On Distributions in Representation Space

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The concept of representation distributions is introduced for quantum systems. The quantum generalization is found for certain relations in the theory of correlated random points. The general formulas are illustrated in a concrete way for the cases of distribution of spin orientation.

THE statistical nature of quantum theory is manifested in the process of physical "measurement." From this it follows that the "pre-observation" state of a quantum system, which exists before the "measurement" and is independent of it (one can speak of such a state, to be understood in a definite sense), is a statistical state.

In the classical theory the "pre-observation" state of a statistical system is described by distribution functions in a certain space (we denote this space by M). We assume that in the quantum theory such a "pre-observation" state is described by distribution functions in this same "representation" space M, which accordingly has a classical meaning. But owing to the fact that a quantum "measurement" is more complicated than a macroscopic measurement, and is inevitably associated with an integral operation in the representation space, in contrast to the classical situation, negative values of the distribution function are possible at particular points of this space. The "representation distributions" of course do not give an entirely classical interpretation of quantum theory, but they provide a basis for that interpretation of the quantum theory which has maximum closeness to classical ideas and thus has the greatest physical-intuitive meaning.

1. We define the "representation distribution" by the following requirements.

1) The space in which it is defined has a classical meaning, for example, phase space or the space of directions.

2) The distribution can be expressed linearly in terms of the density matrix ρ. This requirement is directly related to the linearity of the whole apparatus of quantum theory, i.e., it is connected with the statistical interpretation of the theory. The density matrix, like any other operator A, has associated wtih it a function in the representation space

$$\rho(M) = \mathrm{Tr}\,\rho L(M); \quad A(M) = \mathrm{Tr}\,AL(M) \tag{1}$$

by means of the kernel $L(M)$, which is an operator depending on the point M as a parameter.

3) The distribution must be a real function; in general, to a Hermitian operator there must correspond a real function $A(M)$. This requirement amounts to the condition of Hermiticity of the operators $L(M)$ for all points M.

4) Statistical averaging of the classical functions $A(M)$ must give the same results as the rule for averaging of operators

$$\int A(M)\,\rho(M)\,dM = \mathrm{Tr}\,A\rho. \tag{2}$$

From Eqs. (1) and (2) there follows the formula

$$A = \int A(M)\,L(M)\,dM. \tag{3}$$

Thus the condition (4) is equivalent to the requirement that the direct and inverse transformations are accomplished by means of the same kernel.

Regarding the operators $A, B, \ldots$ as elements of a complex Euclidean space with scalar product—the trace $(A,B) = \mathrm{Tr}\,AB^*$ (B^* is the Hermitian adjoint to B)—we introduce an orthonormal basis $A_1, A_2, \ldots$ ($\mathrm{Tr}\,A_i A_j^* = \delta_{ij}$). The kernel $L(M)$ is represented by the expansion

$$L(M) = \sum_i A_i^* A_i(M), \quad (A_i(M) = \mathrm{Tr}\,A_i L(M)). \tag{4}$$

It is readily proved that the requirement (4) is equivalent to the condition

$$\int A_i(M)\,A_j^*(M)\,dM = \delta_{ij}. \tag{5}$$

According to Eq. (2) the normalization condition $\mathrm{Tr}\,\rho = 1$ takes the form

$$\int \rho(M)\,\lambda(M)\,dM = 1; \quad (\lambda(M) = \mathrm{Tr}L(M)). \tag{6}$$

Consequently, as the normalized density of the distribution we must take

$$w(M) = \rho(M)\,\lambda(M) \tag{7}$$

and the rule for averaging must be written in the form

$$\bar{A} = \int A(M)\,\lambda^{-1}(M)\,w(M)\,dM = \overline{A(M)\,\lambda^{-1}(M)}. \tag{8}$$

The definition presented above, together with simple transformation principles that give the connection with the specific physical case, can be applied to derive concrete distributions.

An important special case is the Wigner distribution in phase space

$$w(p, q) = h^{-n}\int e^{ivp/\hbar}\,\rho_{q+v/2,\ q-v/2}\,d^n v \tag{9}$$

(n is the number of degrees of freedom), which is frequently considered in connection with the statistical interpretation of quantum mechanics.[1] For this case $L(M)$ has the form

$$L_{x'x''} = \pm h^{-n/2}\,e^{i(x'-x'')p/\hbar}\,\delta\left(q - \frac{x'+x''}{2}\right). \tag{10}$$

The Wigner distribution can be derived from the definition of the "representation" distribution by applying the principles of homogeneity and equivalence of directions (invariance under translations and reflection).

2. Another example of a "representation" distribution is the peculiar distribution in the space of orientations of the spin, which corresponds to the spin degrees of freedom. We deal first with the nonrelativistic case of spin $s = 1/2$, with the spin variable taking two values and the corresponding operators being two-rowed matrices. For simplicity we shall not at once take into account the dependence of the elements of these matrices on other variables.

First of all we note the failure of Moyal's attempt[1] to generalize the Wigner distribution to the cases of the spin-variable type. In this case it is natural to take as the noncommuting operators r and s two components of the spin, for example s_y and s_z. Since the "possible values" of these operators comprise only the values $s_{yj} = \pm 1/2, s_{zh} = \pm 1/2$, the characteristic function

$$\sum_{j,h} F(s_{yj}, s_{zh}) \exp\{i(\tau s_{yj} + \theta s_{zh})\}$$

[$F(\pm 1/2, \pm 1/2)$ are probabilities] defined by Moyal must be periodic in τ and θ. On the other hand, the function

$$\mathrm{Tr}\exp\{i(\tau s_y + \theta s_z)\}\rho$$

is in general not periodic, and consequently these functions cannot be equated.

The cause for the failure of Moyal's approach consists in the fact that with a discrete set of characteristic values of the "basis" operators one must not restrict the distribution to these discrete values only, but must include the continuous spectrum in the treatment. Therefore, unlike Moyal, we shall consider a continuous manifold of possible values, taken two-dimensional as before. Moreover, we choose this manifold symmetrically with respect to the three coordinate axes. The direct physical meaning of the concept "spin" suggests that as such a manifold one should take the manifold on points of a sphere

$$s_x = {}^1/_2 \cos\varphi \sin\vartheta;$$

$$s_y = {}^1/_2 \sin\varphi \sin\vartheta; \quad s_z = {}^1/_2 \cos\vartheta,$$

or the manifold of directions of the spin, i.e., a manifold invariant under space rotations. Thus by a point M of the representation space we shall mean a point of the sphere, so that $M = (\varphi, \theta)$.

The Euclidean space of two-rowed matrices is four-dimensional. As basis elements we can take the orthonormal matrices

$$A_i = \sigma_i/\sqrt{2} \quad (i = 0, 1, 2, 3), \tag{11}$$

where σ_1, σ_2, σ_3 are the Pauli matrices and σ_0 is the unit matrix.

In consequence of Eq. (4) the choice of L is equivalent to the choice of the functions $A_i(M)$. In order to have a representation distribution, it is sufficient to choose them orthonormal, real, and with $A_0 = 0$. But there will be no room left for multiple values if we take account of the transformation principle, i.e., require that the correspondence of matrix to function be of a character invariant under rotations. Using Eqs. (1), (4), and (11), we have

$$\sqrt{2}\,\rho(M) = (\rho, \sigma_0)\,A_0(M) + \sum_{j=1}^{3}(\rho, \sigma_j)\,A_j(M). \tag{12}$$

Taking note of the fact that $\rho(M)$, like $w(M)$ must be a scalar function, we examine the transformation properties of the terms appearing in Eq. (12). According to the normalization condition (ρ, σ_0) is an invariant; consequently, $A_0(M)$ must also remain unchanged by rotations, i.e., must be constant over the whole sphere. Furthermore, as is well known, (ρ, σ_j), like $\psi^* \sigma_j \psi$, are the components of a three-dimensional vector (that it is axial is immaterial). From this it follows that the functions $A_j(M)$ also transform on rotations like the components of a vector. This means that they must be proportional to the functions

$$n_1(\varphi, \vartheta) = \cos\varphi \sin\vartheta; \quad n_2(\varphi, \vartheta) = \sin\varphi \sin\vartheta; \quad n_3(\varphi, \vartheta) = \cos\vartheta. \tag{13}$$

Taking into account the normalization condition, we have

$$A_0(M) = \pm 1/\sqrt{4\pi}; \quad \mathbf{A}(M) = \pm\sqrt{3/4\pi}\,\mathbf{n}(\varphi, \vartheta). \tag{14}$$

These functions satisfy all of the conditions: they are orthonormal and real, and $\lambda = \pm A_0 \operatorname{tr}\sigma_0/2^{1/2} = \pm 1/(2\pi)^{1/2} \neq 0$. With this choice the desired functions have, according to Eqs. (4) and (7), the form

$$L(\varphi, \vartheta) = \pm(8\pi)^{-1/2}[\sigma_0 + \sqrt{3}\,\sigma\mathbf{n}(\varphi, \vartheta)]; \tag{15}$$

$$w(\varphi, \vartheta) = (1/4\pi)[1 + \sqrt{3}\,\mathbf{n}(\varphi, \vartheta)\operatorname{tr}(\sigma\rho)]. \tag{16}$$

Let us consider the special case of a diagonal density matrix in the z-representation:

$$\rho = \| c_k\delta_{kl} \| = {}^1/_2\sigma_0 + {}^1/_2(c_1 - c_2)\sigma_3.$$

In this case, by Eq. (16),

$$w(\varphi, \vartheta) = (1/4\pi)[1 + \sqrt{3}(c_1 - c_2)\cos\vartheta]. \tag{17}$$

In the other special case of a "pure" state $\rho = \psi\psi^*$ we have

$$w(\varphi, \vartheta) = (1/4\pi)[1 + \sqrt{3}\,\mathbf{n}(\varphi, \vartheta)\psi^*\sigma\psi] = (1/4\pi)[1 + \sqrt{3}\cos\alpha], \tag{18}$$

where α is the angle between the direction considered and the average, most probable, direction of the spin.

3. An analogous method for the derivation of the representation distribution can be applied in the case of an arbitrary spin s. The wave function consists of a column of the $2s + 1$ components ψ^i $(i = -s, \ldots, s)$, which are the components of a contravariant irreducible s-tensor; the complex conjugate quantities are the components of the covariant tensor. The products $^{i*}\psi^j = \psi^* B^j_i \psi$ $(i, j = -s, \ldots, s)$ are therefore the components of a reducible tensor of the second rank, covariant in the index i and contravariant in j. Here B^j_i denotes the matrix with the elements

$$(B^j_i)_{kr} = \delta_{ki}\delta_{jr}. \tag{19}$$

According to the definition of the density matrix its elements $\rho^j_i = (B^j_i, \rho) = B^j_i$ have the same transformation properties. We go over from $\rho_i^{\ j}$ to the set of irreducible l-tensors $(l = 0, \ldots, 2s)$. This is done by the formula

$$\bar{A}_{lm} = \sum_{i,j} b^{si}_{lm;sj}\rho^j_i \quad (l = 0, \ldots, 2s; \; m = -l, \ldots, l). \tag{20}$$

Here $b^{si}_{lm;sj} = \rho_l\, c^{si}_{lm;sj}$; $c^{si}_{lm;sj}$ are any appropriate tabulated values, and ρ_l are numbers chosen not only from considerations of the unitary property of the entire transformation (20), but also in such a way that the $\bar{A}_{lm}$ form an l-tensor in the strict sense, i.e., transform covariantly with the spherical functions

$$Y_{lm}(\varphi, \vartheta) = (-1)^{\beta_m}\left(\frac{(2l+1)(l-|m|)}{4\pi(l+|m|)}\right)^{1/2} P_l^{|m|}(\cos\vartheta)\, e^{im\varphi}; \quad (\beta_{2\mu} = 0; \; \beta_{2\mu+1} = \mu).$$

Then the sums $\sum_{m=-l}^{l} Y_{lm}\bar{A}^*_{lm}$ $(l = 0, \ldots, 2s)$ are scalars. The scalar function $\rho(M)$, which by Eq. (4) is given by the expansion

$$\rho(M) = \sum_{l,m} A_{lm}(M)\bar{A}^*_{lm}, \tag{21}$$

must be the sum of the scalars in question.

From the orthonormal property of the matrices (19)

and the unitary nature of the transformation (20) there follows the orthonormal property of the basis

$$A_{lm} = \sum_{i,j} b^{si}_{lm,\,sj} B^j_i \quad (l = 0, \ldots, 2s; \quad m = -l, \ldots, l). \tag{22}$$

Since together with this the relations (5) must be satisfied, the functions $A_{lm}(M)$ must be taken in the form

$$A_{lm}(M) = \pm Y_{lm}(\varphi, \vartheta), \tag{23}$$

where for a particular value of l the sign must be the same for all values $m = -l, \ldots, l$. Consequently

$$L(M) = \sum_{l=0}^{2s} (\pm 1) \sum_{m=-l}^{l} Y_{lm}(\varphi, \vartheta) A^*_{lm}; \tag{24}$$

$$(\lambda = \pm Y_{00}\, \mathrm{tr}\, A^*_{00} = \pm \sqrt{(2s+1)/4\pi}).$$

The expression (24) satisfies all of the requirements, since it is Hermitian in virtue of the relations

$$Y^*_{lm} = Y^{lm} = (-1)^m Y_{l,-m}; \qquad A^*_{lm} = A^{lm} = (-1)^m A_{l,-m}. \tag{25}$$

One can convince oneself of the last of these relations by obtaining

$$b^{*si}_{lm,\,sj} = (-1)^m b^{sj}_{l,-m;\,si} = b^{lm;\,sj}_{si}$$

from Eq. (20), taking account of the facts that $\overline{A}^*_{lm} = \overline{A}^{lm}$ and $\rho^{i*}_j = \rho^i_j$. Thus by means of Eq. (24) we indeed obtain the representation distribution

$$w(\varphi, \vartheta) = \sqrt{\frac{2s+1}{4\pi}} \sum_{l=0}^{2s} (\pm 1) \times \sum_{m=-l}^{l} \sum_{i,j=-s}^{s} Y_{lm}(\varphi, \vartheta)\, b^{lm;\,sj}_{si} \rho^i_j. \tag{26}$$

The signs + and – in Eqs. (24) and (26) are readily determined from the physical meaning. For this it suffices to consider special forms of distributions, for example those having axial symmetry, i.e., not depending on φ.

In the particular case when $s = 1$, for axial symmetry we have from the general formula

$$w(\varphi, \vartheta) = \sqrt{3/4\pi}\,[\pm Y_{00}(A_{00}, \rho) \pm Y_{10}(A_{10}, \rho) \pm Y_{20}(A_{20}, \rho)] = (1/4\pi)\Big[1 \pm (3/\sqrt{2})(\rho^1_1 - \rho^{-1}_{-1}) \times \cos\vartheta \pm \frac{\sqrt{10}}{8}(\rho^1_1 + \rho^{-1}_{-1} - 2\rho^0_0)(3\cos 2\vartheta + 1)\Big]. \tag{27}$$

According to the physical meaning, in the case of the z-representation an increase of ρ^1_1 at the expense of ρ^{-1}_{-1} must lead to an increase of the probability of the positive z-orientation of the spin. Analogously, an increase of ρ^0_0 at the expense of $\rho^1_1 + \rho^{-1}_{-1}$ must lead to an increase of the probability of the equatorial orientation. From this we find that in both terms of the last equation we must take the sign +. For the "pure" state with positive z-orientation of the spin ($\rho^1_1 = 1$; $\rho^0_0 = \rho^{-1}_{-1} = 0$) we accordingly have

$$w(\varphi, \vartheta) = (1/4\pi)[1 + (3/\sqrt{2})\cos\vartheta + (\sqrt{10}/8)(3\cos 2\vartheta + 1)]. \tag{28}$$

This last formula gives a sharper maximum than Eq. (18). With increase of the spin it is natural to expect a tendency of the corresponding expressions toward a δ-like distribution.

4. Up to now we have considered the distribution in dynamical and spin variables separately. No difficulty is presented by the generalization to the case when the elements of the density matrix $\rho_{x,x'}$ depend on both types of variables: $x = (q, \sigma)$ (for conveneince we shall use a form of writing as if all variables were continuous).

As the basic operators of the extended operator space we may take the products $A_i A_{lm}$ of basis operators of the original bases. In the space of representation functions the products $A_i(p, q)$ $A_{lm}(\varphi, \theta)$ of the original basis functions will form an orthonormal system. The fulfillment of the other requirements for the extended system also follows from their fulfillment for the original systems. Therefore the combined representation distribution can be found by means of the kernel

$$L(p, q; \varphi, \vartheta) = L_1(p, q) L_2(\varphi, \vartheta); \qquad \lambda = \lambda_1 \lambda_2 = \pm h^{-n/2} \sqrt{(2s+1)/4\pi}. \tag{29}$$

For the same reason the same multiplicative law holds in thecase of N particles:

$$L(M_1, \ldots, M_N) = L_1(M_1) \ldots L_N(M_N); \quad \operatorname{Tr} L(M_1, \ldots, M_N) = \lambda^N. \tag{30}$$

Here M_i are the "representation" variables of the ith particle, for example $p_i, q_i, \varphi_i, \theta_i$, and $L_i(M_i) = [L(M_i)]_{x_i, x'_i}$.

The corresponding "distribution density"

$$w_N(M_1, \ldots, M_N) = \lambda^N (L_1(M_1) \ldots L_N(M_N), \rho_N) \tag{31}$$

can be understood in two ways. On one hand, (31) is the distribution in the N-fold representation space. On the other hand, $w_N(M_1, \ldots, M_N)$ can be understood as an N-fold "distribution density" in a one-fold space, while $M_1, \ldots, M_N$ are different points of this space. The second point of view has a great advantage over the first in the case of an unspecified (chance) number of particles.

5. In thecase of a chance number of particles it is convenient to make use of the apparatus of second quantization. For simplicity we shall restrict ourselves as before to the nonrelativistic case, for which the operator wave functions satisfy the commutation relations

$$\Psi(x)\Psi^*(x') - \Psi^*(x')\Psi(x) = \delta(x - x') \tag{32}$$

(for definiteness we shall confine ourselves to bosons). The state of the system is described by an operator for state $R = \mathrm{M}\,\Phi\Phi^*$. The vacuum state operator R_0 is defined by the equations

$$\Psi(x) R_0 \equiv 0; \quad R_0 \Psi^*(x) \equiv 0. \tag{33}$$

We consider the operators

$$(F_r)_{x_1 \ldots x_r; x'_1 \ldots x'_r} = \Psi^*(x'_r) \ldots \Psi^*(x'_1)\Psi(x_1) \ldots \Psi(x_r), \tag{34}$$

which, using the concept of an "ordered" N-product introduced by Wick,[2] , we can write

$$F_r = N(F_1 \ldots F_1). \tag{35}$$

Expanding them in terms of an orthonormal basis,

$$(r!\,s!)^{-1/2}\Psi^*(x'_s) \ldots \Psi^*(x'_1) R_0 \Psi(x_1) \ldots \Psi(x_r) = (r!\,s!)^{-1/2} E_{rs}(x_1 \ldots x_r; x'_1 \ldots x'_s) \tag{36}$$

by interchanges using Eq. (32) wefind

$$(F_r)_{x_1 \ldots x_r; x'_1 \ldots x'_r} = \sum_{k=0}^{\infty} \frac{1}{k!} \times \int \ldots \int (E_{r+k})_{x_1 \ldots x_r y_1 \ldots y_k; x'_1 \ldots x'_r y_1 \ldots y_k} dy_1 \ldots dy_k \quad (E_m \equiv E_{mm}). \tag{37}$$

From this, going over to the M-representation and writing

$$\lambda^r \bar{F}_r(M_1, \ldots, M_r) = f_r(M_1, \ldots, M_r); \quad \lambda^r \bar{E}_r(M_1, \ldots, M_r) = e_r(M_1, \ldots, M_r), \tag{38}$$

we obtain

$$f_r(M_1, \ldots, M_r) = \sum_{k=0}^{\infty} \frac{1}{k!} \int \ldots \int e_{r+k}(M_1, \ldots, M_{r+k}) \times dM_{r+1} \ldots dM_{r+k}. \tag{39}$$

The identical relation occurs in the theory of correlated random points,[3] if by f_r we understand the distribution functions defining the probability

$$dP = f_r(M_1, \ldots, M_r)\, dM_1 \ldots dM_r \tag{40}$$

for the occurrence of at least one particle (point) in each of theelementary regions (of volumes dM_i) near the points $M_1, \ldots, M_r$, and by e_r we understand the conditional probability functions giving the probability of the analogous event under the supplementary conditions that no further point falls in any other place ($e_N = N!w_N$ when the number N of particles is not a matter of chance). In the quantum case, to be sure, "probability" is not to be understood literally. But in some sense one can preserve the "probability" interpretation: in this connection the equation

$$\mathrm{Tr}\, R = \sum \frac{1}{N!} \int \ldots \int e_N (M_1, \ldots, M_N) \times dM_1 \ldots dM_N = 1 \tag{41}$$

will play the part of the normalization condition of the "probability" in the summation over the whole set of mutually exclusive events. The operator function in the representation space $\xi(M) = \lambda F_1(M)$ is the quantum analogue of the "stochastic density function" $\sum_i \delta(M - M_i)$.

The generating functional is

$$L[u(M)] = \left\langle \prod_j (1 + u(M_j)) \right\rangle = \sum_{r=0}^{\infty} \frac{1}{r!} \int \ldots \int f_r (M_1, \ldots, M_r)\, u(M_1) \ldots u(M_r) \times dM_1 \ldots dM_r \tag{42}$$

and the characteristic functional of the function $\xi(M)$ can be regarded as the average ϑ of the operator functionals

$$\mathbf{L}[u] = N(e^{(F_1 U)}); \quad \vartheta[u] = e^{i(F_1 U)}; \quad \left((F_1 U) = \mathrm{Tr}\, F_1 U = \int \xi(M)\, u(M)\, dM\right). \tag{43}$$

In contrast with the classical case, the functionals (43) are defined only over the class of functions $u(M)$ that can be represented in the form $\lambda^{-1}\,\mathrm{Tr} UL(M)$ by means of the operator U. The formula relating these functionals,

$$\mathbf{L}[u] = N(\vartheta[-iu]) \tag{44}$$

is the quantum analogue of the formula

$$L[u] = \vartheta[-i \ln(1 + u)]. \tag{45}$$

By means of the symbol N the relations (37) and their inverses can be written in the form

$$F_r = N(E_r e^{\mathrm{Tr}\, F_1}); \quad E_r = N(F_r e^{-\mathrm{Tr}\, F_1}), \quad \left(\mathrm{Tr}\, F_1 = \int \xi(M)\, dM\right). \tag{46}$$

If we compare Eq. (46) with the definition (36) of the operators E_r, we can obtain the following expression for the vacuum state operator:

$$R_0 = N(e^{-\mathrm{Tr}\, F_1}) = \mathbf{L}[-1]. \tag{47}$$

This result is connected with the formula of the theory of random points, according to which $L[-1]$ is equal to the probability that not a single point (particle) occurs.

6. The operator moments $\xi(M_1) \ldots \xi(M_r)$ of the operator density function $\xi(M)$ are expressible in terms of the operator distribution function. In contrast with the classical case, all that need be done to obtain these formulas is to carry out the "ordering" of the wave functions, using Eq. (32). As the result we obtain the following formula:

$$\lambda^r F_1(M_1) \ldots F_1(M_r) = \sum \mathrm{S} \int Q_2^{(\alpha_1)} Q_3^{(\alpha_2)} \ldots \lambda^s F_s(M_1'', \ldots, M_s'') \times dM_1'' \ldots dM_s'' \quad (s = \alpha_1 + \alpha_2 + \ldots). \tag{48}$$

Here the first summation Σ is taken over all possible ways of separating r into summands, $r = 1\alpha_1 + 2\alpha_2 + \ldots$ (α_1 are positive integers). In the second summation S the numbers $\alpha_1, \alpha_2, \ldots$ are fixed, and the sum is taken over all ways of distributing the arguments $M_1, \ldots, M_r$ into s groups ($s = \alpha_1 + \alpha_2 + \ldots$), there being α_1 groups with one argument, α_2 groups of two arguments, etc. The number of terms in the latter sum is $r!/\alpha_1!\,(1!)^{\alpha_1}\alpha_2!\,(2!)^{\alpha_2}\ldots$, and each term is of the form

$$\int Q_2(M_1', M_1'') \ldots Q_2(M_{\alpha_1}', M_{\alpha_1}'')\, Q_3(M_{\alpha_1+1}', M_{\alpha_1+2}', M_{\alpha_1+1}'') \ldots \ldots Q_3(M_{\alpha_1+2\alpha_2-1}', M_{\alpha_1+2\alpha_2}', M_{\alpha_1+\alpha_2}'') \ldots F_s(M_1'', \ldots, M_s'')\, dM_1'' \ldots dM_s'',$$

where $M_1', \ldots, M_r'$ is some rearrangment of the arguments $M_1, \ldots, M_r$ and

$$Q_m(M_1, \ldots, M_m) = \lambda^{m-2} \mathrm{Tr}(L(M_1) \ldots L(M_m)) \tag{49}$$

(the quantity whose trace is taken is a product in the sense of matrix multiplication; do not confuse with the case of Eq. (30), where $L_1(M_1)\ldots L_N(M_N)$ denotes an operator of a greater number of dimensions than $L(M)$]. Formulas of analogous structure are encountered in the theory of random functions and random points. The formulas (48) (for averages) go over into the formulas of the classical theory if we set the quantities $Q_m(M_1, \ldots M_m)$ equal to δ-functions,

$$\dot{Q}_m(M_1, \ldots, M_m) = \delta(M_1, \ldots, M_m) \qquad (50)$$

$$\equiv \delta(M_2 - M_1)\ldots\delta(M_m - M_1).$$

Although each function $Q_m(M_1, \ldots, M_m)$ is cyclically symmetrical, nevertheless in the general case it does not possess complete symmetry. This manifests itself in the fact that the operator moments (48), unlike the F_r, are not completely symmetrical. If from all the moments $\langle \xi(M_1)\ldots(M_r)\rangle$ we form the symmetrized moment, according to Eq. (48) it will be equal to

$$m^s_{(r)\xi}(M_1, \ldots, M_r) \qquad (51)$$

$$= \sum S \int \ldots Q^{s(\alpha_i)}_{i+1} \ldots f_k(M''_1, \ldots, M''_k)$$

$$\times dM''_1 \ldots dM''_k,$$

where Q^s_{i+1} is the function Q_{i+1} symmetrized in i arguments, and thus also fully symmetrized (because of the previously existing cyclic symmetry).

The difference between the $Q_m(M_1, \ldots, M_m)$ and δ-functions is equivalent to the presence of specifically quantum correlations. Generally speaking, in the quantum case f_k cannot be equal to every function. Thus, the function

$$f_k(M_1, \ldots, M_k) \qquad (52)$$

$$= \delta(M_1 - M^*)\ldots\delta(M_k - M^*),$$

corresponding to a Poisson distribution concentrated in the limit at a single point, cannot be among the possible ones. But if as a trial procedure we admit (52) and substitute it into (51), then $m^s_{(r)\xi}$ will be expressible in terms of $Q_{i+1}(M_1, \ldots, M^*)$ in just the way that the moments of an ordinary random function are expressible in terms of its correlation functions $k_{(r)\xi}(M_1, \ldots, M_r)$. Consequently, instead of the classical formula $k_{(r)\xi} = \delta(M_1, \ldots, M_2, M^*)$ the formula

$$k^s_{(r)\xi}(M_1, \ldots, M_r) \qquad (53)$$

$$= Q^s_{r+1}(M_1, \ldots, M_r, M^*),$$

will hold, from which it follows that $Q^s_{r+1}(M_1, \ldots M_{r+1})$ describes the effects of the quantum correlation of the $(r+1)$st order between the points $M_1, \ldots, M_r, M_{r+1}$. It can be maintained that the difference between quantum theory and classical theory is comprised in the difference between the functions Q_m and δ-functions.*

If the functions $a(M) = [(A, L, (M)]/\lambda$ and $b(M) = [B, L(M)]/\lambda$ are given in the representation space, then according to the rule for multiplication of operators one must take as the product of these functions in the quantum case not $a(M)\,b(M)$ but

$$(AB, L(M))/\lambda$$

$$= \int Q_3(M, M_1, M_2)\, a(M_1)\, b(M_2)\, dM_1\, dM_2.$$

The difference between this expression and the ordinary product again reduces to the difference between Q_3 and a δ-function.

In view of the special part played by the functions Q_m, we shall calculate them for the special cases considered above. For spinless particles, using the kernel (10) and taking the sign + to make it a definite case, we find.

$$Q_{2m+1} \qquad (54)$$

$$= (\pi\hbar)^{-2mn} \exp\left\{\frac{2i}{\hbar}\sum_{k>l}(-1)^{k+l}(p_k q_l - p_l q_k)\right\};$$

$$Q_{2m+2} = \delta(q_1 - q_2 + \ldots - q_{2m+2})\,\delta(p_1 - p_2$$

$$+ \ldots - p_{2m+2})\, Q_{2m+1}.$$

*The possibility of negative values of the distribution functions is due to this same feature.

In particular, along with the previously known equation

$$Q_2 = \delta(q_1 - q_2)\,\delta(p_1 - p_2),$$

we have

$$Q_3 = (\pi\hbar)^{-2n} e^{2i\Delta/\hbar}; \tag{55}$$

$$Q_3^S = (\pi\hbar)^{-2n}\cos(2\Delta/\hbar),$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix}.$$

For the spin kernel (15) ($s = \frac{1}{2}$) we find

$$Q_2 = (1/4\pi)(1 + 3\mathbf{n}_1\mathbf{n}_2); \tag{56}$$

$$Q_3 = {}^1/_4(2\pi)^{-3/2}(1 + 3\mathbf{n}_1\mathbf{n}_2 + 3\mathbf{n}_2\mathbf{n}_3 + 3\mathbf{n}_3\mathbf{n}_1 + 3\sqrt{3}\,i\mathbf{n}_1[\mathbf{n}_2\mathbf{n}_3]);$$

$$Q_3^S = {}^1/_4(2\pi)^{-3/2}(1 + 3\mathbf{n}_1\mathbf{n}_2 + 3\mathbf{n}_2\mathbf{n}_3 + 3\mathbf{n}_3\mathbf{n}_1).$$

Equations (54) and (55) are characterized by a lesser degree of quantum degeneracy (correlation) than Eq. (56), since Q_2, and also the reduced functions $\int Q_{r+1}\,dq_1 \ldots dq_r$ and $\int Q_{r+1}\,dp_1 \ldots dp_r$, have the "classical" form (are equal to δ-functions), which is not true of the functions with spin.

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On the Derivation of the Fokker-Planck Equation for a Plasma

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The Fokker-Planck equation for a many-component plasma is derived by the method of N. N. Bogoliubov, and the coefficients are calculated in explicit form.

THE Fokker-Planck equation is usually derived from Smoluchowski's equation for stochastic processes,[1] and thus the dependence of the coefficients in the Fokker-Planck equation on the law of interaction between the particles is left undetermined. For a plasma the Fokker-Planck equation can be obtained from a known kinetic equation of a form given by Landau.[2] In this case divergences appear for large and small distances, owing to the long-range nature of the Coulomb forces, so that in Ref. 2 the integrals are cut off at the limits of small and large distances.

The method of Bogoliubov[3] makes it possible to derive the Fokker-Planck equation on the basis of the mechanics of an assembly of molecules and to calculate the coefficients in explicit form for a given interaction law. In the case of a plasma the divergence of the Fokker-Planck coefficients at large distances is disposed of by cutting off at the Debye radius, which is not introduced from outside, as in Ref. 2, but follows automatically from Bogoliubov's method. In the present paper we give a derivation of the Fokker-Planck equation for a many-component plasma with uniform spatial distribution, and study the asymptotic cases of the behavior of plasma particles at large and small energies of motion.

We consider the plasma in a state of statistical equilibrium and investigate the behavior of a certain individual particle belonging to the plasma (or a foreign charged particle projected into the plasma). In the derivation of the equation for the distribution function of such a particle we assume that its in-