The Three Body Problem for Short Range Forces II

G. V. SKORNIAKOV

Leningrad Physico-Technical Institute, Acad. Sci. USSR

(Submitted to JETP editor October 31, 1955)
J. Exptl. Theoret. Phys. (USSR) 31, 1046-1054 (December, 1956)

An iteration method is proposed for finding the wave function and binding energy of a three body system for the case of short range forces. The method is applied to the ground state of H^3.

1. METHOD

IN Ref. 1, an integral equation was obtained for the Fourier-transformed wave function of three identical particles

$$F(\mathbf{k}_1, \mathbf{r}_{23}) = \int e^{-i\mathbf{k}_1\mathbf{s}_1}\Psi(\mathbf{s}_1, \mathbf{r}_{23})\, d\mathbf{s}_1,$$

which, for the case of a bound state, can be written in the form

$$F(\mathbf{k}_1, \mathbf{r}_{23}) = \int \frac{\exp\{-\sqrt{\alpha^2 + 3k_1^2/4}\,|\mathbf{r}_{23}-\mathbf{r}|\}}{|\mathbf{r}_{23}-\mathbf{r}|} V(r) F(\mathbf{k}_1\mathbf{r})\, d\mathbf{r} + 8\pi \int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2+\mathbf{k})}{\alpha^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} V(r) \times \cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2) F(\mathbf{k}, \mathbf{r})\, d\mathbf{r} \frac{d\mathbf{k}}{(2\pi)^3}. \quad (1)$$

Here

$$\mathbf{s}_1 = \mathbf{r}_1 - {}^1/_2(\mathbf{r}_2 + \mathbf{r}_3); \quad \mathbf{r}_{23} = \mathbf{r}_2 - \mathbf{r}_3$$

are the variables describing the relative motion;

$$V(r) = -MU(r)/4\pi\hbar^2;$$

$U(r)$ is the interaction potential between any pair of particles; $\alpha^2 = -ME/\hbar^2$, E is the energy of the system.

From Eq. (1) it follows that if the potential $V(r)$ has a finite range r_0, then to determine the wave function over all space it is sufficient to know $F(\mathbf{k}, \mathbf{r})$ for $r < r_0$. This makes possible the development of an iteration method for finding the eigenfunction F and the eigenvalue α for the case where the characteristic dimensions of the system far exceed the force range r_0.

We shall assume that $V(r)$ is practically zero for $r > r_0$, while for $r < r_0$, $V(r) \gg \alpha^2$, where $\alpha r_0 \ll 1$. The quantity $1/\alpha$ determines the effective linear dimensions of the system.

In addition, we shall assume that there is a single bound state for two particles with small binding energy ϵ (as for the deuteron). The wave function φ_0 of such a system satisfies the integral equation

$$\varphi_0(r) = \int \frac{\exp\{-\gamma|\mathbf{r}-\mathbf{r}'|\}}{|\mathbf{r}-\mathbf{r}'|} V(r')\varphi_0(r')\, d\mathbf{r}', \quad (2)$$

where $\gamma = \sqrt{M\varepsilon}/\hbar$,

where $\gamma r_0 \ll 1$.

It is not hard to see that with these assumptions concerning the structure of Eq. (1), the second term on the right is smaller than the first by a factor of order αr_0 when $r_{23} < r_0$. This enables us to apply to the solution of Eq. (1) a method of successive approximations analogous to the usual perturbation theory.

In fact, let us rewrite Eq. (1) in the following form:

$$F(\mathbf{k}_1, \mathbf{r}_{23}) = \int \frac{\exp\{-\gamma|\mathbf{r}_{23}-\mathbf{r}|\}}{|\mathbf{r}_{23}-\mathbf{r}|} V(r) F(\mathbf{k}_1, \mathbf{r})\, d\mathbf{r} + \int \frac{\exp\{-\sqrt{\alpha^2+3k_1^2/4}\,|\mathbf{r}_{23}-\mathbf{r}|\} - \exp\{-\gamma|\mathbf{r}_{23}-\mathbf{r}|\}}{|\mathbf{r}_{23}-\mathbf{r}|} \times V(r) F(\mathbf{k}_1, \mathbf{r})\, d\mathbf{r} \quad (3)$$
$$+ 8\pi \int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2+\mathbf{k})}{\alpha^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \times V(r)\cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2) F(\mathbf{k}, \mathbf{r})\, d\mathbf{r} \frac{d\mathbf{k}}{(2\pi)^3}.$$

From this expression it is clear that for $r_{23} < r_0$ the last two terms on the right are smaller than the first by a factor of order αr_0, since

$$\int V(r)\, d\mathbf{r} \sim r_0, \quad ^{1,3}$$

and the last two terms unlike the first do not contain a denominator of order r_0. In addition we must take into account the fact that in the integral over $\mathbf{k}$ the essential region of integration should be the region $k \lesssim \alpha$, since for large k the function $F(\mathbf{k}, \mathbf{r})$, $(r < r_0)$ decreases rapidly.

Therefore the solution of Eq. (1) when $r_{23} < r_0$ must coincide, to terms of order αr_0, with the solution of the equation

$$F^0(\mathbf{k}_1, \mathbf{r}_{23}) = \int \frac{\exp\{-\gamma |\mathbf{r}_{23} - \mathbf{r}|\}}{|\mathbf{r}_{23} - \mathbf{r}|} V(r) F^0(\mathbf{k}_1, \mathbf{r})\, d\mathbf{r}. \tag{4}$$

The solution of this equation, corresponding to an S-state of the system, will be

$$F^0(\mathbf{k}_1, \mathbf{r}_{23}) = \chi(k_1)\varphi_0(r_{23}), \tag{5}$$

where χ is an as yet unknown function and φ_0 is determined by Eq. (2).

Thus the wave function in zeroth approximation has the form (5) in the region $r_{23} < r_0$. Using this function, we can determine the zeroth approximation wave function $F_0(\mathbf{k}_1, \mathbf{r}_{23})$ over all space by substituting (5) into the right side of Eq. (1):

$$F_0(\mathbf{k}_1, \mathbf{r}_{23}) = \chi(k_1) \times \int \frac{\exp\{-\sqrt{\alpha_0^2 + 3k_1^2/4}\, |\mathbf{r}_{23} - \mathbf{r}|\}}{|\mathbf{r}_{23} - \mathbf{r}|} V(r)\varphi_0(r)\, d\mathbf{r} + 8\pi \int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2 + \mathbf{k})}{\alpha_0^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \times V(r)\cos(\mathbf{r}_1, \mathbf{k}_1 + \mathbf{k}/2)\varphi_0(r)\, d\mathbf{r}\frac{d\mathbf{k}}{(2\pi)^3}. \tag{6}$$

For $r_{23} < r_0$, the function $F_0(\mathbf{k}_1, \mathbf{r}_{23})$ thus determined must reduce (up to terms of order $\alpha_0 r_0$) to the product $\chi(k_1)\varphi_0(r_{23})$.

It is easy to show that the difference

$$F_0(\mathbf{k}_1, \mathbf{r}_{23}) - \chi(k_1)\varphi_0(r_{23})|_{r_{23}<r_0} = \Delta(\mathbf{k}_1, \mathbf{r}_{23}) \tag{7}$$
$$= \chi(k_1) \times \int \frac{\exp\{-\sqrt{\alpha_0^2 + 3k_1^2/4}\, |\mathbf{r}_{23} - \mathbf{r}|\} - \exp\{-\gamma|\mathbf{r}_{23} - \mathbf{r}|\}}{|\mathbf{r}_{23} - \mathbf{r}|} \times V(r)\varphi_0(r)\, d\mathbf{r} + 8\pi \int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2 + \mathbf{k})}{\alpha_0^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \times V(r)\cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2)\varphi_0(r)\, d\mathbf{r}\frac{d\mathbf{k}}{(2\pi)^3}$$

when expanded in powers of $(\alpha_0 r_0)$, contains a term proportional to $(\alpha_0 r_0)$

$$\left\{(\gamma - \sqrt{\alpha_0^2 + 3k_1^2/4})\chi(k_1) + 8\pi \int \frac{\chi(k)}{\alpha_0^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \frac{d\mathbf{k}}{(2\pi)^3}\right\} \int V(r)\varphi_0(r)\, d\mathbf{r}, \tag{7a}$$

which does not depend on $\mathbf{r}_{23}$, and also terms proportional to higher powers of $(\alpha_0 r_0)$. $\chi(k_1)$ and α_0 (the eigenvalue in zeroth approximation) are as yet undetermined. To determine them we demand that, for $r_{23} < r_0$, the difference

$$\Delta(\mathbf{k}_1, \mathbf{r}_{23}) = F_0(\mathbf{k}_1, \mathbf{r}_{23}) - \chi(k_1)\varphi_0(r_{23})$$

shall vanish on the average:

$$\int V(r_{23})\varphi_0(r_{23})\Delta(\mathbf{k}_1, \mathbf{r}_{23})\, d\mathbf{r}_{23} = 0. \tag{8}$$

Since for $r_{23} < r_0$ the term in $\Delta(\mathbf{k}_1, \mathbf{r}_{23})$ proportional to $(\alpha_0 r_0)$ does not depend on $\mathbf{r}_{23}$, and can be taken out from under the integral sign, this requirement means that when $r_{23} < r_0$ the function $F_0(\mathbf{k}_1, \mathbf{r}_{23})$ reproduces itself up to terms of order $(\alpha_0 r_0)$ inclusive. Since the function $F(\mathbf{k}_1, \mathbf{r}_{23})$ is determined over all space by $F(\mathbf{k}_1, \mathbf{r}_{23})$ for $r_{23} < r_0$, the function $F_0(\mathbf{k}_1, \mathbf{r}_{23})$ determined in this way will satisfy the integral Eq. (1) to this same degree of accuracy.

However, we cannot simply equate to zero the term (7a) in the difference $\Delta(\mathbf{k}_1, \mathbf{r}_{23})$, which is proportional to $(\alpha_0 r_0)$, since such a requirement leads to the equation for $\chi(k_1)$ which was obtained[1] on the assumption of zero range of the forces, in which case it is known[2] that a bound state of three particles cannot exist.

It is therefore necessary even when we determine the zeroth approximation to the eigenfunction and eigenvalue, to treat the force range r_0 as finite; we then effectively include higher terms of the expansion of $\Delta(\mathbf{k}_1, \mathbf{r}_{23})$ in powers of r_0.

The condition (8) gives an integral equation for determining $\chi(k_1)$ and α_0:

$$\chi(k_1) \times \int \frac{\exp\{-\sqrt{\alpha_0^2 + 3k_1^2/4}\, |\mathbf{r}_{23} - \mathbf{r}|\} - \exp\{-\gamma|\mathbf{r}_{23} - \mathbf{r}|\}}{|\mathbf{r}_{23} - \mathbf{r}|} \times V(r_{23})V(r)\varphi_0(r_{23})\varphi_0(r)\, d\mathbf{r}d\mathbf{r}_{23} + 8\pi \int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2 + \mathbf{k})\chi(k)}{\alpha_0^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \times \cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2) V(r_{23})V(r)\varphi_0(r_{23})\varphi_0(r)\, d\mathbf{r}d\mathbf{r}_{23}\frac{d\mathbf{k}}{(2\pi)^3} = 0. \tag{9}$$

We note that for k_1, k small the kernel of the integral equation (9) does not depend on the specific form of the potential, while for large k_1, k, because of the cosines in the integral, the kernel decreases rapidly. The manner in which the kernel decreases also depends very little on the specific form of the potential and is determined only by the effective range of the forces.

According to formula (6), the determination of the eigenfunction $\chi(k_1)$ and eigenvalue α_0 of Eq. (9) gives us the zeroth approximation to the eigenfunction $F(\mathbf{k}_1, \mathbf{r}_{23})$. α_0 is the zeroth approximation to the eigenvalue α. Thus the solution of the multidimensional Eq. (1) reduces in zeroth approximation to the solution of the one-dimensional integral Eq. (9) and the quadriture in (6).

In finding the higher approximations, we can make use of the following iteration method. The higher approximations to the eigenvalue α will be determined from the equation

$$\int |F_{n-1}(\mathbf{k}_1, \mathbf{r}_{23})|^2 d\mathbf{k}_1 d\mathbf{r}_{23} = \int F^*_{n-1}(\mathbf{k}_1, \mathbf{r}_{23}) \Bigg\{ \int \frac{\exp\left\{-\sqrt{\alpha_n^2 + 3k_1^2/4}\,|\mathbf{r}_{23} - \mathbf{r}|\right\}}{|\mathbf{r}_{23} - \mathbf{r}|} \times V(r) F_{n-1}(\mathbf{k}_1, \mathbf{r})\, d\mathbf{r} + 8\pi \int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2 + \mathbf{k})}{\alpha_n^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2) V(r) \times F_{n-1}(\mathbf{k}, \mathbf{r})\, d\mathbf{r} \frac{d\mathbf{k}}{(2\pi)^3} \Bigg\} d\mathbf{k}_1 d\mathbf{r}_{23}, \tag{10}$$

where α_n is the nth approximation to the eigenvalue, and F_{n-1} is the $(n-1)$st approximation to the eigenfunction.

It is easy to see that this determination of α_n (and consequently of the energy of the system in nth approximation) is equivalent to computing the matrix element of the Hamiltonian using the wave function of the $(n-1)$st approximation.*

After having determined the nth approximation to the eigenvalue α, by substituting the $(n-1)$st approximation to the eigenfunctiom and the nth approximation to the eigenvalue on the right of Eq. (1), we get the nth approximation to the eigenfunction

$$F_n(\mathbf{k}_1, \mathbf{r}_{23}) = \int \frac{\exp\left\{-\sqrt{\alpha_n^2 + 3k_1^2/4}\,|\mathbf{r}_{23} - \mathbf{r}|\right\}}{|\mathbf{r}_{23} - \mathbf{r}|} \times V(r) F_{n-1}(\mathbf{k}_1, \mathbf{r})\, d\mathbf{r} + 8\pi \int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2 + \mathbf{k})}{\alpha_n^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \times \cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2) V(r) F_{n-1}(\mathbf{k}, \mathbf{r})\, d\mathbf{r} \frac{d\mathbf{k}}{(2\pi)^3}. \tag{11}$$

Thus each successive approximation is obtained from the preceding one by a multiple integration. If $\alpha_0 r_0 \ll 1$, we can limit ourselves to the zeroth approximation for a practical solution of the problem.

2. APPLICATION TO H^3

The method we have presented is easily generalized to the case of a system of three nucleons (H^3). The operator for the interaction of the nucleons in the H^3 nucleus will be assumed to be a sum of interaction potentials between pairs of nucleons.

In addition we shall assume that the interaction potential between a pair of nucleons is central and that the nuclear forces are charge independent. The most general expression for such a potential is

$$\hat{U}_{ik} = U_1(|\mathbf{r}_i - \mathbf{r}_k|) + (\boldsymbol{\sigma}_i\boldsymbol{\sigma}_k) U_2(|\mathbf{r}_i - \mathbf{r}_k|) + (\boldsymbol{\tau}_i\boldsymbol{\tau}_k) U_3(|\mathbf{r}_i - \mathbf{r}_k|) + (\boldsymbol{\sigma}_i\boldsymbol{\sigma}_k)(\boldsymbol{\tau}_i\boldsymbol{\tau}_k) U_4(|\mathbf{r}_i - \mathbf{r}_k|). \tag{12}$$

Under these assumptions concerning the nuclear forces, the spin S and isotopic spin T of the three-nucleon system are conserved. In the ground state of H^3, $S = T = \frac{1}{2}$. The wave function of such a state can be written as[1,4]

$$\Psi_{1/2,1/2} = \Psi_a\Phi_c - \Psi_s\Phi_a + \Psi_1\Phi_2 - \Psi_2\Phi_1.$$

The spin functions Φ were determined in Ref. 1, which also gives a detailed description of the transformation properties of the space functions Ψ under permutations.

Using the Green's function[1]

* It can be shown that the difference between the binding energies ϵ_0 and ϵ_1, calculated in zeroth and first approximations, respectively, is a quantity of order $\epsilon_0(\alpha_0 r_0)^2$.

$$G=\int\frac{\exp\{-\sqrt{\alpha^2+3k^2/4}\;|\mathbf{r}-\mathbf{r}'|\}}{|\mathbf{r}-\mathbf{r}'|}\,e^{i\mathbf{k}(\mathbf{s}-\mathbf{s}')}\frac{d\mathbf{k}}{(2\pi)^3}\tag{13}$$

we write the Schrodinger equation in integral form:

$$\Psi_{1/2,\,1/2}=\int G\,\{\hat{V}_{12}+\hat{V}_{13}+\hat{V}_{23}\}\,\Psi_{1/2,\,1/2}d\tau,\tag{14}$$

$$\hat{V}_{ik}=-\,(M/4\pi h^2)\,\hat{U}_{ik}.$$

If we multiply Eq. (14) by $\Phi^*_\alpha(\alpha = s,\ a,\ 1,\ 2)$ and sum over the spin variables, then by making use of the transformation properties of the space wave function under permutations we find as the result of very messy but (in principle) simple calculations, the following system of integral equations for the Fourier transforms

$$f_\alpha(\mathbf{k}_1,\mathbf{r}_{23})=\int e^{-i\mathbf{k}_1\mathbf{s}_1}\,\Psi_\alpha(\mathbf{s}_1,\ \mathbf{r}_{23})\,d\mathbf{s}_1,$$

of the space functions:

$$\begin{aligned}
f_s(\mathbf{k}_1,\mathbf{r}_{23})=&\int\frac{\exp\left\{-\sqrt{\alpha^2+3k_1^2/4}\,|\mathbf{r}_{23}-\mathbf{r}|\right\}}{|\mathbf{r}_{23}-\mathbf{r}|}\left\{\frac{1}{2}[V_t(r)+V_s(r)]\,f_s(\mathbf{k}_1,\mathbf{r})\right.\\
&\left.+\frac{1}{2}[V_s(r)-V_t(r)]\,f_2(\mathbf{k}_1,\mathbf{r})\right\}d\mathbf{r}+8\pi\int\frac{\cos(\mathbf{r}_{23},\,\mathbf{k}_1/2+\mathbf{k})}{\alpha^2+k_1^2+k^2+\mathbf{k}_1\mathbf{k}}\cos(\mathbf{r},\,\mathbf{k}_1+\mathbf{k}/2)\\
&\times\left\{\frac{1}{2}[V_t(r)+V_s(r)]\,f_s(\mathbf{k},\,\mathbf{r})+\frac{1}{2}[V_s(r)-V_t(r)]\,f_2(\mathbf{k},\mathbf{r})\right\}d\mathbf{r}\,\frac{d\mathbf{k}}{(2\pi)^3};\\
f_a(\mathbf{k}_1,\mathbf{r}_{23})=&\int\frac{\exp\left\{-\sqrt{\alpha^2+3k_1^2/4}\,|\mathbf{r}_{23}-\mathbf{r}|\right\}}{|\mathbf{r}_{23}-\mathbf{r}|}\left\{\frac{1}{2}[V_q(r)+V_r(r)]\,f_a(\mathbf{k}_1,\mathbf{r})\right.\\
&\left.+\frac{1}{2}[V_q(r)-V_r(r)]\,f_1(\mathbf{k}_1,\mathbf{r})\right\}d\mathbf{r}-8\pi\int\frac{\sin(\mathbf{r}_{23},\,\mathbf{k}_1/2+\mathbf{k})}{\alpha^2+k_1^2+k^2+\mathbf{k}_1\mathbf{k}}\sin(\mathbf{r},\,\mathbf{k}_1+\mathbf{k}/2)\\
&\times\left\{\frac{1}{2}[V_q(r)+V_r(r)]\,f_a(\mathbf{k},\mathbf{r})+\frac{1}{2}[V_q(r)-V_r(r)]\,f_1(\mathbf{k},\mathbf{r})\right\}d\mathbf{r}\,\frac{d\mathbf{k}}{(2\pi)^3};\\
f_2(\mathbf{k}_1,\mathbf{r}_{23})=&\int\frac{\exp\left\{-\sqrt{\alpha^2+3k_1^2/4}\,|\mathbf{r}_{23}-\mathbf{r}|\right\}}{|\mathbf{r}_{23}-\mathbf{r}|}\left\{\frac{1}{2}[V_s(r)-V_t(r)]\,f_s(\mathbf{k}_1,\mathbf{r})\right.\\
&\left.+\frac{1}{2}[V_t(r)+V_s(r)]\,f_2(\mathbf{k}_1,\mathbf{r})\right\}d\mathbf{r}+8\pi\int\frac{\cos(\mathbf{r}_{23},\mathbf{k}_1/2+\mathbf{k})}{\alpha^2+k_1^2+k^2+\mathbf{k}_1\mathbf{k}}\left\{\cos(\mathbf{r},\,\mathbf{k}_1+\mathbf{k}/2)\right.\\
&\times\left[\frac{1}{4}(V_t(r)-V_s(r))\,f_s(\mathbf{k},\mathbf{r})-\frac{1}{4}(V_t(r)+V_s(r))\,f_2(\mathbf{k},\mathbf{r})\right]\\
&-i\sin(\mathbf{r},\mathbf{k}_1+\mathbf{k}/2)\left[\frac{\sqrt{3}}{4}(V_r(r)-V_q(r))\,f_a(\mathbf{k},\mathbf{r})-\frac{\sqrt{3}}{4}(V_r(r)+V_q(r))\right.\\
&\left.\left.\times\ f_1(\mathbf{k},\mathbf{r})\right]\right\}d\mathbf{r}\,\frac{d\mathbf{k}}{(2\pi)^3};\\
f_1(\mathbf{k}_1,\mathbf{r}_{23})=&\int\frac{\exp\{-\sqrt{\alpha^2+3k_1^2/4}\,|\mathbf{r}_{23}-\mathbf{r}|\}}{|\mathbf{r}_{23}-\mathbf{r}|}\left\{\frac{1}{2}[V_q(r)-V_r(r)]\,f_a(\mathbf{k}_1,\mathbf{r})\right.\\
&\left.+\frac{1}{2}[V_q(r)+V_r(r)]\,f_1(\mathbf{k}_1,\mathbf{r})\right\}d\mathbf{r}+8\pi i\int\frac{\sin(\mathbf{r}_{23},\,\mathbf{k}_1/2+\mathbf{k})}{\alpha^2+k_1^2+k^2+\mathbf{k}_1\mathbf{k}}\left\{\cos(\mathbf{r},\,\mathbf{k}_1+\mathbf{k}/2)\right.\\
&\times\left[\frac{\sqrt{3}}{4}(V_t(r)-V_s(r))\,f_s(\mathbf{k},\mathbf{r})-\frac{\sqrt{3}}{4}(V_t(r)+V_s(r))\,f_2(\mathbf{k},\mathbf{r})\right]\\
&-i\sin(\mathbf{r},\,\mathbf{k}_1+\mathbf{k}/2)\left[\frac{1}{4}(V_q(r)-V_r(r))\,f_a(\mathbf{k},\mathbf{r})\right.\\
&\left.\left.+\frac{1}{4}(V_q(r)+V_r(r))\,f_1(\mathbf{k},\mathbf{r})\right]\right\}d\mathbf{r}\,\frac{d\mathbf{k}}{(2\pi)^3}.
\end{aligned}\tag{15}$$

Equation (15) is the generalization of Eq. (1) to the case of three nucleons. The potentials V_t, V_s, V_r, and V_q are directly related to processes occurring in the two-nucleon system: the potential V_t is responsible for the existence of the deuteron as well as for the scattering of two nucleons in a state of spin 1 and even orbital angular momentum; the potential V_s causes the scattering of nucleons with spin 0 and even orbital angular momentum, and V_q the scattering for spin 0 and odd orbital angular momentum.

3. METHOD OF APPROXIMATE SOLUTION OF THE EQUATION

For an approximate solution of the system (15), we shall make use of the fact that the potentials V_t, V_s, V_r and V_q are short-range, and that the range of the nuclear forces, r_0, can be regarded as approximately the same for all four potentials in making rough estimates. From the transformation properties of the wave functions in coordinate representation it follows that $\Psi_s(\mathbf{s}_1, \mathbf{r}_{23})$ and $\Psi_2(\mathbf{s}_1 \mathbf{r}_{23})$ are symmetric with respect to the permutation P_{23}, while $\Psi_a(\mathbf{s}_1, \mathbf{r}_{23})$ and $\psi_1(\mathbf{s}_1, \mathbf{r}_{23})$ are antisymmetric with respect to this permutation.[1] Since the Fourier transformation of the wave function does not affect the variable $\mathbf{r}_{23}$, the functions f_α have these same properties.

Using the antisymmetry of the functions $f_a(\mathbf{k}_1, \mathbf{r}_{23})$ and $f_1(\mathbf{k}_1, \mathbf{r}_{23})$ with respect to a change in sign of $\mathbf{r}_{23}$, it is not difficult to show that, to terms of order $(\alpha r_0)^3$, the system of integral equations (15) reduces to the following:

$$f_s(\mathbf{k}_1, \mathbf{r}_{23}) = \int \frac{\exp\left\{-\sqrt{\alpha^2 + 3k_1^2/4}\,|\mathbf{r}_{23} - \mathbf{r}|\right\}}{|\mathbf{r}_{23} - \mathbf{r}|} \left\{\frac{1}{2}[V_t(r) + V_s(r)] f_s(\mathbf{k}_1, \mathbf{r}) + \frac{1}{2}[V_s(r) - V_t(r)] f_2(\mathbf{k}_1, \mathbf{r})\right\} d\mathbf{r} + 8\pi \int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2 + \mathbf{k})}{\alpha^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2) \times \left\{\frac{1}{2}[V_t(r) + V_s(r)] f_s(\mathbf{k}, \mathbf{r}) + \frac{1}{2}[V_s(r) - V_t(r)] f_2(\mathbf{k}, \mathbf{r})\right\} d\mathbf{r} \frac{d\mathbf{k}}{(2\pi)^3}; \quad f_a(\mathbf{k}_1, \mathbf{r}_{23}) \equiv 0; \tag{16}$$

$$f_2(\mathbf{k}_1, \mathbf{r}_{23}) = \int \frac{\exp\left\{-\sqrt{\alpha^2 + 3k_1^2/4}\,|\mathbf{r}_{23} - \mathbf{r}|\right\}}{|\mathbf{r}_{23} - \mathbf{r}|} \left\{\frac{1}{2}[V_s(r) - V_t(r)] f_s(\mathbf{k}_1, \mathbf{r}) + \frac{1}{2}[V_t(r) + V_s(r)] f_2(\mathbf{k}_1, \mathbf{r})\right\} d\mathbf{r} + 8\pi \int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2 + \mathbf{k})}{\alpha^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2) \times \left\{\frac{1}{4}[V_t(r) - V_s(r)] f_s(\mathbf{k}, \mathbf{r}) - \frac{1}{4}[V_t(r) + V_s(r)] f_2(\mathbf{k}, \mathbf{r})\right\} d\mathbf{r} \frac{d\mathbf{k}}{(2\pi)^3};$$

$$f_1(\mathbf{k}_1, \mathbf{r}_{23}) = 8\pi i \int \frac{\sin(\mathbf{r}_{23}, \mathbf{k}_1/2 + \mathbf{k})}{\alpha^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2) \left\{\frac{\sqrt{3}}{4}[V_t(r) - V_s(r)] \times f_s(\mathbf{k}, \mathbf{r}) - \frac{\sqrt{3}}{4}[V_t(r) + V_s(r)] f_2(\mathbf{k}, \mathbf{r})\right\} d\mathbf{r} \frac{d\mathbf{k}}{(2\pi)^3}.$$

Thus in this approximation we have obtained a system of two integral equations for determining the functions f_s and f_2, while f_1 is determined from f_s and f_2 by integration. It is an essential point that only the two potentials $V_t(r)$ and $V_s(r)$ appear in (16), while it is precisely these potentials which determine all processes in a two-nucleon system for low energies of relative motion.

It is convenient to introduce in place of the functions f_s and f_2 the functions

$$\eta_1 = f_s + f_2; \quad \eta_2 = f_s - f_2, \tag{17}$$

for whose determination we obtain the following systems of integral equations:

$$\eta_1(\mathbf{k}_1, \mathbf{r}_{23}) = \int \frac{\exp\left\{-\sqrt{\alpha^2 + 3k_1^2/4}\,|\mathbf{r}_{23} - \mathbf{r}|\right\}}{|\mathbf{r}_{23} - \mathbf{r}|} \times V_s(r)\,\eta_1(\mathbf{k}_1, \mathbf{r})\, d\mathbf{r} + 8\pi \int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2 + \mathbf{k})}{\alpha^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \times \cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2) \left\{\frac{1}{4} V_s(r)\,\eta_1(\mathbf{k}, \mathbf{r}) + \frac{3}{4} V_t(r)\,\eta_2(\mathbf{k}, \mathbf{r})\right\} d\mathbf{r} \frac{d\mathbf{k}}{(2\pi)^3}; \tag{18}$$

$$\eta_2(\mathbf{k}_1, \mathbf{r}_{23}) = \int \frac{\exp\left\{-\sqrt{\alpha^2 + 3k_1^2/4}\,|\mathbf{r}_{23} - \mathbf{r}|\right\}}{|\mathbf{r}_{23} - \mathbf{r}|} V_t(r)\,\eta_2(\mathbf{k}_1, \mathbf{r})\, d\mathbf{r} + 8\pi \int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2 + \mathbf{k})}{\alpha^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2) \times \left\{\frac{3}{4} V_s(r)\,\eta_1(\mathbf{k}, \mathbf{r}) + \frac{1}{4} V_t(r)\,\eta_2(\mathbf{k}, \mathbf{r})\right\} d\mathbf{r} \frac{d\mathbf{k}}{(2\pi)^3}.$$

To solve the equation system (18) it is necessary to know the functions only within the range of the nuclear forces. In complete analogy to the case of three identical particles, we may assume approximately that, within the range of action of the nuclear forces, the wave functions η_1 and η_2 can be represented in the form of products

$$\eta_1(\mathbf{k}_1, \mathbf{r}_{23}) \approx \chi_1(k_1)\,\varphi_1(r_{23}); \qquad \eta_2(\mathbf{k}_1, \mathbf{r}_{23}) \approx \chi_2(k_1)\,\varphi_2(r_{23}) \tag{19}$$

for $r_{23} < r_s$ and $r_{23} < r_t$, respectively; φ_1 and φ_2 are the wave functions within the range of nuclear forces for two nucleons in singlet and triplet states.

The wave functions φ_1 and φ_2 satisfy the following integral equations:

$$\varphi_1(r_{23}) = \int \frac{\exp\{-\alpha_s |\mathbf{r}_{23} - \mathbf{r}|\}}{|\mathbf{r}_{23} - \mathbf{r}|} V_s(r)\,\varphi_1(r)\,d\mathbf{r}, \tag{20a}$$

$$\varphi_2(r_{23}) = \int \frac{\exp\{-\alpha_t |\mathbf{r}_{23} - \mathbf{r}|\}}{|\mathbf{r}_{23} - \mathbf{r}|} V_t(r)\,\varphi_2(r)\,d\mathbf{r}, \tag{20b}$$

where α_s and α_t are defined in terms of the logarithmic derivatives at the interaction radius*:

$$\alpha_s = -\frac{d}{dr}\ln\{r\varphi_1(r)\}_{r=r_s}; \qquad \alpha_t = -\frac{d}{dr}\ln\{r\varphi_2(r)\}_{r=r_t}. \tag{21}$$

From here on the procedure is completely analogous to the case of three identical particles.

Using the approximate expressions (19) for the wave functions within the range of the nuclear forces, we construct the zeroth approximation wave function:

$$\begin{aligned}
\eta_1^0(\mathbf{k}_1, \mathbf{r}_{23}) &= \chi_1(k_1)\int \frac{\exp\left\{-\sqrt{\alpha_0^2 + 3k_1^2/4}\,|\mathbf{r}_{23} - \mathbf{r}|\right\}}{|\mathbf{r}_{23} - \mathbf{r}|} V_s(r)\,\varphi_1(r)\,d\mathbf{r} \\
&+ 2\pi\int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2 + \mathbf{k})}{\alpha^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2)\{\chi_1(k) V_s(r)\,\varphi_1(r) \\
&+ 3\chi_2(k) V_t(r)\,\varphi_2(r)\}\,d\mathbf{r}\,\frac{d\mathbf{k}}{(2\pi)^3}; \\
\eta_2^0(\mathbf{k}_1, \mathbf{r}_{23}) &= \chi_2(k_1)\int \frac{\exp\left\{-\sqrt{\alpha_0^2 + 3k_1^2/4}\,|\mathbf{r}_{23} - \mathbf{r}|\right\}}{|\mathbf{r}_{23} - \mathbf{r}|} V_t(r)\,\varphi_2(r)\,d\mathbf{r} \\
&+ 2\pi\int \frac{\cos(\mathbf{r}_{23}, \mathbf{k}_1/2 + \mathbf{k})}{\alpha^2 + k_1^2 + k^2 + \mathbf{k}_1\mathbf{k}} \cos(\mathbf{r}, \mathbf{k}_1 + \mathbf{k}/2)\{3\chi_1(k) V_s(r)\,\varphi_1(r) \\
&+ \chi_2(k) V_t(r)\,\varphi_2(r)\}\,d\mathbf{r}\,\frac{d\mathbf{k}}{(2\pi)^3}.
\end{aligned} \tag{22}$$

As before, we determine χ_1, χ_2, and α_0 so that the differences $\eta_1^0(\mathbf{k}_1, \mathbf{r}_{23}) - \chi_1(k_1)\,\varphi_1(r_{23})$ and $\eta_2^0(\mathbf{k}_1, \mathbf{r}_{23}) - \chi_2(k_1)\,\varphi_2(r_{23})$ vanish on the average within the range of the forces:

$$\begin{aligned}
&\int V_s(r_{23})\,\varphi_1(r_{23})\{\eta_1^0(\mathbf{k}_1, \mathbf{r}_{23}) - \chi_1(k_1)\,\varphi_1(r_{23})\}\,d\mathbf{r}_{23} = 0, \\
&\int V_t(r_{23})\,\varphi_2(r_{23})\{\eta_2^0(\mathbf{k}_1, \mathbf{r}_{23}) - \chi_2(k_1)\,\varphi_2(r_{23})\}\,d\mathbf{r}_{23} = 0.
\end{aligned} \tag{23}$$

These conditions give a system of two integral equations for the determination of χ_1, χ_2, and α_0:

* From experiments on neutron scattering in ortho- and parahydrogen, we know that α_s is negative. Therefore for $r_{23} \to \infty$, equation (20a) has no physical meaning, since the wave function $\varphi_1(r_{23})$ then tends to infinity. However, it is not difficult to show that for small $r_{23} < r_s$, the function satisfying Eq. (20a) coincides with the wave function for two nucleons which are in a singlet state with low energy of relative motion.

$$\chi_1(k_1)\int\frac{\exp\left\{-\sqrt{\alpha_0^2+3k_1^2/4}\,|\mathbf{r}_{23}-\mathbf{r}|\right\}-\exp\{-\alpha_s\,|\mathbf{r}_{23}-\mathbf{r}|\}}{|\mathbf{r}_{23}-\mathbf{r}|}$$
$$\times V_s(r_{23})V_s(r)\varphi_1(r_{23})\varphi_1(r)\,d\mathbf{r}_{23}d\mathbf{r}$$
$$+2\pi\int\frac{\cos(\mathbf{r}_{23},\mathbf{k}_1/2+\mathbf{k})}{\alpha_0^2+k_1^2+k^2+\mathbf{k}_1\mathbf{k}}V_s(r_{23})\varphi_1(r_{23})\cos(r,\mathbf{k}_1+\mathbf{k}/2)\{\chi_1(k)V_s(r)\varphi_1(r)$$
$$+3\chi_2(k)V_t(r)\varphi_2(r)\}\,d\mathbf{r}d\mathbf{r}_{23}\frac{d\mathbf{k}}{(2\pi)^3}=0; \tag{24}$$

$$\chi_2(k_1)\int\frac{\exp\left\{-\sqrt{\alpha_0^2+3k_1^2/4}\,|\mathbf{r}_{23}-\mathbf{r}|\right\}-\exp\{-\alpha_t|\mathbf{r}_{23}-\mathbf{r}|\}}{|\mathbf{r}_{23}-\mathbf{r}|}$$
$$\times V_t(r_{23})V_t(r)\varphi_2(r_{23})\varphi_2(r)\,d\mathbf{r}_{23}d\mathbf{r}$$
$$+2\pi\int\frac{\cos(\mathbf{r}_{23},\mathbf{k}_1/2+\mathbf{k})}{\alpha_0^2+k_1^2+k^2+\mathbf{k}_1\mathbf{k}}V_t(r_{23})\varphi_2(r_{23})\cos(\mathbf{r},\mathbf{k}_1+\mathbf{k}/2)\{3\chi_1(k)V_s(r)\varphi_1(r)$$
$$+\chi_2(k)V_t(r)\varphi_2(r)\}\,d\mathbf{r}d\mathbf{r}_{23}d\mathbf{k}\,|\,(2\pi)^3=0.$$

Thus in this case we obtain in zero approximation a system of two integral equations for two functions which depend on a single variable. The wave functions f_α^0 of the zeroth approximation are determined, according to Eqs. (16) and (17), from the eigenvalue α_0 and the eigenfunctions χ_1 and χ_2 of Eq. (24), in the following form:

$$f_s^0(\mathbf{k}_1,\mathbf{r}_{23})=\int\frac{\exp\left\{-\sqrt{\alpha_0^2+3k_1^2/4}\,|\mathbf{r}_{23}-\mathbf{r}|\right\}}{|\mathbf{r}_{23}-\mathbf{r}|}\left\{\frac{1}{2}V_s(r)\varphi_1(r)\chi_1(k_1)\right. \tag{25}$$
$$\left.+\frac{1}{2}V_t(r)\varphi_2(r)\chi_2(k_1)\right\}d\mathbf{r}+8\pi\int\frac{\cos(\mathbf{r}_{23},\mathbf{k}_1/2+\mathbf{k})}{\alpha_0^2+k_1^2+k^2+\mathbf{k}_1\mathbf{k}}\cos(\mathbf{r},\mathbf{k}_1+\mathbf{k}/2)$$
$$\times\left\{\frac{1}{2}V_s(r)\varphi_1(r)\chi_1(k)+\frac{1}{2}V_t(r)\varphi_2(r)\chi_2(k)\right\}d\mathbf{r}\frac{d\mathbf{k}}{(2\pi)^3};\quad f_a^0(\mathbf{k}_1,\mathbf{r}_{23})\equiv 0;$$
$$f_2^0(\mathbf{k}_1,\mathbf{r}_{23})=\int\frac{\exp\left\{-\sqrt{\alpha_0^2+3k_1^2/4}\,|\mathbf{r}_{23}-\mathbf{r}|\right\}}{|\mathbf{r}_{23}-\mathbf{r}|}\left\{\frac{1}{2}V_s(r)\varphi_1(r)\chi_1(k_1)\right.$$
$$\left.-\frac{1}{2}V_t(r)\varphi_2(r)\chi_2(k_1)\right\}d\mathbf{r}+8\pi\int\frac{\cos(\mathbf{r}_{23},\mathbf{k}_1/2+\mathbf{k})}{\alpha_0^2+k_1^2+k^2+\mathbf{k}_1\mathbf{k}}\cos(\mathbf{r},\mathbf{k}_1+\mathbf{k}/2)$$
$$\times\left\{\frac{1}{4}V_t(r)\varphi_2(r)\chi_2(k)-\frac{1}{4}V_s(r)\varphi_1(r)\chi_1(k)\right\}d\mathbf{r}\frac{d\mathbf{k}}{(2\pi)^3};$$
$$f_1^0(\mathbf{k}_1,\mathbf{r}_{23})=8\pi i\int\frac{\sin(\mathbf{r}_{23},\mathbf{k}_1/2+\mathbf{k})}{\alpha_0^2+k_1^2+k^2+\mathbf{k}_1\mathbf{k}}\cos(\mathbf{r},\mathbf{k}_1+\mathbf{k}/2)$$
$$\times\left\{\frac{\sqrt{3}}{4}V_t(r)\varphi_2(r)\chi_2(k)-\frac{\sqrt{3}}{4}V_s(r)\varphi_1(r)\chi_1(k)\right\}d\mathbf{r}\frac{d\mathbf{k}}{(2\pi)^3}.$$

To find the successive approximations, one can use a procedure which is completely analogous to that which was presented for finding higher approximations for the eigenvalue and the wave function in the case of three identical particles.

In conclusion I express my deep gratitude to K. A. Ter-Martirosian for his direction and constant interest in the work. I also sincerely thank Academician L. D. Landau, O. B. Firsov and I. M. Shmushkevich for helpful discussions.

[1] G. V. Skorniakov and K. A. Ter-Martirosian, Dokl. Akad. Nauk SSSR **106**, 425 (1956); J. Exptl. Theoret. Phys. (USSR) **31**, 775 (1956); Soviet Phys. JETP **4**, **648** (**1957**).

[2] L. H. Thomas, Phys. Rev. **47**, 903 (1935).

[3] A. Akhiezer and I. Pomeranchuk, *Some Problems of Nuclear Theory*, Moscow, 1950.

[4] M. Verde, Helv. Phys. Acta. **22**, 339 (1949).

Translated by M. Hamermesh
219