35}{x} = 0. \quad (29)$$

We now set  $n = z'/z$ ,  $z = x + \psi$ ,  $\psi \ll x$ . Then we get the equation

$$\frac{d}{dx} x^3 \psi'' - \frac{4A}{x} \psi' \ln \frac{2.35}{x} = 0 \quad (30)$$

for  $\psi'$ .

This can be integrated approximately (by the WKB method) if we consider the coefficient for  $\psi'$  to be a large number. With this accuracy,

$$\psi' = C \exp \left\{ -\frac{2\sqrt{A}}{x} \sqrt{\ln \frac{2.35}{x}} \right\}. \quad (31)$$

Such a solution is valid, strictly speaking, only when the exponent is large in comparison with unity. But if we choose  $C = -1$ , then we get an interpolation formula for the distribution function, which possesses the necessary properties and for  $x \gg x_0$ :

$$n = \frac{1}{x} \left( 1 - \exp \left\{ -\frac{2\sqrt{A}}{x} \sqrt{\ln \frac{2.35}{x}} \right\} \right). \quad (32)$$

For large  $x/x_0$ , the number of quanta is much less than the equilibrium number. Adding the factor  $(1 - nx)$  under the integral (27), a factor which takes into account induced emission and absorption of quanta, we get a converging expression which can be integrated from  $\omega = 0$ :

$$\left(\frac{dE}{dt}\right)_C = 3kT \frac{B}{\pi^2 c^3 h} \int_0^{\infty} \frac{1}{\tau} \omega^2 d\omega \quad (33)$$

$$\times \exp \left\{ -\frac{2kT\sqrt{A}}{h\omega} \sqrt{\ln \frac{2.35kT}{h\omega}} \right\}.$$

The function under the integral vanishes for sufficiently small  $\omega$ . The effective limit of integration lies at about  $\omega \sim \omega_0$ .

#### 4. GENERAL FORMULAS FOR THE AVERAGE FREQUENCY IN THE COMPTON PROCESS

If the photon gas is still far from statistical equilibrium with the electrons, then the number of photons with frequency  $\omega > \omega_0$  is small in comparison with unity. In this phase of the process, we can write down the following linearized kinetic equation for quanta with frequency  $\omega > \omega_0$ :

$$\frac{\partial n}{\partial y} - \frac{1}{x^2} \frac{\partial}{\partial x} x^4 \left( \frac{\partial n}{\partial x} + n \right) = 4\tau_C / \tau_B^0. \quad (34)$$

Let us first consider the corresponding homogeneous equation. For this we introduce a new unknown function by the formula

$$n = e^{-x/2} \varphi(x) / x. \quad (35)$$

$\varphi$  satisfies the equation (we are considering the homogeneous equation)

$$\frac{\partial \varphi}{\partial y} = x \frac{\partial^2}{\partial x^2} x \varphi + \left( -\frac{x}{2} + 2 - \frac{2}{x} \right) x \varphi \equiv \hat{A}(x) \varphi. \quad (36)$$

\* It is further assumed that  $\ln(2.35/x) \gg 1$ .

Here  $\hat{A}(x)$  is an operator on the right-hand side. Some simplifications of the calculations arise because of the fact that  $\hat{A}(x)$  is a Hermitian operator\*.

Let us assume that the initial photon distribution is given in the form of some function  $n_0(x)$ ; then the distribution at any later moment is written symbolically as

$$\varphi(x, y) = e^{\hat{A}(x)y} x e^{x/2} n_0(x); \quad (37)$$

$$n = \frac{e^{-x/2}}{x} e^{\hat{A}(x)y} x e^{x/2} n_0(x).$$

We assume that a quantum with frequency  $\omega_1$  is emitted at the initial moment. Then  $n_0(x) = \delta(x - x_1)/x_1^2$ . We also compute (in dimensionless form) the mean frequency which this quantum will have over some time interval as a result of the quantum process:

$$\begin{aligned} \bar{x} &= \int_0^\infty x n x^2 dx = \int_0^\infty x^2 e^{-x/2} e^{\hat{A}(x)y} e^{x/2} \frac{\delta(x - x_1)}{x_1^2} dx \\ &= \int_0^\infty e^{x/2} \frac{\delta(x - x_1)}{x_1} e^{\hat{A}y} x^2 e^{-x/2} dx \\ &= \frac{1}{x_1} e^{x_1/2} e^{\hat{A}(x_1)y} x_1^2 e^{-x_1/2} \equiv \frac{1}{x_1} e^{x_1/2} \chi(x_1). \end{aligned} \quad (38)$$

Here we have made use of the fact that a function of a Hermitian operator is also a Hermitian operator. In Eq. (38), there enters the function  $\chi(x_1)$ , which clearly satisfies, relative to  $x_1$ , the same equation (36) as  $\varphi(x)$  relative to  $x$ . In contrast to  $\varphi$ , the function  $\chi$  is subject to a different initial condition:

$$\chi(y = 0) = x_1^2 e^{-x_1/2}. \quad (39)$$

In what follows, the index 1 will be omitted from  $x_1$ , because the "current frequency"  $x$  enters no further.

Thus, in order to find the mean frequency of the quantum, which is emitted with an initial frequency  $x$ , we must solve Eq. (36) with the initial condition (39). For this purpose, we must first determine the spectrum of the Hermitian operator  $\hat{A}(x)$ . We set

\* We have introduced the operator  $\hat{A}(x)$  and found its spectrum by making use of the results of I. M. Gel'fand.

$$z = \ln x, \quad \chi = e^{-z/2} f_\mu(z) e^{-(\mu^2 - 1/4)y}. \quad (40)$$

The function  $f_\mu(z)$  satisfies an ordinary differential equation of the same form as the Schrödinger equation for the vibration levels of the diatomic molecules, if the potential function of the nuclei is taken in the form of a Morse potential:

$$-d^2 f_\mu / dz^2 + (-2e^z + 1/4 e^{2z}) f_\mu = \mu^2 f_\mu. \quad (41)$$

Equation (41) is integrated with the help of the degenerate hypergeometric functions, which we denote by the variable  $x$ :

$$\chi = C x^{-1} W_{2, i\mu}(x) \quad (42)$$

(Ref. 3, Sec. 16). Here  $W_{2, i\mu}(x)$  is the Whittaker function, which is a combination of two ordinary degenerate hypergeometric functions:

$$\begin{aligned} W_{2, i\mu}(x) &= \frac{\Gamma(-2i\mu)}{\Gamma(-3/2 - i\mu)} M_{2, i\mu}(x) \\ &+ \frac{\Gamma(2i\mu)}{\Gamma(-3/2 + i\mu)} M_{2, -i\mu}(x). \end{aligned} \quad (43)$$

Here  $M_{2, i\mu}(x)$  is determined by the well-known series

$$\begin{aligned} M_{2, i\mu}(x) &= x^{1/2 + i\mu} e^{-x/2} \left( 1 + \frac{(-3/2 + i\mu)x}{1!(2i\mu + 1)} \right. \\ &+ \left. \frac{(-3/2 + i\mu)(-1/2 + i\mu)x^2}{2!(2i\mu + 1)(2i\mu + 2)} \right). \end{aligned} \quad (44)$$

The function  $W$  falls off at infinity as  $e^{-x}$ , while  $M$  does not possess this property. Therefore,  $\chi$  is also expressed by  $W$ .

We now find the function for the discrete spectrum, analogous to the vibrational states of the diatomic molecule (functions of the continuous spectrum correspond to a dissociating molecule). The discrete spectrum is possible for negative  $\mu^2$  [see Eq. (41)] or for purely imaginary  $\mu$ . The corresponding eigenfunctions should be quadratically integrable. According to Eqs. (42)-(44), the functions  $W$ , for small  $x$  and purely imaginary  $\mu$ , consist of two components of the form  $x^{-1/2 + |\mu|}$ . However, we cannot integrate the square of the function for this upper sign. Consequently, the eigenvalues are numbers for which  $W$  consists only of functions with an integrable square. Here the numbers are  $|\mu| = 3/2$  and  $|\mu| = 1/2$ , since the first component of Eq. (43) vanishes [ $\Gamma(-2) = \infty$ ,  $\Gamma(-3) = \infty$ ].

The series for the functions  $M_{2,1/2}(x)$  and  $M_{2,3/2}(x)$  are broken off. Elementary functions are obtained which we at once write with normalized coefficients

$$\chi_{1/2}(x) = \sqrt{2}(1-x/2)e^{-x/2}, \quad \chi_{3/2} = 2^{-1/2}xe^{-x/2}. \quad (45)$$

The functions for the continuous spectrum are normalized by transition to their normalized asymptotic expansion in a way entirely analogous to that used in the problem of the hydrogen atom. The expansions are referred, not to large but to small  $x$ , because for large  $x$  the function  $W$  decays exponentially. It is seen from Eq. (41) that for  $z \rightarrow -\infty$ , i.e., for small  $x$ , the function  $f_\mu$  is proportional to  $\cos(\mu z + \lambda)$ . The normalized factor of the cosine is  $\sqrt{2/\pi}$ . Expressing  $W$  in terms of  $\cos(\mu z + \beta) = \cos(\mu \ln x + \lambda)$ , we get the normalized functions of the continuous spectrum

$$\chi_\mu = \frac{1}{\sqrt{2\pi}} \left| \frac{\Gamma(-3/2 + i\mu)}{\Gamma(2i\mu)} \right| \frac{W_{2,i\mu}(x)}{x}. \quad (46)$$

It is now easy to write down the general formula for  $\bar{x}$  ( $x$  on the right-hand side gives the initial frequency of the quantum, previously called  $x_1$ )

$$\bar{x} = \frac{1}{x} e^{x/2} \left[ e^{-9y/4} \int_0^\infty e^{-\mu^2 y} \zeta_\mu \chi_\mu d\mu \right. \\ \left. + e^{-2y} \zeta_{1/2} \chi_{1/2} + \zeta_{3/2} \chi_{3/2} \right], \quad (47)$$

where  $\zeta_\mu$ ,  $\zeta_{1/2}$  and  $\zeta_{3/2}$  are the expansion coefficients of the function  $x^2 e^{-x/2}$  [see Eq. (38)] in the orthogonal set of functions  $\chi_\mu$ ;  $\zeta_{1/2}$  and  $\zeta_{3/2}$  are found in elementary fashion. For the determination of  $\zeta_\mu$  we make use of the integral representation of the function  $W_{2,i\mu}(x)$  (Ref. 3, p. 345):

$$W_{2,i\mu}(x) = \frac{x^2 e^{-x/2}}{2\pi i} \\ \times \int_{-i\infty}^{i\infty} \frac{\Gamma(\sigma) \Gamma(-\sigma - i\mu - 3/2) \Gamma(-\sigma + i\mu - 3/2) x^6}{\Gamma(-i\mu - 3/2) \Gamma(i\mu - 3/2)} d\sigma.$$

With the aid of Eq. (46), we find

$$\zeta_\mu = \frac{1}{\sqrt{2\pi}} \left| \frac{\Gamma(-3/2 + i\mu)}{\Gamma(2i\mu)} \right| \frac{1}{2\pi i} \\ \times \int_{-i\infty}^{i\infty} \frac{\Gamma(\sigma) \Gamma(\sigma+4) \Gamma(-\sigma - i\mu - 3/2) \Gamma(-\sigma + i\mu - 3/2)}{\Gamma(-i\mu - 3/2) \Gamma(i\mu - 3/2)} d\sigma \\ = \frac{1}{\sqrt{2\pi}} \left| \frac{\Gamma(-3/2 + i\mu)}{\Gamma(2i\mu)} \right| \Gamma(5/2 + i\mu) \Gamma(5/2 - i\mu),$$

where the complex integral is taken according to Barnes' formula<sup>3</sup>. Substituting this in Eq. (47), making use of the equation  $\Gamma(u) \Gamma(1-u) = \pi \csc \pi u$ , and the fact that  $W_{2,i\mu}(x)$  is an even function of  $\mu$ , we get the expression for  $\bar{x}$  in the form of a complex integral

$$\bar{x} = \frac{i}{x^2} e^{-x/2} e^{-9y/4} \int_{-i\infty}^{i\infty} e^{s^2 y} W_{2,-s}(x) \operatorname{tg} \pi s \times s ds \quad (48) \\ - 2 \left( \frac{1}{x} - \frac{1}{2} \right) e^{-2y} + 3.$$

It is therefore evident that, as a result of the Compton process, the mean frequency of any quantum tends toward  $3kT/h$ , independently of its initial frequency. Application of Eq. (48) will be given in the Appendix.

In conclusion, I wish to express my sincere gratitude to Ia. B. Zel'dovich who set up the present problem and displayed a constant interest in it, and also to L. D. Landau and I. M. Gel'fand who made a series of important suggestions. Much of the results of Sec. 3 were obtained with the participation of the late S. P. D'iakov.

## APPENDIX

### THE COMPTON PROCESS IN A BODY OF FINITE DIMENSIONS

We can compute the average frequency which is possessed by quanta in a body of finite dimensions as a result of the Compton process. Since the Thomson cross section does not depend on the energy, the diffusion coefficient of the quantum is a constant. Here the probability of emission of a quantum at the instant of time  $y$  depends exponentially on the time only if the initial distribution corresponds to one of the eigenfunctions of the diffusion problem. It is natural to take a distribution which corresponds to the ground state eigenvalue since only it of all the functions is constant in sign. In dimensionless units, the probability of emission of a quantum from the system is given by

$$dw(y) = \beta e^{-\beta y} dy. \tag{I}$$

The mean frequency of the emitted quantum in this case is less than  $3kT/h$ . It can be determined if we multiply (I) by  $\bar{x}$  and integrate over  $y$ :

$$\bar{x} = \int \bar{x} dw(y) = \frac{i\beta e^{x/2}}{x^2} \int_{-i\infty}^{i\infty} \text{tg } \pi s W_{2,-s}(x) \times \frac{s ds}{9/4 + \beta - s^2} - \frac{2\beta(1/x - 1/2)}{\beta + 2} + 3. \tag{II}$$

The energy transferred to the quanta from the electrons under these conditions is equal to the integral (II), taken over the bremsstrahlung spectrum. We introduce the notation

$$H(x_0, s) = \int_{x_0}^{\infty} M_{2,-s}(x) K_0\left(\frac{x}{2}\right) \frac{dx}{x^3}. \tag{III}$$

We then have

$$\begin{aligned} \bar{x} &= \int \bar{x} e^{-x/2} K_0\left(\frac{x}{2}\right) \frac{dx}{x} \tag{IV} \\ &= i\beta \int_{-i\infty}^{i\infty} \text{tg } \pi s \left\{ \frac{\Gamma(2s)}{\Gamma(-3/2 + s)} H(x_0, s) \right. \\ &\quad \left. + \frac{\Gamma(-2s)}{\Gamma(-3/2 + s)} H(x_0, -s) \right\} \frac{s ds}{\beta + 9/4 - s^2} \\ &\quad - 2\beta \int_{x_0}^{\infty} \frac{(1/x - 1/2)}{\beta + 2} e^{-x/2} K_0\left(\frac{x}{2}\right) \frac{dx}{x} \\ &\quad + 3 \int_{x_0}^{\infty} e^{-x/2} K_0\left(\frac{x}{2}\right) \frac{dx}{x}. \end{aligned}$$

The complex integral of the first component is completed by the integral over an infinite semicircle in the left half-plane. Then the entire integral is expressed in the form of the sum of the residues of the expression under the integral from poles located to the left of the imaginary axis. These poles are located at the points  $s = -1/2, -3/2, -5/2$  and, furthermore, at the point  $s = \sqrt{\beta + 9/4}$ . The residues at the first two points are reduced with the last two integrals in Eq. (IV). The residues at the points  $s = -5/2, -7/2, -9/2$  remain finite even for  $x_0 = 0$ . Therefore, we can replace  $x_0$  in them by 0. All these residues give only a small contribution to the integral. The principal contribution is made by the residue for  $s = -\sqrt{9/4 + \beta} = -(3/2 + u)$ . (If the body has dimensions that are not too small, then  $u$  is a small number.) For  $\bar{x}$ , we get the expression

$$\begin{aligned} \bar{x} &= -2\pi\beta \text{tg } \pi \sqrt{\beta + 9/4} \frac{\Gamma(-2\sqrt{\beta + 9/4})}{\Gamma(-3/2 - \sqrt{\beta + 9/4})} \tag{V} \\ &\quad \times H(x_0, -\sqrt{\beta + 9/4}) \\ &\quad + \sum_{k=2}^{\infty} \frac{(-)^k (k+2)!}{(2k+1)!} \frac{\beta}{\beta + 9/4 - (2k+1)^2/4} \\ &\quad \times H\left(0, -\frac{2k+1}{2}\right). \end{aligned}$$

The quantities  $H(0, -(2k+1)/2)$  are decomposed into the converging numerical series  $H(0, -5/2) = 2.27, H(0, -7/2) = 2.31$ ; the remaining expressions of this type are multiplied by small coefficients. We put  $H(x_0, -\sqrt{\beta + 9/4})$  in the form of the difference between two integrals

$$\begin{aligned} H(x_0, -\sqrt{\beta + 9/4}) &= \int_0^{\infty} M_{2,-(3/2+u)}(x) K_0\left(\frac{x}{2}\right) \frac{dx}{x^3} \tag{VI} \\ &\quad - \int_0^{x_0} M_{2,-(3/2+n)}(x) K_0\left(\frac{x}{2}\right) \frac{dx}{x^3}. \end{aligned}$$

The first integral [we call it  $F(u)$ ] is computed by expansion of  $M$  in a power series, The numerical values of  $F(u)$  are the following:

$u = 0.01$	$0.03$	$0.1$	$0.2$	
$F(u) = 10080$	$1137$	$107$	$28.8$	
$u = 0.3$	$0.4$	$0.5$	$0.75$	$1.0$
$F(u) = 13.7$	$8.2$	$5.7$	$3.3$	$2.2$

In the second integral in (VI), we must use the expressions  $M$  and  $K_0$  for small  $x$ . This gives

$$\int_0^{x_0} x^{u-1} \ln\left(\frac{x\gamma}{4}\right) dx = \frac{x_0^u}{u} \left(-\ln \frac{4}{\gamma u} - \frac{1}{u}\right).$$

As  $u$  tends to zero, we obtain the results for an unbounded medium that we already know.

<sup>1</sup> W. Heitler, *Quantum theory of radiation*.

<sup>2</sup> G. N. Watson, *Theory of Bessel functions*.

<sup>3</sup> E. T. Whittaker and G. N. Watson, *Modern Analysis*, 4th ed.