SOVIET PHYSICS

JETP

A translation of the Journal of Experimental and Theoretical Physics of the USSR.

SOVIET PHYSICS—JETP VOL. 4, NO. 6, PP 777-979 JULY, 1957

Electrodynamics of Charged Scalar Particles

L. P. GOR'KOV AND I. M. KHALATNIKOV

Institute for Physical Problems, Academy of Sciences, USSR

(Submitted to JETP editor November 23, 1955)

J. Exptl. Theoret. Phys. (U.S.S.R.) 31, 1062-1078 (December, 1956)

The behavior of the Green's function in the region of high momenta for the electrodynamics of spin-zero charged particles is investigated on the basis of the Kemmer formalism.

THE asymptotic behavior of the Green's function for high momenta has been found for spin-1/2 electrodynamics by Landau, Abrikosov, and Khalatnikov[1] by means of the direct solution of the integral equations. The same problem has been treated by Gell-Mann and Low[2], who made use of the renormalizability of the theory. The present work is a similar attempt for the electrodynamics of spin-zero particles in the Kemmer formalism.

1. β-FORMALISM

Two equivalent formulations can be used for the description of particles with spin zero and one: the second order Klein-Gordon equation and the matrix formalism of Kemmer. The latter is characterized by a deep analogy with the Dirac equation for the electron. In the β-formalism, the wave function of the spin-zero or spin-one particles satisfies a first order equation

$$(\hat{p} - m_1)\Psi(x) = 0,$$

where* $p = p_k\beta_k$, p is the momentum operator, and the β-matrices satisfy the following commutation relations

$$\beta_\mu\beta_\sigma\beta_\nu + \beta_\nu\beta_\sigma\beta_\mu = \beta_\nu\delta_{\mu\sigma} + \beta_\mu\delta_{\sigma\nu}. \qquad (1)$$

The algebra determined by Eq. (1) has two non-trivial irreducible representations, a five-dimensional and a ten-dimensional one; the first corresponds to spin-zero particles, and the second to spin-one particles. From the point of view of the Klein-Gordon equation, the five-component function for the spin-zero particles consists of one scalar component and its four derivatives with respect to space and time. The components of the wave function are therefore not dynamically independent. This is related to the fact that the β-matrices determined by relations (1) have no inverses.

The interaction of mesons with an electromagnetic field can be described both in the Klein Gordon formalism and in the β-formalism. A very significant disadvantage of the scalar formulation is the extremely complicated structure of the perturbation theory series, due to diagrams with vertices connected by two photon lines. The Kemmer formalism is more convenient, since the class of diagrams possible in its perturbation theory is the same as that in the electrodynamics of electrons, so that the rules for constructing the matrix elements from the diagrams, as has been shown by Peasly,[4] are similar to Feynman's rules for electrodynamics[3] with the difference that the γ-matrices are everywhere replaced by the β-matrices, and the Feynman propagation factor

* In the following we shall use the notation of Feynman[3].

$(p - m_1)^{-1}$ is equal to

$$(\hat{p} - m_1)^{-1} = \{\hat{p} + m_1 + (\hat{p}^2 - p^2) / m_1\} (p^2 - m_1^2)^{-1}. \tag{2}$$

In addition, a meson closed loop enters the matrix element with the sign opposite to that in ordinary electrodynamics. The whole formalism is equally applicable to spin zero and to spin one. The difference between the two spins occurs in considering the convergence of the perturbation theory integrals.

It is well known[5] that all existing field theories may be divided into the renormalizable and nonrenormalizable ones. In the renormalizable theories there exist several types of irreducible divergent diagrams, and the infinite expressions due to these can be included in the physical constants of the theory in a unique way. In a nonrenormalizable theory the number of different kinds of divergent diagrams is infinite. The electrodynamics of vector mesons belongs to this class, which can be seen immediately from expression (2) for $G_0(p)$, since for large values of the momentum p this factor is approximately unity. We will therefore consider only spin-zero mesons, using the five-dimensional representation of the β-matrices. For spin zero the situation is simplified, since a large number of the matrix expressions vanish due to the properties of the five-dimensional representation of the β-matrices.

We shall define two matrices[6]

$$X = (\beta_\lambda \beta_\lambda - 1) / 3; \qquad Y = (4 - \beta_\lambda \beta_\lambda) / 3; \quad X + Y = 1. \tag{3}$$

For spin zero

$$XY = 0, \tag{4}$$

$$X \beta_\nu \beta_\mu = \delta_{\mu\nu} X. \tag{5}$$

We present several useful expressions which follow immediately from (1), (3)-(5):

$$X \beta_\mu = \beta_\mu Y; \qquad \beta_\mu (\hat{p}^2 - p^2) \beta_\nu = X (p_\mu p_\nu - \delta_{\mu\nu} p^2); \tag{6}$$

$$X (\hat{p}^2 - p^2) = 0; \quad \beta_\mu \beta_\mu = 3X + 1; \quad \beta_\mu \beta_\sigma \beta_\mu = \beta_\sigma;$$
$$\beta_\mu \beta_\sigma \beta_\tau \beta_\mu = \delta_{\sigma\tau}; \quad \beta_\mu \beta_\sigma \beta_\tau \beta_\rho \beta_\mu = X \beta_\sigma \delta_{\rho\tau} + \beta_\rho X \delta_{\sigma\tau}.$$

In calculating traces, we must bear in mind that the trace of a product of an odd number of β-matrices vanishes. In addition,

$$\mathrm{Sp}\, 1 = 5; \quad \mathrm{Sp}\, X = 1; \quad \mathrm{Sp}\, \beta_\lambda \beta_\mu = 2\delta_{\lambda\mu}; \quad \mathrm{Sp}\, \beta_\lambda \beta_\mu \beta_\nu \beta_\rho = \delta_{\lambda\mu} \delta_{\nu\rho} + \delta_{\lambda\rho} \delta_{\mu\nu}. \tag{7}$$

Consideration of the perturbation theory expressions shows[6] that in the electrodynamics of a spin-zero meson there exist in general five irreducibly divergent diagrams: 1) and 2) the self-energy diagrams of the meson and photon, which are quadratically divergent; 3) the vertex part, which is logarithmically divergent; 4) a Compton type diagram, which is logarithmically divergent; 5) the diagram corresponding to scattering of a meson by a meson, which is logarithmically divergent.

Scattering of light by light gives a finite expression when summed over all permutations of the emitted quanta.

2. FUNDAMENTAL EQUATIONS

The character of the divergence which appears in the theory is determined by the asymptotic behavior of the Green's function for high momenta. Two of the articles already cited[1,2] are devoted to the problem of finding the asymptotic behavior of the Green's functions in electrodynamics and in pseudoscalar theory. In solving the corresponding problem in the electrodynamics of spin-zero particles, we shall henceforth base our considerations on the idea of a smeared out interaction[1], according to which the interaction which is usually described by a δ-function is assumed smeared out and nonvanishing in some finite region whose linear dimensions are about a. In calculating radiation effects, integration over virtual quanta is cut off at momenta $\sim \Lambda$ where ($\Lambda \sim 1/a$), and the integrals are then finite. The transition to the exact interaction corresponds to $\Lambda \to \infty$.

Let us write the integral equations which are satisfied by the quantities entering into the theory. The equation for the meson Green's function $G(p)$ can be written, in complete analogy with the usual electrodynamics, in the following way[7]

$$G(p) \Big\{ \hat{p} - m_1 - (e_1^2 / \pi i) \times \int B_\mu (p, p - k, k)\, G(p - k)\, \beta_\nu D_{\mu\nu}(k)\, d^4 k \Big\} = 1. \tag{8}$$

Here m_1 is the "bare" mass, e_1 is the "bare" meson charge, and $B_\mu(p, p - k, k)$ is the operator of the vertex part. The photon propagation factor $D_{\mu\nu}(k)$ as has previously been noted[1], can be written as the sum of the transverse and longitudinal parts

$$D_{\mu\nu}(k)=\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{d_t(k^2)}{k^2}+\frac{k_\mu k_\nu}{k^2}\frac{d_l(k^2)}{k^2}, \quad (9)$$

of which only the transverse part has physical meaning for the interaction; it must therefore be determined from the equations, whereas the longitudinal part may be chosen arbitrarily. The equation for the transverse part of the photon Green's function is of the form

$$D^t_{\mu\sigma}(k)\left\{k^2\delta_{\sigma\nu}-\frac{e_1^2}{\pi i}\int \mathrm{Sp}\,[G(p)B_\sigma(p,p-k,k) \times G(p-k)\beta_\nu]\,d^4p\right\}=\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}. \quad (10)$$

The gauge invariance of the theory should follow directly from the transverse nature of the polarization tensor in this equation. Let us also note that the different sign for the integral term in Equation (10), as compared with the electro-dynamics of an electron, is related to the Bose statistics for spin-zero particles. Equations (8) and (10) for the Green's functions of an interacting meson and photon are exact and contain no approximations. The situation with respect to the equations for the operators of the vertex parts and the Compton diagrams is much more complicated. It is in general impossible to obtain exact integral equations for them in closed form. It turns out, however, that for certain conditions it is possible to write an approximate integral equation for the operator of the vertex part. In order to do this, let us go over to a consideration of the expressions furnished by perturbation theory. We shall write the expression for the meson Green's function obtained from perturbation theory to first order in e_1^2 in the high momentum region (for the space-components, $-p^2 \gg m^2$) up to terms containing a large logarithm

$$G^{-1}(p)=\hat{p}-m_1+\left(\frac{3X}{m_1}\right)\left(\frac{e_1^2}{\pi i}\right)\int\frac{d^4k}{k^2} -\left(\frac{e_1^2}{8\pi}\right)(\hat{p}+2m_1)\ln\left(\frac{\Lambda^2}{-p^2}\right) +\left(\frac{e_1^2}{16\pi}\right)\left[\hat{p}-m_1 +\left(\frac{3X}{m_1}\right)(p^2-m_1^2)\right]\ln\left(\frac{\Lambda^2}{-p^2}\right)(d_l-1). \quad (11)$$

Calculation of the matrix elements for the first order correction to the vertex part leads to the following asymptotic expression for $B_\sigma(p, p-l, l)$ for large space components, the largest of which we shall call f:

$$B_\sigma(p,p-l,l)=\beta_\sigma-\frac{e_1^2}{8\pi}\beta_\sigma\ln\left(\frac{\Lambda^2}{-f^2}\right) +\frac{e_1^2}{16\pi}\left[\beta_\sigma+\frac{3X}{m_1}(2p-l)_\sigma\right]\ln\left(\frac{\Lambda^2}{-f^2}\right)(d_l-1). \quad (12)$$

As can be seen from these expressions, not only the longitudinal, as for ordinary electrodynamics, but also the transverse part of the photon Green's function leads to terms logarithmic in the meson self-energy and vertex parts. The last (third) term in Eq. (12) shows, in view of the Ward identity, that the logarithmic divergence in the radiation corrections to the Compton effect occur only in combination with the factor $d_l - 1$. This result can also be obtained by considering the matrix element of the first correction to the Compton effect (Appendix I). Thus with the choice $d_l = d_t$ the Compton diagrams are finite in first order perturbation theory, and give no logarithmic contribution. It is interesting to note that this situation follows from the use of the β-formalism; to one diagram for the correction to the Compton effect in the Kemmer formalism corresponded several diagrams in the second order equations, each of which diverges separately. Only for the special choice $d_l = d_t$ is their β-matrix sum finite.

Except for the quadratically diverging integrals in the meson and photon mass, the perturbation theory series thus contains only logarithmic terms in the limiting momentum. Use of perturbation theory effectively implies a condition $e_1^2\ln(\Lambda^2/m^2) \ll 1$. We shall, following an earlier work[1], use the much weaker condition $e_1^2 \ll 1$. Subjecting the theory to such a limitation means that in the asymptotic region we must sum only over those diagrams whose contribution to the matrix elements contains the large logarithm raised to a power equal to the perturbation theory order. In this approximation the Compton diagrams in all perturbation theory orders are the same as in the zeroth approximation (for $d_l = d_t$; this assumption, based at this point on calculations to the first approximation in e_1^2 is later verified by the expression obtained for the operator of the vertex part). For the vertex part $B_\sigma(p, p-l, l)$ we can write the approximate integral (so-called three-vertex) equation

$$B_\sigma(p,p-l,l)=\beta_\sigma+(e_1^2/\pi i) \times\int B_\mu(p,p-k,k)G(p-k)B_\sigma(p-k,p-k-l,l) \times G(p-k-l)B_\mu(p-k-l, p-l,-k)D(k)\,d^4k. \quad (13)$$

This equation expresses the fact that in summing the radiation corrections to the vertex part, only diagrams with nonintersecting photon lines are accounted for, and is derived on the assumption that the longitudinal part of the photon Green's function is chosen equal to its transverse part ($d_l = d_t$). In other words

$$D_{\mu\nu}(k) = \delta_{\mu\nu} D(k) = \delta_{\mu\nu} d_t(k^2)/k^2.$$

Direct calculations show that the second order diagram in e_1^2 with two intersecting photon lines gives no contribution in this approximation (Appendix I). Let us note also that the matrix element for scattering of a meson by a meson is not "linked" with the fundamental equations and can be considered separately.

In the following sections Equations (8), (10), and (13) will be used to find the asymptotic forms of the photon and meson Green's functions.

3. PHOTON GREEN'S FUNCTION

Let us now go on to a solution of the above equations. We should note that the matrix composition of the exact Green's functions for the meson and vertex parts will not in general, as can be seen, for instance, from expression (11), be the same as the corresponding expressions for these quantities in the zeroth approximation for noninteracting fields. Considerations of relativistic invariance, although they narrow the possible matrix combinations, still leave a sufficient number of different invariant combinations both in the meson Green's function and, especially, in the vertex part. The reason for this is the triple (and not double, as in ordinary electrodynamics) permutation rules, so that a large number of invariants are independent. For $d_l = d_t$, however, a consideration of the perturbation theory formulas (11) and (12) allow us to assume that in the high-momentum region the meson Green's function $G(p)$ and the vertex part $B_\sigma(p, p-l, l)$ can be written in the following form:

$$G^{-1}(p) = \left\{\hat{p} - m_1(p^2) - \frac{X\varepsilon}{m_1}\right\}\left(\frac{1}{\beta(p^2)}\right); \tag{14}$$

$$B_\sigma(p, p-l, l) = \beta_\sigma \alpha(f^2).$$

Here $\beta(p^2)$, $\alpha(f^2)$, and $m_1(p^2)$ are slowly (logarithmically) varying functions of their arguments, $m_1(p^2)$ is proportional to the "bare" mass m_1 of the meson. and f is the largest of the three arguments in $B_\sigma(p, p-l, l)$. The quantity ϵ in the expression for $G(p)$ is of order $\sim e_1^2\Lambda^2$, where Λ is the upper limit of the interaction, and is the quadratic correction to the mass. We note that the operator which is the inverse of $(\hat{p} - m_1 - XA/m_1)$ is equal to

$$\left(\hat{p} - m_1 - \frac{XA}{m_1}\right)^{-1} \tag{14a}$$

$$= \left\{\hat{p} + m_1 + \frac{1}{m_1}(\hat{p}^2 - p^2) + \frac{YA}{m_1}\right\}\frac{1}{p^2 - m_1^2 - A}.$$

We have already seen in Eq. (11) that ϵ comes from the transverse part of the photon Green's function. It is significant that the quantity ϵ drops out of all further calculations, so that it effects only a change in mass. In our approximation ϵ may be considered constant. We make the following additional remark: in integral equations (8), (10), and (13) the high momentum region is of fundamental importance. Therefore in order to calculate these integrals we must know the behavior of all the functions in the high momentum region. This makes it possible to determine the variation of all quantities in the asymptotic region, for this approximation, without considering regions of momentum small compared to the meson mass. In addition, after all intermediate operations the integrands can be written as the sum of various combinations of matrices multiplied by a scalar function whose argument is of the form $(k-a)^2$, where a may be any of the vectors in Eqs. (8), (10), and (13). These functions have no singularities in the upper half plane. If a is a spacelike vector, then integration over k may be replaced by integration over four-dimensional Euclidean space by replacing k_0 by ik_0. Therefore in solving the equations which determine $G(p)$, $B_\sigma(p, p-l, l)$, $D(k)$, we may limit ourselves to the region of spacelike vectors. For timelike arguments, these functions may be obtained by analytic continuation through the upper half plane. In those regions of integration over k which give the main contribution (logarithmic) to arguments of the form $(k-a)^2$, we may neglect a in comparison with k, and therefore all the functions will depend on k^2; the transition to integration over Euclidean space is accomplished particularly simply:

$$d^4k \rightarrow i/4\,(-k^2)\,d(-k^2). \tag{15}$$

The importance of this transformation was first noted by Landau and co-workers[1]. Turning to Equation (13), let us replace $G(p)$ and $B_\sigma(p, p-l, l)$ by the corresponding expressions from (14). It is easy to see that the integral diverges logarithmically in the region $-f^2 \ll -k^2 \ll \Lambda^2$, where f is the largest of the arguments of the vertex part.

Bringing similar terms into the numerator, going over to integration over Euclidean space, and averaging over angles, the integral on the right side becomes

$$-\beta_\sigma \frac{e_1^2}{8\pi} \int_{-k^2 \gg -f^2}^{-k^2 \ll \Lambda^2} \alpha^3(-k^2)\,\beta^2(-k^2)\,d(-k^2)\frac{d-k^2}{-k^2}$$

in agreement with the assumption that the right side of Eq. (13) is again proportional to B_σ. This also verifies the validity of both expressions (14), since $G(p)$ and $B_\sigma(p, p-l, l)$ are related by the Ward identity

$$\partial G^{-1}(p)/\partial p_\sigma = B_\sigma(p, p, 0).$$

The function α satisfies the following equation:

$$\alpha(\xi) = 1 - \frac{e_1^2}{8\pi}\int_\xi^L \alpha^3(z)\,\beta^2(z)\,d(z)\,dz. \tag{16}$$

Here we have introduced the new variable $z = \ln(-k^2/m^2)$ so that $\epsilon = \ln(-f^2/m^2)$ and $L = \ln(\Lambda^2/m^2)$.

The equation which determines the function $\beta(p)$ should be obtained from (8); however, since the integral contains a quadratically diverging part, in order to obtain the logarithmic terms in it, it is necessary to have not only expressions (14) for $B_\mu(p, p-k, k)$, but the contributions to it up to terms of the order of about p^2/k^2 inclusive. For simplicity, we shall use the Ward identity, which we write in the form

$$\alpha(\xi)\,\beta(\xi) = 1. \tag{17}$$

With the aid of this expression, Eq. (16) can be written

$$\alpha(\xi) = 1 - \frac{e_1^2}{8\pi}\int_\xi^L \alpha(z)\,d(z)\,dz. \tag{18}$$

Eq. (10) for the photon Green's function contains a quadratically diverging integral. The quadratically diverging part can be separated, and represents the photon mass. Because of the gauge invariance of the theory, the interaction should be smeared out in such a way that this quantity remain zero. The remaining part again contains logarithmic integrals, and in order to calculate these it is necessary to know the logarithmic corrections to the vertex parts $B_\sigma(p, p-l, l)$ in the region $p \gg l$ accurately to terms of second order in l/p inclusive. To obtain this information we again refer to Eq. (3), replacing the variable $p-k$ by k:

$$\begin{aligned} &B_\sigma(p, p-l, l) \\ &= \beta_\sigma + \frac{e_1^2}{\pi i}\int B_\mu(p, k, p-k)\,G(k)\,B_\sigma(k, k-l, l) \\ &\times G(k-l)\,B_\mu(k-l, p-l, k-p)\,D(p-k)\,d^4k. \end{aligned} \tag{19}$$

Inserting Eq. (14) for the appropriate factors, and multiplying out the numerator of the integrand, we see that the appropriate terms occur on the right side for the region of integration $p \gg k \gg l$ in expanding $B_\sigma(k, k-l, l)$ and the denominators of $G(k)$ and $G(k-l)$. Expanding the rest of the factors, as can be easily seen, leads to higher order terms. In order to clarify the form of the corrections in this region, let us consider the additions which come from expanding the denominators in $G(k)$ and $G(k-l)$ in powers of l/k, $(m_1^2+\epsilon)/k^2$. It turns out that the corresponding term is of the form

$$\frac{e_1^2}{24\pi}(l\hat{l}_\sigma - \beta_\sigma l^2)\,(p^{-2})\,\alpha^2(\xi)\,d(\xi)\int_\eta^\xi \beta^2(z)\,\alpha(z)\,dz,$$

$$\xi = \ln(-p^2/m^2), \quad \eta = \ln(-l^2/m)^2.$$

We shall try to obtain $B_\sigma(p, p-l, l)$ in the region $p \gg l$ in the form

$$\begin{aligned} B_\sigma(p, p-l, l) &= \beta_\sigma \alpha(p^2) \\ &+ (l\hat{l}_\sigma - \beta_\sigma l^2)\,(p^{-2})\,\alpha^2(p^2)\,d(p^2)\,S_0(p^2, l^2). \end{aligned} \tag{20}$$

We then obtain the following equation (in closed form) for the function $S_0(\xi, \eta)$:

$$\begin{aligned} S_0(\xi, \eta) = &-\frac{e_1^2}{8\pi}\int_\eta^\xi \alpha^2(z)\,\beta^2(z)\,d(z)\,S_0(z, \eta)\,dz \\ &+ \frac{e_1^2}{24\pi}\int_\eta^\xi \alpha(z)\,\beta^2(z)\,dz. \end{aligned} \tag{21}$$

Corrections of other forms do not occur.

If we insert (20) into expression (10) for the polarization tensor, we obtain an equation which determines the function $d_t = d_l$

$$d_t^{-1}(\eta) = 1 + 2S_0(L, \eta), \tag{22}$$

where L is the logarithm of the cut-off parameter.

Equations (18), (21) and (22) must be solved simultaneously. With the aid of the Ward identity, Eq. (21) can be written

$$S_0(\xi, \eta) = -\frac{e_1^2}{8\pi}\int_\eta^\xi d(z)\, S_0(z, \eta)\, dz + \frac{e_1^2}{24\pi}\int_\eta^\xi \beta(z)\, dz.$$

Differentiating this by ξ, we obtain

$$\partial S_0(\xi, \eta)/\partial\xi = -(e_1^2/8\pi)\, d(\xi)\, S_0(\xi, \eta) + (e_1^2/24\pi)\,\beta(\xi).$$

But

$$d\beta(\xi)/d\xi = -(e_1^2/8\pi)\, d(\xi)\,\beta(\xi), \tag{23}$$

and therefore

$$\partial[S_0(\xi, \eta)\,\beta^{-1}(\xi)]/\partial\xi = e_1^2/24\pi$$

and with the conditions $S_0(\xi,\xi) = 0$, we obtain

$$S_0(\xi, \eta) = (e_1^2/24\pi)(\xi - \eta)\,\beta(\xi).$$

Since $\beta(L) = 1$, we have $S_0(1, \eta) = (e_1^2/24\pi) \times (L - n)$. It is interesting to note that $S_0(L, \eta)$ is exactly identical with the same function in spinor electrodynamics. We note in this respect that in addition to the polarization of spin-zero particles by the photon, there actually occurs also a polarization of the spin 1/2 particles. Therefore, instead of (10), we should have written the photon Green's function equation in the form

$$D^t_{\mu\sigma}(k)\Big\{k^2\delta_{\sigma\nu} - \Big(\frac{e_1^2}{\pi i}\Big) \times \int \mathrm{Sp}\,[G^0(p)\, B_\sigma(p, p-k, k)\, G^0(p-k)\,\beta_\nu]\, d^4p + \Big(\frac{e_1^2}{\pi i}\Big)\int \mathrm{Sp}[G^{1/2}(p)\,\Gamma_\sigma(p, p-k, k) \times G^{1/2}(p-k)\,\gamma_\nu]\, d^4p\Big\} = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}.$$

Making use of previously derived results [1], we can write down immediately

$$d_t^{-1}(\eta) = 1 + 8\nu S(L, \eta).$$

The quantity ν, here introduced, plays the role of the effective number of different particles. Spin-1/2 gives 1 for ν, and spin zero, 1/4. Thus

$$d_t(\eta) = [1 + (e_1^2/3\pi)\,\nu\,(L - \eta)]^{-1}$$

or

$$d_t(k^2) = [1 + (e_1^2/3\pi)\,\nu \ln(\Lambda^2/-k^2)]^{-1}. \tag{24}$$

We note that this formula is valid not only for $k \gg m$, but also for $k \gtrsim m$, since in the calculation both m_1 and ϵ drop out. For $-k^2 \ll m^2$

$$d_t(k^2) = [1 + (e_1^2/3\pi)\,\nu \ln(\Lambda^2/m^2)]^{-1}$$

which corresponds to the fact that the logarithmic region of integration in Eq. (10) lies* above m.

The physical charge is given by

$$e^2 = e_1^2[1 + (e_1^2/3\pi)\,\nu \ln(\Lambda^2/m^2)]^{-1}. \tag{25}$$

The expression we obtain for the "bare" charge e_1^2 is then

$$e_1^2 = e^2[1 - (e^2/3\pi)\,\nu \ln(\Lambda^2/m^2)]^{-1}. \tag{26}$$

For a sufficiently large radius of interaction $(e^2/3\pi)\nu\ln\Lambda^2/m^2 \sim 1$, and the bare charge e_1^2 becomes of the order of unity in contradiction to the original assumption of weak interaction ($e_1^2 \ll 1$). As has been shown by Pomeranchuk[8], this restriction is not of importance, as Eq. (25) is the exact result for large Λ^2. From this we obtain the fundamental conclusion that in the point-interaction limit ($\Lambda \to \infty$) the physical charge vanishes (for more details see Pomeranchuk[8]).

The interaction $e_1^2 d_t$ can be written in a form containing only renormalized quantities

$$e_1^2 d_t(k^2) \equiv e^2 d_{tc}(k^2) = e^2[1 - (e^2/3\pi)\,\nu \ln(-k^2/m^2)]^{-1}. \tag{27}$$

4. MESON GREEN'S FUNCTION

With the function $d(p^2)$ obtained, it is easy to derive the formula for $\beta(p^2)$ from Eq. (18):

$$\beta(p^2) = [1 + (e_1^2/3\pi)\,\nu \ln(\Lambda^2/-p^2)]^{3/8\nu}. \tag{28}$$

The function $\alpha(p^2) = 1/\beta(p^2)$. We note that the function $\beta(p^2)$ can be written as the product of a constant and a function independent of the cut-off parameter,

$$\beta(p^2) = (e_1/e)^{3/4\nu} \times [1 - (e^2/3\pi)\,\nu \ln(-p^2/m^2)]^{3/8\nu} = [e_1^2 d_t(p^2)/e_1^2]^{-3/8\nu}. \tag{29}$$

* We are here neglecting the difference between the meson and electron masses.

In this form $\beta(p^2)$ decreases as $-p^2$ increases, and becomes small at the same point where the denominator of expression (27) vanishes. Equation (29) expresses the so-called renormalizability of the function $\beta(p^2)$.

Let us write down the meson Green's function $G(p)$ for large momenta

$$G(p) = \left\{\hat{p} + m_1(p^2) + \frac{1}{m_1(p^2)}(\hat{p}^2 - p^2) - \frac{Y\varepsilon}{m_1}\right\}\frac{\beta(p^2)}{p^2}.$$

In the β-formalism $G(p)$ consists of a scalar, a vector, and a tensor component. The dependence of each of these components on the radius of cut-off Λ is different. Using a special and simply derivable representation of the β-matrices, which establishes the relation between the Klein-Gordon equation and the Kemmer equation*, we can separate out the scalar component

$$G_{55}(p) \approx \beta(p^2)\, m_1(p^2)/p^2, \tag{30a}$$

The vector component

$$G_{i5}(p) \approx \beta(p^2)(p_i/p^2) \tag{30b}$$

and the tensor component

$$G_{ik}(p) \approx \frac{\beta(p^2)}{m_1(p^2)}\left(\frac{p_i p_k}{p^2} - \delta_{ik}\right) \tag{30c}$$

in the region $-p^2 \gg m^2$. In these expressions, in addition to $\beta(p^2)$, the function $m_1(p^2)$ also enters. We shall not try to find an equation for $m_1(p^2)$, but shall make use of the fact that, as can be seen from the expression for the meson Green's function in coordinate space[9],

$$G(x, x') = \langle T(\Psi(x), \overline{\Psi}(x'))\rangle_0 - (i/m_1)(1 - \beta_0^2)\,\delta(x - x')$$

the component $G_{55}(p)/m_1$ for our representation of the β-matrices is the meson propagation kernel in the scalar formulation of the theory. The asymptotic behavior of this quantity can be determined by the method of Gell-Mann and Low[2]. The applicability of this method to one or another theory of interacting fields depends on whether or not this theory refers to quantities renormalized in the sense of Z-factors. Let us bear in mind that the requirement of the renormalizability means the following. Assume that certain functions, say $f_i(p^2)$, referring to the particle Green's function and the vertex parts appear in the theory. We construct some combination $H(p^2)$ of these functions, such that $g_1 H(p^2)$ represents an interaction, where g_1 is the bare coupling constant. Let Λ be the interaction cut-off radius, chosen so that $f_i(\Lambda^2) = 1$. The requirement of renormalizability means that the function $f_i(p^2)$ differs only by some factor from a function independent of the cut-off radius

$$f_i(p^2/\Lambda^2, g_1) = f_{ic}(p^2/m^2, g)/f_{ic}(\Lambda^2/m^2, g) \tag{31}$$

(for large values of $-p^2$, the function f_i depends only on the ratio $-p^2/\Lambda^2$). The renormalized interaction constant is determined by the condition

$$g_1 = gH_c(\Lambda^2/m^2, g).$$

It can then be shown[2] that the following relation holds:

$$g_1 H(-p^2/\Lambda^2, g_1) = F(-\varphi(g_1)\,p^2/\Lambda^2),$$

where F and φ are inverse functions: $F[\varphi(x)] = x$.

Equation (31) is not trivial. We shall see later that in the Kemmer formalism for $d_l \neq d_t$ by no means do all functions satisfy Eq. (31). In the scalar formulation, however, this relation holds; Eq. (29) shows that this is valid both with relation to the function $\beta(p^2)$ in our approximation and for the condition $d_l = d_t$. We shall present all the calculations anew in this case, using only (31). We then have

$$e_1^2 d_t(k^2) = F(-\varphi(e_1^2)\,k^2/\Lambda^2). \tag{32}$$

Inverting this expression and taking logarithms, we obtain

$$\ln\varphi(e_1^2 d_t) - \ln\varphi(e_1^2) = \ln(-k^2/\Lambda^2).$$

Expanding the left side of this equation in powers of $e_1^2(d_t - 1)$ and taking only the first term, we obtain

$$\frac{1}{\varphi(e_1^2)}\frac{d\varphi(e_1^2)}{de_1^2}\, e_1^2(d_t - 1) = \ln\frac{-k^2}{\Lambda^2}. \tag{33}$$

The quantity $d_t - 1$ may be found by perturbation theory. Vacuum polarization for spin 1/2 particles gives the following value for $d_t - 1$ in the first approximation: $(e_1^2/3\pi)\ln(-k^2/\Lambda^2)$,

---

* This representation is $(\beta_\mu)_{\alpha\tau} = \delta_{\mu\alpha}\delta_{5\tau} + \delta_{\mu\tau}\delta_{5\alpha}$; $(\mu = 1\ldots 4;\ \alpha\tau = 1\ldots 5)$.

vacuum polarization for spin zero particles:

$$(e_1^2/12\pi)\ln(-k^2/\Lambda^2);$$

introducing again the quantity ν, we obtain $d_t - 1 = (e_1^2/3\pi)\nu\ln(-k^2/\Lambda^2)$. Inserting this expression in (32) gives a differential equation for the function $\varphi(x)$

$$\varphi^{-1}(x)\,d\varphi(x)/dx = (3\pi/\nu)\,x^{-2},$$

whose solution is $\varphi(x) = \exp(-3\pi/\nu x)$, and for the inverse function $F(x)$

$$F(x) = -(3\pi/\nu\ln x),$$

which leads again to Eq. (24), which was obtained by solving the integral equations.

If the functions $\beta(p^2)$ and $m_1(p^2)/m_1$ are renormalizable, then it follows from Eq. (32) that they can be written in the following form:

$$\beta(p^2) = H[e_1^2 d_t]/H[e_1^2]; \tag{34}$$

$$m_1(p^2)\beta(p^2)/m_1 = Q[e_1^2 d_t]/Q[e_1^2].$$

At the same time, perturbation theory, according to (11), leads to the following expression:

$$\beta(p^2) \approx 1 - (e_1^2/8\pi)\ln(-p^2/\Lambda^2); \tag{35}$$

$$m_1(p^2)\beta(p^2)/m_1 \approx 1 - (e_1^2/2\pi)\ln(-p^2/\Lambda^2).$$

Taking logarithms in expression (34), expanding in powers of $e_1^2(d_t - 1)$ and comparing with (35), we obtain the following equations for the functions $H(x)$ and $Q(x)$, $H^{-1}(x)\,dH(x)/dx = -(3/8\nu)\,x^{-1}$; $Q^{-1}(x)\,dQ(x)/dx = -(3/2\nu)\,x^{-1}$, whose solutions are

$$H(x) = x^{-3/8\nu}; \quad Q(x) = x^{-3/2\nu},$$

which gives the previous expression (28) for the function $\beta(p^2)$, and for $m_1(p^2)$ gives the following result:

$$m_1(p^2) = m_1[1 + (e_1^2/3\pi)\nu\ln(\Lambda^2/-p^2)]^{9/8\nu} \tag{36}$$

or

$$m_1(p^2) = [e_1^2 d_t/e_1^2]^{-9/8\nu}$$

$$= (e_1/e)^{9/4\nu}[1 - (e_1^2/3\pi)\nu\ln(-p^2/m^2)]^{9/8\nu}.$$

Knowing the functions $m_1(p)$ and $\beta(p^2)$ we can write an expression for all components of the meson Green's function for $-p^2 \gg m^2$.

For the scalar component

$$G_{55}(p) \approx m_1[1 + (e_1^2/3\pi)\nu\ln\Lambda^2/-p^2]^{3/2\nu}(1/p^2), \tag{37a}$$

For the vector component

$$G_{i5}(p) \approx [1 + (e_1^2/3\pi\nu)\ln\Lambda^2/-p^2]^{3/8\nu}(p_i/p^2) \tag{37b}$$

and, finally, for the tensor component

$$G_{ik}(p) \approx \frac{1}{m_1}[1 + (e_1^2/3\pi)\nu\ln\Lambda^2/-p^2]^{-3/4\nu}\left(\frac{p_i p_k}{p^2} - \delta_{ik}\right). \tag{37c}$$

Thus when used for all the components of the meson Green's function, method (2) leads to the same results as the direct solution of the integral equations. We emphasize, however, that in the β-formalism, the use of this method requires preliminary proof of the renormalizability by solving the integral equations. Equation (31) is satisfied for all the functions only in the approximation we are considering and for the special choice $d_l = d_t$. For arbitrary d_l, Eq. (31) is not satisfied even in the first approximation. We are not, of course, referring to the function $m_1(p^2)$.

5. GAUGE TRANSFORMATION FOR THE MESON GREEN'S FUNCTION

The equations of electrodynamics remain invariant under the following transformation of the four-dimension potential

$$A_\mu(x) \to A_\mu(x) + \partial\varphi(x)/\partial x_\mu$$

with the simultaneous transformation of the particle wave function according to

$$\psi(x) \to e^{ie\varphi(x)}\psi(x),$$

and therefore an electromagnetic field operator can be separated into its transverse and longitudinal parts. The longitudinal part of the electromagnetic field operator may be arbitrary in view of gauge invariance, and does not interact (commutes) with all other quantities. For the photon Green's function this separation leads to expression (9). Gauge transformation of charged particle Green's functions has been studied by Landau and Khalatnikov[10] and is given by the following formula

$$G(x, x') = G_0(x, x')\exp\{ie_1^2(\Delta_F(0) - \Delta_F(x - x'))\},$$

where $G_0(x, x')$ is the corresponding Green's function in the absence of scalar quanta, and

$$\Delta_F(x-x') = i\langle T(\varphi(x), \varphi(x'))\rangle_0.$$

In momentum space it is more convenient to make use of another result of the same work[10], which was obtained for the infinitesimal gauge transformation

$$\delta G(p) = \frac{ie_1^2}{\pi}\int\{G(p) - G(p-k)\}\frac{\delta d_l(k^2)\, d^4k}{(k^2)^2}.$$

Restricting ourselves to slowly varying functions $d_l(k^2)$, we see that this integral diverges logarithmically in the region $-k^2 \gg -p^2$. Let us write this relation for the scalar component $G_{55}(p)$:

$$\delta G_{55}(p) = \frac{ie_1^2}{\pi}\int_{\gg p^2} G_{55}(p)\frac{\delta d_l(k^2)\, d^4k}{(k^2)^2}.$$

Transforming this to integration over Euclidean space and writing $Z = \ln(-k^2/m^2)$, we obtain

$$\delta G_{55}(p) = -\frac{e_1^2}{4\pi}G_{55}(p)\int_{\ln(-p^2/m^2)}^{\ln(\Lambda^2/m^2)}\delta d_l(z)\,dz.$$

This relation should be integrated with the boundary condition that for $d_l = d_t\, G_{55}(p)$ it gives Eq. (37a). The result is then

$$G_{55}(p) \approx m_1[d_t(-p^2)]^{-9/8\nu} \times \exp\left\{-\frac{e_1^2}{4\pi}\int_{\ln(-p^2/m^2)}^{\ln(\Lambda^2/m^2)} d_l(z)\,dz\right\}(1/p^2). \qquad (38a)$$

In exactly the same way we can obtain expressions for the vector components $G_{i5}(p)$ for $-p^2 \gg m^2$

$$G_{i5}(p) \approx [d_t(-p^2)]^{-9/8\nu} \times \exp\left\{-\frac{e_1^2}{4\pi}\int_{\ln(-p^2/m^2)}^{\ln(\Lambda^2/m^2)} d_l(z)\,dz\right\}(p_i/p^2). \qquad (38b)$$

This shows that the difference between the exact functions and the interacting field functions does not arise only from the gauge transformation, which is not the case in electron electrodynamics. We note that according to (30), (36), and (38) the function $m_1(p^2)$ does not depend on d_l, since the vector and scalar components of $G(p)$ transform in the same way under a gauge transformation. This, furthermore should have been expected, since the function $m_1(p^2)$ gives a contribution to the experimental meson mass m

$$m^2 = m_1^2(m^2) + \varepsilon.$$

The tensor component of the meson Green's function for arbitrary d_l cannot now be written in the form of Eq. (30c). For $-p^2 \gg m^2$, however, it can always be represented in the form

$$G_{ik}(p) \approx \frac{1}{m_1}\left[\left(\frac{p_ip_k}{p^2} - \frac{1}{4}\delta_{ik}\right)\lambda(p^2) - \frac{3}{4}\delta_{ik}\mu(p^2)\right]. \qquad (38c)$$

Let us write the equation for an infinitesimal gauge transformation of the tensor component

$$\left[\left(\frac{p_ip_k}{p^2} - \frac{1}{4}\delta_{ik}\right)\delta\lambda(p^2) - \frac{3}{4}\delta_{ik}\delta\mu(p^2)\right] = \frac{ie_1^2}{\pi}\int\left\{\left(\frac{p_ip_k}{p^2} - \frac{1}{4}\delta_{ik}\right)\lambda(p^2) - \frac{3}{4}\delta_{ik}\mu(p^2) - \left(\frac{k_ik_k}{k^2} - \frac{1}{4}\delta_{ik}\right)\lambda(k^2) + \frac{3}{4}\delta_{ik}\mu(k^2)\right\}\frac{\delta d_l(k^2)\,d^4k}{(k^2)^2},$$

from which we obtain, after averaging over angles

$$\delta\lambda(\xi) = -\frac{e_1^2}{4\pi}\lambda(\xi)\int_\xi^L \delta d_l(z)\,dz, \qquad (39a)$$

$$\delta\mu(\xi) = -\frac{e_1^2}{4\pi}\int_\xi^L[\mu(\xi) - \mu(z)]\,\delta d_l(z)\,dz, \qquad (39b)$$

$$\xi = \ln(-p^2/m^2), \quad L = \ln(\Lambda^2/m^2).$$

With the boundary condition (37c) we obtain the following expression for the function $\lambda(p)$:

$$\lambda(p^2) = \exp\left\{-\frac{e_1^2}{4\pi}\int_{\ln(-p^2/m^2)}^{\ln(\Lambda^2/m^2)} d_l(z)\,dz\right\}. \qquad (40)$$

Differentiating Eq. (39) by ξ

$$\delta\frac{d\mu(\xi)}{d\xi} = -\frac{e_1^2}{4\pi}\frac{d\mu(\xi)}{d\xi}\int_\xi^L \delta d_l(z)\,dz$$

and making use of boundary condition (37c) for $d_l = d_t$, we obtain

$$\mu(\xi) = 1 - \frac{e_1^2}{4\pi}\int_\xi^L \lambda(z)\,d_t(z)\,dz. \qquad (41)$$

For $d_l = d_t$ as follows from Eq. (41), $\mu(\xi) = \lambda(\xi)$; for $d_l = 0$, the function $\lambda(\xi)$ becomes unity, and $\mu(\xi)$ is given by

$$\mu(\xi) = 1 - (3/4\nu)\ln\{1 + (e_1^2/3\pi)\nu(L-\xi)\}. \qquad (42)$$

Thus for $d_l \neq d_t$, the function $\mu(\xi)$ no longer satisfies the renormalizability relation

6. CONCLUSION

The above considerations show that spin-zero electrodynamics in the β-formalism exhibit certain formal similarities with spin 1/2 electrodynamics. Beyond this, the general situation is more complicated. In order to derive the integral equations, it is necessary to make wide use of the results of perturbation theory. The problem is simplified with the choice $d_l = d_t$, in which case it is possible to write the three-vertex equation for the vertex parts. For this choice of d_l in the approximation we are considering it becomes possible to restrict ourselves to finding only the photon and meson Green's functions, and the vertex part; the Compton diagrams give contributions equal to their zeroth approximation. In this case, use of the three-vertex equation leads to the correct expression for the photon Green's function, which follows from a comparison with a result obtained by a separate method using only perturbation theory and the requirement of renormalizability. Both methods give the same results for the meson Green's function, since expression (28), which is found directly from the equations, satisfies the renormalizability relation (31).

For $d_l \neq d_t$ Eq. (13) for the vertex part is not applicable; the expression for the meson Green's function in this case can be obtained by a gauge transformation of the expression obtained for the condition $d_l = d_t$ to arbitrary d_l. It is interesting that the difference between the exact meson Green's function and its zero-approximation form does not arise only from the guage transformation, as it does in ordinary electrodynamics.

The authors are deeply grateful to Academician L. D. Landau for many valuable suggestions.

APPENDIX I

a) Correction to the Compton effect in the first approximation in e_1^2. Let us consider the matrix element for the Compton type diagram (Fig. 1)

$$C_{\sigma\tau}(p,\ p-q_1-q_2;\ q_1,\ q_2) = -\frac{e_1^2}{\pi i}\int \beta_\mu G(p-k)\,\beta_\sigma G(p-k-q_1) \times \beta_\tau G(p-k-q_1-q_2)\,\beta_\nu D_{\mu\nu}(k)\,d^4k.$$

Here $(q_1;\sigma)$ and $(q_2;\tau)$ are the momenta and polarizations of the two omitted quantities. Writing out the products of the various factors according to Eq. (6), we obtain

$$C_{\sigma\tau}(p,\ p-q_1-q_2;\ q_1,\ q_2)$$
$$= -\frac{e_1^2}{\pi i}\int\Big\{\beta_\mu(\hat{p}_1+m_1)\beta_\sigma(\hat{p}_2+m_1)\beta_\tau(\hat{p}_3+m_1)\beta_\nu$$
$$+(p_{1\mu}p_{1\sigma}-\delta_{\mu\sigma}p_1^2)\,X\beta_\tau(\hat{p}_3+m_1)\,\beta_\nu$$
$$+\beta_\mu X(p_{2\sigma}p_{2\tau}-\delta_{\sigma\tau}p_2^2)(\hat{p}_3+m_1)\,\beta_\nu$$
$$+\frac{X}{m_1}(p_{1\mu}p_{1\sigma}-\delta_{\mu\sigma}p_1^2)\,p_{2\tau}(\hat{p}_3+m_1)\,\beta_\nu$$
$$+\frac{X}{m_1}p_{1\mu}(p_{2\sigma}p_{2\tau}-\delta_{\sigma\tau}p_2^2)(\hat{p}_3+m_1)\,\beta_\nu$$
$$+\beta_\mu(\hat{p}_1+m_1)\beta_\sigma(\hat{p}_2+m_1)\frac{X}{m_1}(p_{3\tau}p_{3\nu}-\delta_{\tau\nu}p_3^2)$$
$$+\frac{X}{m_1}(p_{1\mu}p_{1\sigma}-\delta_{\mu\sigma}p_1^2)(p_{3\tau}p_{3\nu}-\delta_{\nu\tau}p_3^2)\Big\}$$
$$\times\frac{[\delta_{\mu\nu}+(k_\mu k_\nu/k^2)(d_l-1)]\,d^4k}{(p_1^2-m_1^2)(p_2^2-m_1^2)(p_3^2-m_1^2)\,k^2}$$

(for simplicity we have written $p_1 = p-k;\ p_2 = p-k-q_1;\ p_3 = p-k-q_1-q_2$).

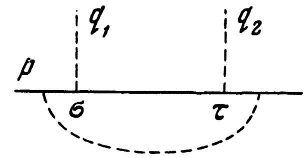

FIG. 1

The highest power of the virtual momentum k in the numerator is equal to 4, and therefore the integral diverges logarithmically in the region of large k. In this region $p_1 = p_2 = p_3 = k \gg m$:

$$-\frac{e_1^2}{\pi i}\frac{X}{m_1}\int_{-k^2\gg -f^2}[(k_\mu k_\sigma-\delta_{\mu\sigma}k^2)\,k_\tau k_\nu$$
$$+k_\mu k_\nu(k_\sigma k_\tau-\delta_{\sigma\tau}k^2)+k_\mu k_\sigma(k_\tau k_\nu-\delta_{\tau\nu}k^2)$$
$$+(k_\mu k_\sigma-\delta_{\mu\sigma}k^2)(k_\tau k_\nu-\delta_{\tau\nu}k^2)]$$
$$\times[\delta_{\mu\nu}+(k_\mu k_\nu/k^2)(d_l-1)]\,d^4k/(k^2)^4.$$

The nonzero result gives, in the numerator, the product of the longitudinal part of the photon function by the second term in the square brackets. Averaging over angles and integrating, we find

$$\frac{e_1^2}{16\pi}\frac{3X}{m_1}\delta_{\sigma\tau}\int_{\ln(-f^2/m^2)}^{\ln(\Lambda^2/m^2)}(d_l(z)-1)\,dz$$

in complete agreement with (12) (the expression obtained should be doubled to take account of exchanging the positions of quanta (q_1 and q_2). Therefore for $d_l = 1$ the Compton diagrams are finite.

b) Vertex part with two intersecting photon lines. The vertex diagram with two intersecting photon lines (Fig. 2) can be treated similarly. The matrix element for this diagram is of order e_1^4 and is proportional to the integral

$$\Lambda_\sigma (p, p-l, l) \qquad \text{(A1)}$$

$$\sim \int \{\beta_\mu G(p-k)\,\beta_\rho G(p-k_1-k_2)\,\beta_\sigma \times G(p-l-k_1-k_2)\,\beta_\nu\, G(p-l-k_2)\,\beta_\tau\}\, D_{\mu\nu}(k_1) \times D_{\rho\tau}(k_2)\, d^4k_1 d^4k_2.$$

We shall not write out the whole expression in the brackets in the integrand, which is extremely complicated, and shall consider only the denominators

$$(p-k_1)^2 (p-k_1-k_2)^2 \times (p-l-k_1-k_2)^2 (p-l-k_2)^2 k_1^2 k_2^2. \qquad \text{(A2)}$$

If we exclude integration in those regions of k_1 and k_2 which can give double logarithmic integrals, then it can be seen that there are two possibilities.

1. $k_1 \gg k_2$ (or $k_2 \gg k_1$). Then the numerator of the integrand should contain terms in the fourth power of k_1. Withthe choice $d_l = 1$, no such terms appear, as has been shown above.

2. The case in which $k_1 + k_2$ is small in comparison with k_1 (although large in comparison with p and l), which corresponds to a change of variables $k_2 = q - k_1$ or $k_1 = q - k_2$). The denominator of Eq. (A2) is about equal to

$$(k_1^2)^2 [(k_1-q)^2]^2 (p-l-q)^2 (p-q)^2.$$

We must therefore look for terms in the expression for $\Lambda_\sigma(p, p-l, l)$ which can give the power k_1^4 in the numerator. Such terms are easily found. The corresponding double logarithmic integral (for $d_l = d_t$) will be

$$\frac{X}{m_1}\int \{(k_\mu k_\rho - \delta_{\mu\rho}k^2)(k_\mu k_\rho - \delta_{\mu\rho}k^2)(2p-l-2q)_\sigma\} \times \frac{d^4k\, d^4q}{(k^2)^4 q^4 (1-2(p,q)/q^2)(1-2(p-l,q)/q^2)}.$$

It thus follows that the logarithmic term gives only integration over k, and integration over q after averaging over angles vanishes.

With respect to the diagram being considered, we must bear in mind the following. In calculating the polarization tensor

$$P_{\mu\nu}(l) = (e_1^2/\pi i) \times \int \mathrm{Sp}\,[B_\mu(p-l, p, l)\, G(p)\, \beta_\nu G(p-l)]\, d^4p$$

it turns out that the combination of factors $G(p)\,\beta_\nu G(p-l)$ contains the following powers of p in the numerator: p^2, p^3/m_1. Therefore, in general, in $B_\sigma(p-l, p, l)$ in the region $p \gg l$ we must know, in addition to the term proportional to β_σ, also the corrections of order l^2/p^2 and $m_1 l^2/p^3$. Terms proportional to β_σ, as we have seen above, do not occur. Terms $\sim l^2/p^2$ and ml^2/p^3 occur in (A1) from the integration region

$$p \gg k_1' \gg k_1' + k_2 \gg l\ (k_1' = p - k_1).$$

Their total contribution,however, to the polarization tensor vanishes. The same is true about the diagram of Fig. 1, for all corrections having a matrix structure other than expression (20). Consideration of higher order diagrams in the vertex part is complicated by the increasing complexity of the matrix expressions and the integrals. The above consideration of the matrix element of the vertex diagram with two intersecting photon lines allows us, however, to assume that for $d_l = d_t$ the higher approximations also give no contribution to the equation for the vertex part. The validity of this assumption is demonstrated by the final expression for the photon Green's function (24).

APPENDIX II

Gauge transformation of the vertex part. We shall find the vertex part $B_\mu(p, p-l, l)$ for arbitrary d_l on the basis of the relation [10]

$$G(p)\,\delta B_\mu(p, p-l, l)\, G(p-l) \qquad \text{(A3)}$$

$$= -\frac{ie_1^2}{\pi}\int \{G(p)\, B_\mu(p, p-l, l)\,[G(p-l) - G(p-l-k)] + G(p-k) \times [B_\mu(p-k, p-l-k, l)\, G(p-l-k) - B_\mu(p, p-l, l)\, G(p-l)]\}\, \delta d_l(k^2)\, d^4k/(k^2)^2,$$

which gives the form of the infinitesimal gauge transformation of $B_\mu(p, p-l, l)$. Considering only slowly varying $d_l(k^2)$, we see that the integral on the right side of (A3) is logarithmic in the region $|-k^2| \gg |(p-l)^2|$.

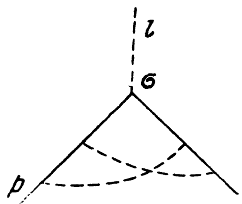

FIG. 2

We note, first of all, that the expressions (38) for the meson Green's function can be written in the matrix form

$$G(p) \approx \beta(p^2)\left\{\hat{p} + m_1(p^2) + \frac{\hat{p}^2 - p^2}{m_1(p^2)} - \frac{Yp^2}{m_1(p^2)}[\varphi(p^2) - 1]\right\}\frac{1}{p^2}, \quad \text{(A4)}$$

where $m_1(p^2)$ is given by (36),

$$\beta(p^2)/m_1(p^2) = \lambda(p^2)/m_1, \; \varphi(p^2) = \frac{1}{4\lambda(p^2)}[\lambda(p^2) + 3\mu(p^2)].$$

From (14a) and (A4) it follows that

$$G^{-1}(p) = \left\{\hat{p} - m_1(p^2) + \frac{Xp^2}{m_1(p^2)}[\varphi(p^2) - 1]\right\}\frac{1}{\beta(p^2)\varphi(p^2)}. \quad \text{(A5)}$$

In the region $p \gg l$ or $p \sim l$, we shall attempt to find the vertex part in the form

$$B_\mu(p, p-l, l) = \beta_\mu \alpha[(p-l)^2] + (2p-l)_\mu X\rho[(p-l)^2]. \quad \text{(A6)}$$

Turning to (A3), we see that the first term in the second square bracket can be dropped, and Eq. (A3) can be written in the form

$$\delta B_\mu(p, p-l, l) = -\frac{ie_1^2}{\pi}\left\{B_\mu(p, p-l, l)\int\frac{\delta d_l(k^2)\, d^4k}{(k^2)^2} - B_\mu(p, p-l, l)\int G(p-l-k) \times G^{-1}(p-l)\frac{\delta d_l(k^2)\, d^4k}{(k^2)^2} - G^{-1}(p)\int G(p-k) \times \frac{\delta d_l(k^2)\, d^4k}{(k^2)^2} B_\mu(p, p-l, l)\right\}. \quad \text{(A3a)}$$

In order to obtain the logarithmic expressions in the second and third terms, we need consider only the quadratic terms in the numerator

$$(\beta(k^2)/m_1(k^2))\{\hat{k}^2 - k^2 - Yk^2[\varphi(k^2) - 1]\}.$$

After averaging over angles this expression becomes

$$-(3/4m_1)\mu(k^2)Yk^2.$$

Going over to Euclidean space $k_0 \to ik_0$, introducing the new variable $\xi = \ln[(p-l)^2/m^2]$, and comparing corresponding terms in both parts of Equation (A3a), we obtain the following equations for the functions $\alpha(\xi)$ and $\rho(\xi)$:

$$\delta\alpha(\xi) = \frac{e_1^2}{4\pi}\alpha(\xi)\left\{\int_\xi^L \delta d_l(z)\, dz - \frac{3}{\lambda(\xi) + 3\mu(\xi)}\int_\xi^L \mu(z)\,\delta d_l(z)\, dz\right\}; \quad \text{(A7a)}$$

$$\delta\rho(\xi) = \frac{e_1^2}{4\pi}\left\{\rho(\xi)\int_\xi^L \delta d_l(z)\, dz + \frac{3\alpha(\xi)}{m_1(\xi)(\lambda(\xi) + 3\mu(\xi))}\int_\xi^L \mu(z)\,\delta d_l(z)\, dz\right\}, \quad \text{(A7b)}$$

where $L = \ln(\Lambda^2/m^2)$.

By comparing (A7a) with Eqs. (39a) and (39b), we obtain

$$\delta\alpha(\xi)/\alpha(\xi) = -\delta[\lambda(\xi) + 3\mu(\xi)]/[\lambda(\xi) + 3\mu(\xi)].$$

Using the boundary condition

$$\alpha(\xi)|_{d_l = d_t} = \beta^{-1}(\xi)|_{d_l = d_t}$$

and the fact that the function $m_1(\xi)$ does not depend on d_l, we obtain

$$\alpha(\xi) = [\varphi(\xi)\beta(\xi)]^{-1},$$

which is in agreement with (A5) and the Ward identity. Dividing (A7a) by $m_1(\xi)$ and adding it to (A7b), we find

$$\delta\left[\rho(\xi) + \frac{\alpha(\xi)}{m_1(\xi)}\right] = \frac{e_1^2}{4\pi}\left[\rho(\xi) + \frac{\alpha(\xi)}{m_1(\xi)}\right]\int_\xi^L \delta d_l(z)\, dz.$$

We have $\rho(\xi)|d_l = d_t = 0$. Integrating the previous equation, we obtain the following expression for $\rho(\xi)$:

$$\rho(\xi) = [1 - \alpha(\xi)\beta(\xi)]/m_1(\xi)\beta(\xi), \qquad (A9)$$

which again is in agreement with (A5).

In the other case, when $l \gg p$, expression (A6) for the vertex part becomes inexact. Instead of this, $B_\mu(p, p-l, l)$ is represented in the following form:

$$B_\mu(p, p-l, l) = \beta_\mu \alpha[(p-l)^2] + (2p-l)_\mu X\rho[(p-l)^2] + p_\mu X\chi_1(p-l, p) + \beta_\mu X\chi_2(p-l, p). \qquad (A10)$$

The logarithmic functions $\chi_1(\xi, \eta)$ and $\chi_2(\xi, \eta)$ vanish for $d_l = 0$ and when $\xi = \eta$. Insertion of (A10) into Eq. (A3a) leads to equations for the functions χ_1 and χ_2, which can be transformed to the following expressions:

$$\delta[\chi_1(\xi, \eta) + \rho(\xi)] = \frac{e_1^2}{4\pi}\left\{[\chi_1(\xi, \eta) + \rho(\xi)]\int_\xi^L \delta d_l(z)\, dz + \frac{3[\alpha(\xi) + \chi_2(\xi, \eta)]}{[\lambda(\eta) + 3\mu(\eta)]\, m_1(\eta)}\int_\xi^L \mu(z)\,\delta d_l(z)\, dz\right\};$$

$$\delta[\chi_2(\xi, \eta) + \alpha(\xi)] = \frac{e_1^2}{4\pi}[\chi_2(\xi, \eta) + \alpha(\xi)]\left\{\int_\xi^L \delta d_l(z)\, dz - \frac{3}{\lambda(\eta) + 3\mu(\eta)}\int_\xi^L \mu(z)\,\delta d_l(z)\, dz\right\}.$$

The solutions of these equations cannot be simply expressed in terms of the functions $\lambda(\xi)$, $\mu(\xi)$, and $m_1(\xi)$. We note only that the following relation holds:

$$m_1(\eta)\chi_1(\xi, \eta) + \chi_2(\xi, \eta) = [m_1(\xi) - m_1(\eta)]\,\rho(\xi).$$

The expressions we have obtained for the vertex part show again that for arbitrary d_l the functions entering into this expression do not satisfy the renormalizability condition.

1 Landau, Abrikosov and Khalatnikov, Dokl. Akad. Nauk SSSR **95**, 497, 773, 1177 (1954).

2 M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954)

3 R. Feynman, Phys. Rev. **76**, 749, 769 (1949).

4 D. Peasly, Phys. Rev. **81**, 94 (1951)

5 P. Matthews and A. Salam, Rev. Mod. Phys. **23**, 311 (1951).

6 A. Salam, Proc. Roy. Soc. (London) **211A**, 276 (1952).

7 F. Dyson, Phys. Rev. **75**, 1736 (1949).

8 I. Ia. Pomeranchuk, Dokl. Akad. Nauk SSSR **103**, 1005 (1955).

9 J. Schwinger, Proc. Nat. Acad. Sci. **37**, 452 (1951).

10 L. D. Landau and I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **29**, 89 (1955); Soviet Phys. JETP **2**, 69, (1956).

Translated by E. J. Saletan
221